Existence of a free boundary in a two-dimensional problem modeling the magnetic confinement of a plasma.

J.I. Díaz[1,2] M.B. Lerena[1]

Abstract

We study sufficient conditions for the existence of a free boundary in a two-dimensional elliptic problem modeling the magnetic confinement of a plasma in a Stellarator configuration. The free boundary represents the separation between the plasma and the vacuum region. We use some properties of the relative rearrangement and some comparison principles.

Introduction.

The magnetic confinement of a plasma can be modeled from the ideal magnetohydrodynamics static system with the help of averaging methods and a special coordinates system: the *Boozer vacuum coordinates system* (ρ, θ, ϕ) (see [1], [8]). By averaging in one of the coordinates of the above system and adding a free boundary formulation, the problem (of inverse type) can be stated in the following terms (see [3]): let Ω be an open bounded and regular set of $\mathbb{R}^2$, and let

$$\lambda > 0,\ k_0 > 0,\ F_v > 0,\ a,\ b \in L^\infty(\Omega),\ b > 0 \text{ almost everywhere in } \Omega.$$

Given $\sigma > 1$, find $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\Phi \in \mathcal{C}^0(\mathbb{R}; [0, \infty))$ such that $\Phi(s) = F_v$ for any $s \geq 1$, $\Phi^2 \in W^{1,\infty}_{loc}(\mathbb{R})$ and (ψ, Φ) satisfy

$$(\mathcal{P}^I) \begin{cases} \mathcal{L}\psi = \dfrac{a(x)}{k_0}\Phi(\psi) - \dfrac{1}{k_0^2}\Phi(\psi)\Phi'(\psi) + b(x)\lambda(1-\psi)_+ & \text{in } \Omega, \\ \psi = \sigma & \text{on } \partial\Omega, \\ \int_{\{\psi \leq t\}} \left[\Phi(\psi)\Phi'(\psi) - k_0^2 \lambda b (1-\psi)_+\right](x)dx = 0 & t \in [0, \sup_\Omega \psi]. \end{cases} \tag{1}$$

Here $\mathcal{L}$ is a suitable second order elliptic operator (see [8]) given by

$$\mathcal{L}\psi := \frac{1}{\rho}\{\frac{\partial}{\partial\rho}(a_{\rho\rho}\frac{\partial\psi}{\partial\rho}) + \frac{\partial}{\partial\rho}(a_{\rho\theta}\frac{\partial\psi}{\partial\theta}) + \frac{\partial}{\partial\theta}(a_{\theta\rho}\frac{\partial\psi}{\partial\rho}) + \frac{\partial}{\partial\theta}(a_{\theta\theta}\frac{\partial\psi}{\partial\theta})\} \tag{2}$$

with

$$\begin{aligned} a_{\rho\rho}(\rho,\theta) &:= \rho < g^{\rho\rho} > (\rho,\theta), \\ a_{\rho\theta}(\rho,\theta) &= a_{\theta\rho}(\rho,\theta) :=< g^{\rho\theta} > (\rho,\theta), \\ a_{\theta\theta}(\rho,\theta) &:= \tfrac{1}{\rho} < g^{\theta\theta} > (\rho,\theta), \end{aligned} \tag{3}$$

and where $< g^{i,j} >$, $i, j = \rho, \theta$ are the averaged components of the Riemannian metric associated to the Boozer coordinates system ([1], [4]). For the sake of simplicity in the exposition we shall assume that $\mathcal{L} = \Delta$ (the Laplace operator) and we replace (ρ, θ) by the associated Cartesian coordinate $x \in \Omega \subset \mathbb{R}^2$.
In order to determinate the unknown Φ, the above problem can be reformulated (see [3]) using the notion of *relative rearrangement* of a function. It was proved ([7]) that if $(\psi,\ \Phi)$ is a solution of $(\mathcal{P}^I)$ such that $\psi \in \mathcal{U} \subset C^0(\Omega)$, where

$$\mathcal{U} = \left\{ \psi \in W^{2,p}(\Omega),\ \text{for any } 1 \le p < \infty \text{ and meas}\{x \in \Omega : \nabla\psi(x) = 0\} = 0 \right\},$$

then ψ satisfies the following *non local* problem

$$(\mathcal{P}^{NL}) \begin{cases} \Delta\psi = \dfrac{a(x)}{k_0}\left[F_v^2 - \dfrac{\lambda}{k_0}\displaystyle\int_{|\psi<1-[1-\psi(x)]_+|}^{|\psi<1|} (1-\psi^*(s))_+ b_\psi^*(s)ds\right]_+^{1/2} \\ +\dfrac{\lambda}{k_0}(1-\psi)_+\left[b(x) - b_\psi^*(|\psi < \psi(x)|)\right] & \text{in } \Omega, \\ \psi - \sigma \ \in \ H_0^1(\Omega), \end{cases} \tag{4}$$

and necessarily $\Phi = \mathcal{F}_\psi$ on $\left(-\infty, \left|\psi_+\right|_{L^\infty(\Omega)}\right]$, with

$$\mathcal{F}_\psi(t) := \left[F_v^2 - \frac{\lambda}{k_0}\int_{(1-t)_+}^{1} (1-s)_+ b_\psi^*(|\psi < s|)ds\right]_+^{1/2}, \tag{5}$$

where $|\psi < t|$ denotes meas$\{x \in \Omega : \psi < t\}$, ψ^* represents the increasing rearrangement of ψ and b_ψ^* is the relative rearrangement of b with respect to ψ (see [9]).

Conditions for the existence of a free boundary.

In this section we study some criteria for the formation of the free boundary for problem $(\mathcal{P}^I)$. We establish a new criterion and give an estimate of the size and spacial localization of the set of points where the flux function ψ is bigger than 1.
The free boundary represents the separation between the plasma and the vacuum regions, $\Omega_p = \{\psi < 1\}$and $\Omega_v = \{\psi > 1\}$ respectively, and it is defined as the boundary of the set Ω_p. As ψ is greater than 1 on $\partial\Omega$, the existence of this free boundary is reduced to the study of conditions for which the set Ω_p is not empty. From the physical point of view, this study is equivalent to find a range of parameters (Ω, F_v, σ and λ) for which there exists an identification between the mathematical model and the physics problem.
We start by studying the non existence of plasma case.

Lemma 1 *Let $B \subset \mathbb{R}^2$ be an open ball with center the origin and radius R, and $F_v > 0$, $\sigma > 1$ two constants given. Assume $\hat{a} \in L^\infty(B)$, such that $\hat{a}(x) = \hat{a}(|x|)$ mboxa.e.$x \in B$. Let $\Psi \in W^{2,p}(B)$ (1$\le p < \infty$) be the unique solution of*

$$\begin{cases} \Delta\Psi = \dfrac{\hat{a}}{k_0}F_v & \text{in } B, \\ \Psi = \sigma & \text{on } \partial B. \end{cases} \tag{6}$$

Then

$$\Psi(0) = 1 \text{ (respectively } \Psi(0) > 1, \text{ or } \Psi(0) < 1), \tag{7}$$

if

$$\int_0^R \left(\frac{1}{\xi} \int_0^\xi \frac{\hat{a}(\tau)}{k_0} \tau d\tau \right) d\xi - \frac{\sigma - 1}{F_v} = 0 \text{ (resp. } < 0, \text{ or } > 0). \tag{8}$$

Moreover, if $\int_0^r \tau \hat{a}(\tau) d\tau \geq 0$ *for all* $r \in (0, R]$, *then* Ψ *is increasing along the radius* $r = |x|$.

Proof. The existence, uniqueness and regularity of Ψ are well known. Moreover, this solution has necessarily radial symmetry because Ψ can be taken as $\Psi(x) = Y(r)$ with $r = |x|$ and $Y(r)$ being the unique solution of the boundary problem

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial Y}{\partial r} \right) = \dfrac{\hat{a}(r)}{k_0} F_v \quad 0 < r < R, \\ Y(R) = \sigma, \quad Y'(0) = 0. \end{cases}$$

Therefore

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Psi}{\partial r} \right) = \frac{\hat{a}(r)}{k_0} F_v \text{ for almost every } r \in (0, R).$$

Integrating the above equation twice in r we get

$$\Psi(r) = \sigma - F_v \int_r^R \left(\frac{1}{\xi} \int_0^\xi \frac{\hat{a}(\tau)}{k_0} \tau d\tau \right) d\xi, \quad r \in [0, R]. \tag{9}$$

The first conclusion of the lemma can be got by substituting $r = 0$. The second one is a consequence of the first integration in r. ■

A first criterion for the existence of a free boundary was obtained in [6] in terms of the first eigenfunction of the Laplace operator. Let $\varphi_1 be$ a normalized eigenfunction associated to the first eigenvalue of the operator $-\Delta$ on Ω with Dirichlet boundary condition, i.e. $-\Delta\varphi_1 = \lambda_1 \varphi_1$ in Ω and $\varphi_1 \in H_0^1(\Omega)$. We know that $\varphi_1 > 0$ on Ω. We renormalizate φ_1 such that $\lambda_1 \int_\Omega \varphi_1 dx = 1$. It follows (see [6]) that if we assume

$$\sigma - 1 < \frac{F_v}{k_0} \int_\Omega a(x) \varphi_1(x) dx, \tag{10}$$

then any solution ψ of $(\mathcal{P}^{NL})$ satisfies $(1 - \psi)_+ \not\equiv 0$.

For the study of the existence and spacial localization of the set Ω_p we need some information of the monotonicity of the function $t \longrightarrow (\Phi^2)'(t)/2$. To this purpose we use the characterization of this function in terms of the relative rearrangement of b ((5), [6])

$$\left(\Phi^2\right)'(t) = 2\lambda k_0 (1 - t)_+ b_\psi^*(|\psi < t|).$$

Using the well-known estimate $\left|\left|b_{\psi}^{*}\right|\right|_{\infty} \leq ||b||_{\infty}$, we conclude that there exists $\underline{b} \leq \overline{b}$ such that

$$2\lambda k_0\underline{b}(1-t)_+ \leq \left(\Phi^2\right)'(t) \leq 2\lambda k_0\overline{b}(1-t)_+, \text{ for almost every } t \in \left(-\infty, |\psi|_{L^\infty(\Omega)}\right).$$

Theorem 2 *Assume* $\inf_\Omega a > 0$. *Then, there exists a positive constant* δ *such that if* $0 < \lambda < \delta$ *we get that*

$$\Omega_p \supset \left\{ x \in \Omega : \mathrm{d}(x, \partial\Omega) \geq \left(\frac{4k_0(\sigma-1)}{F_v \inf_\Omega a}\right)^{1/2} \right\}, \tag{11}$$

where d *denotes the Euclidean distance.*

Proof. Let $B_0 = B(x_0, R_0) \subset \Omega$ be an open ball of radius $R_0 = \left(\dfrac{4k_0(\sigma-1)}{F_v \inf_\Omega a}\right)^{1/2}$ and center x_0, for some $x_0 \in \Omega$.

Assume $\partial B_0 \cap \partial\Omega \neq \emptyset$. We get

$$\Delta\psi + f(x,\psi) = -\frac{1}{k_0^2}\left(\frac{\Phi^2}{2}\right)'(\psi) + \lambda b(x)(1-\psi)_+ + \lambda\overline{b}(1-\psi)_+ \geq 0 \text{ in } B,$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is the function given by $f(x,\psi) = -\dfrac{a(x)}{k_0}\Phi(\psi) + \lambda\overline{b}(1-\psi)_+$. This function is non increasing in ψ, due to the fact that $\Phi(\psi)$ is non decreasing in ψ. Moreover, we have

$$\psi \leq \sigma \text{ in } \Omega. \tag{12}$$

To prove (12) we multiply the equation of (1) by $(\psi-\sigma)_+$. Integrating by parts in Ω we get

$$\begin{aligned} -\int_\Omega \left|\nabla(u-\sigma)_+\right|^2 dx &= \int_\Omega \left[\frac{a}{k_0}\Phi(\psi) - \frac{1}{k_0^2}\left(\frac{\Phi^2}{2}\right)'(\psi) + \lambda b(1-\psi)_+\right](\psi-\sigma)_+ dx = \\ &= \int_{\psi>\sigma} \frac{a}{k_0}F_v dx \geq 0, \end{aligned}$$

since $\sigma > 1$ and so $\dfrac{a}{k_0}\Phi(\psi) - \dfrac{1}{k_0^2}\left(\dfrac{\Phi^2}{2}\right)'(\psi) + \lambda b(1-\psi)_+ = \dfrac{a}{k_0}F_v$ in $\{\psi > \sigma\}$. Thus, we get that $(\psi-\sigma)_+$ is constant on $\overline{\Omega}$ and as $\psi = \sigma$ on $\partial\Omega$, that constant must be null, that is, $\psi \leq \sigma$ a.e. $x \in \Omega$.

Now, we consider the solution Ψ of problem (12) when we take $\hat{a} \equiv \inf_\Omega a$ and $B = B(x_0, R_0) = B_0$. That is, Ψ is the solution of the problem

$$\begin{cases} \Delta\Psi = \dfrac{\inf_\Omega a}{k_0}F_v & \text{in } B_0, \\ \Psi = \sigma & \text{on } \partial B_0, \end{cases}$$

and due to the choice of $R = R_0$ by lemma 1 we know that $\Psi(x) > 1$ $\forall x \in B\setminus\{x_0\}$ and in particular $f(x,\Psi) = -\dfrac{a(x)}{k_0}F_v$. Hence

$$\begin{cases} \Delta\Psi + f(x,\Psi) \leq 0 \leq \Delta\psi + f(x,\psi) & \text{in } B, \\ \Psi = \sigma > \psi & \text{on } \partial B. \end{cases}$$

Then, by the weak maximum principle ([7]), we conclude that

$$\psi \leq \Psi \text{ in } \overline{B}.$$

In fact we will show that this inequality is strict in B, and so $\psi(x_0) < \Psi(x_0) = 1$. To prove this fact we define $w := \Psi - \psi \geq 0$ in B. Due to the hypotesis on λ we know that Φ is Lipschitz continuous ([4]). Therefore $f(x,\psi)$ is also Lipschitz continuous in the second variable. Then we arrive to

$$\begin{cases} \Delta w \leq |f(x,\psi) - f(x,\Psi)| \leq Cw & \text{in } B, \\ w \geq 0 & \text{on } \partial B. \end{cases}$$

for some positive constant C. By the Hopf strong maximum principle ([7]), we get that or $w \equiv k$ for some constant $k \geq 0$, or $w > 0$ in B.
The conclusion of the theorem follows trivially when the above k is strictly positive. But, the constant k cannot vanish. In fact, by the strong maximum principle we deduce that

$$\psi < \sigma \text{ if } x \in \Omega \text{ and } \mathrm{d}(x,\partial\Omega) \text{ is small enough.} \tag{13}$$

To see this we use that $\psi \in C^0(\overline{\Omega})$, and thus there exists a neighborhood Ω_{ε_0} of $\partial\Omega$ such that $\sigma > \psi(x) > 1 + \varepsilon_0$, $\forall x \in \Omega_{\varepsilon_0}$. In this case we have

$$\begin{cases} \Delta\psi = \dfrac{a(x)}{k_0}F_v & \text{in } \Omega_{\varepsilon_0}, \\ \psi = \sigma & \text{on } \partial\Omega, \\ \psi = 1 + \varepsilon_0 < \sigma & \text{on } \partial\Omega_{\varepsilon_0}\setminus\partial\Omega, \end{cases}$$

and again, by an application of the strong maximum principle we deduce that $\dfrac{\partial\psi}{\partial n} > 0$ on $\partial\Omega$, and from this it follows (13). If $\partial\Omega \cap \partial B \neq \emptyset$, we consider the set

$$\Omega_{\sigma-\varepsilon} := \{x \in \Omega : \psi(x) < \sigma\} \subset \Omega,$$

which is not empty. Reproducing the above arguments, now with $x_0 \in \Omega_{\sigma-\varepsilon}$, we get the conclusion of the theorem. ∎

Remark 3 *The conclusion obtained in the above theorem is optimal in the sense that if we assume* $\left\{x \in \Omega : \mathrm{d}(x,\partial\Omega) \geq \left(\frac{4k_0(\sigma-1)}{F_v \inf_\Omega a}\right)^{1/2}\right\} = \emptyset$ *and* $\lambda = 0$, *then we get* $\Omega_p = \emptyset$ *(Lemma 1). In fact, using some rearrangement comparisons ([2]), the above affirmation holds when* Ω *is an open no necessarily symmetric set such that* $|\Omega| < \left(\frac{4k_0(\sigma-1)}{F_v \inf_\Omega a}\right)^{1/2}$.

Remark 4 *The criterion for the formation of the free boundary obtained in Theorem 3 improves the obtained in [6] in the sense that if we take $\Omega = B_R$, the coefficient $a \equiv cte > 0$ and $\lambda > 0$, then the hypotesis (10) is more restrictive than that of Theorem 3.*
The first eigenfunction of Δ in B_R is given in terms of the Bessel function of order zero, by $\varphi_1 = J_0(\sqrt{\lambda_1} r)$, where λ_1 is the first zero of this function. Thus, (10) becomes

$$k_0(\sigma - 1) < \frac{aF_v}{\lambda_1}. \tag{14}$$

Using some well-known estimates for λ_1, $(\frac{3\pi}{4R})^2 \leq \lambda_1 \leq (\frac{7\pi}{8R})^2$ (see Watson [10]). Then we have that (14) implies that necessarily

$$R^2 > 5.5517 \left(\frac{k_0(\sigma - 1)}{aF_v} \right).$$

Nevertheless, the assumtions for the existence of the free boundary in Theorem 2 is expressed as

$$R^2 \geq 4 \left(\frac{k_0(\sigma - 1)}{aF_v} \right). \tag{15}$$

The operator $\mathcal{L}$ with radial symmetry.

In this section we will study the criterion obtained in the above section for the operator $\mathcal{L}$ when Ω is a ball of radius R, i.e. $\Omega = B_R$. In the case of radial symmetry we have $\dfrac{\partial u}{\partial \theta} = 0$, and thus

$$\mathcal{L}u := \frac{1}{\rho}\{\frac{\partial}{\partial \rho}(a_{\rho\rho}\frac{\partial u}{\partial \rho}) + \frac{\partial a_{\theta\rho}}{\partial \theta}(\frac{\partial u}{\partial \rho})\}.$$

Replacing the expression for the coefficients $a_{\rho\rho}$ and $a_{\theta\rho}$ (3), and assuming

$$\begin{aligned} &< \ g^{\rho\rho} > (\rho, \theta) \equiv cte > 0, \\ &< \ g^{\rho\theta} > (\rho, \theta) = < g^{\rho\theta} > (\rho), \end{aligned} \tag{16}$$

we arrive to

$$\mathcal{L}\psi = \frac{< g^{\rho\rho} >}{\rho}\frac{\partial}{\partial \rho}(\rho\frac{\partial \psi}{\partial \rho}).$$

Theorem 5 *Let assume (16) and $\inf_\Omega a > 0$. There exists a positive constant δ such that if $0 < \lambda < \delta$, then*

$$\Omega_p \supset \left\{ x \in \Omega : \mathrm{d}(x, \partial\Omega) \geq \left(\frac{4 < g^{\rho\rho} > (\sigma - 1)}{F_v \inf_\Omega a} \right)^{1/2} \right\},$$

where d *denotes the Euclidean distance.*

Proof. The proof is analogous to the one of Theorem 2 but now we take $R_0 = \left(\frac{4<g^{\rho\rho}>(\sigma-1)}{F_v \inf_\Omega a}\right)^{1/2}$ and as a supersolution we take the function $\widehat{\Psi}$, that is the solution of the problem

$$\begin{cases} \Delta\widehat{\Psi} = \dfrac{\inf_\Omega a}{< g^{\rho\rho} > k_0} F_v & \text{in } B_0, \\ \widehat{\Psi} = \sigma & \text{on } \partial B_0, \end{cases}$$

where $B_0 = B(x_0, R_0)$. Now, by the choice of R_0 and lemma 1 we get again that $\widehat{\Psi}(x) > 1 \ \forall x_0 \in B_0 \backslash \{x_0\}$. ∎

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1. Dpto. de Matemática Aplicada. Universidad Complutense de Madrid. Av. Complutense s/n, 28040 Madrid.

2. Proyect PB96/0583 de la DGES.