On the formation of the free boundary for the obstacle and Stefan problems via an energy method

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The main goal of this communication is to extend the, so called, *energy method*, developed since the beginning of the eighties for the study of the free boundaries giving rise by the solutions of nonlinear pdes (see, e.g. the recent monograph [1] by S.N. Antontsev, J.I. Díaz and S.I. Shmarev to the case of multivalued equations, as it is the case, for instance, of *the obstacle problem* and *the Stefan problem.* Let us mention that the study of the qualitative behavior of the coincidence set for the obstacle problem was initiated in 1976 by Brezis and Friedman [3] (see also Tartar [10] and Evans and Knerr [6]) by using the maximum principle. Some general references on the Stefan problem are Friedman [7] and Meirmanov [8]. The energy methods are of special interest in the situations where the traditional methods based on the comparison principles fail. A typical example of such a situation is either a higher-order equation or a system of PDEs. Moreover, even when the comparison principle holds, it may be extremely difficult to construct suitable sub or super-solutions if, for instance, the equation under study contains transport terms and has either variable or unbounded coefficients or the right-hand side.

Here we shall deal with the formation of a free boundary for local solutions of the obstacle problem

$$\frac{\partial \psi(u)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,u,D\,u) + B(x,t,u,D\,u) + C(x,t,u) + \beta(u) \ni f(x,t), \qquad (1)$$

where $\beta(u)$ is the maximal monotone graph given by $\beta(u) = \{0\}$ if $u \leq 0$ and $\beta(u) = \phi$ (the empty set) if $u < 0$. The general structural assumptions we shall made are the following

$$|\mathbf{A}(x,t,r,\mathbf{q})| \leq C_1 |\mathbf{q}|^{p-1}, C_2\,|\mathbf{q}|^p \leq \mathbf{A}(x,t,r,\mathbf{q})\cdot\mathbf{q}, \qquad (2a)$$

$$|B(x,t,r,\mathbf{q})| \leq C_3\,|r|^{\alpha}\,|\mathbf{q}|^{\beta},\ \ 0 \leq C(x,t,r)\,r, \qquad (2b)$$

$$C_6\,|r|^{\gamma+1} \leq G(r) \leq C_5\,|r|^{\gamma+1},\ \text{where} \qquad (2c)$$

$$G(r) = \psi(r)\,r - \int_0^r \psi(\tau)\,d\tau.$$

Here $C_1 - C_6$, p, α, β, σ, γ, k are positive constants which will be specified later on. We shall also consider the Stefan problem

$$\frac{\partial \psi(u)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,u,D\,u) + B(x,t,u,D\,u) + C(x,t,u) \ni f(x,t), \tag{3}$$

where now $\psi(u)$ is the maximal monotone graph $\psi(u) = k_+ u + L$ if $u > 0$, $\psi(u) = k_- u$ if $u < 0$, $\psi(0) = [0, L]$, with k_+, k_- and L positive constants. In both cases, we shall deal with weak solutions satisfying the initial condition

$$u(x,0) = u_0(x) \qquad x \in \Omega. \tag{4}$$

Let us start by considering the obstacle problem.

Definition. A function $u(x,t)$, with $\psi(u) \in C\left([0,T] : L^1_{loc}(\Omega)\right)$, is called weak solution of problem (1), (4) if $u \in L^\infty(0,T;\, L^{\gamma+1}(\Omega')) \cap L^p(0,T;\, W^{1,p}(\Omega'))$, $\overline{\Omega'} \subset \Omega$, $\mathbf{A}(\cdot,\cdot,u,Du)$, $B(\cdot,\cdot,u,Du)$, $C(\cdot,\cdot,u) \in L^1(Q)$; $\liminf_{t\to 0} G(u(\cdot,t)) = G(u_0)$ in $L^1(\Omega)$; $u(x,t) \geq 0$ and $c(x,t) \in \beta(u(t,x))$ a.e. $(t,x) \in (0,T) \times \Omega$ for some $c \in L^1((0,T)\times\Omega)$, and for every test function $\varphi \in L^\infty\left(0,T; W_0^{1,p}(\Omega)\right) \cap W^{1,2}(0,T; L^\infty(\Omega))$,

$$\int_Q \{\psi(u)\varphi_t - \mathbf{A}\cdot D\varphi - B\,\varphi - C\,\varphi - c\varphi\}\, dxdt - \int_\Omega \psi(u)\varphi\, dx\Big|_{t=0}^{t=T} = -\int_Q f\,\varphi\, dx\, dt. \tag{5}$$

In contrast to considerations on the *finite speed of propagation* or the *uniform localization of the support*, we shall use some energy functions defined on domains of a special form. Let us introduce the following notation: given $x_0 \in \Omega$ and the nonnegative parameters ϑ and υ, we define the *energy set*

$$P(t) \equiv P(t;\vartheta,\upsilon) = \{(x,s) \in Q : |x - x_0| < \rho(s) \equiv \vartheta(s-t)^\upsilon, s \in (t,T)\}.$$

The shape of $P(t)$, *the local energy set,* is determined by the choice of the parameters ϑ and υ. Here we shall take $\vartheta > 0$, $0 < \upsilon < 1$ and so $P(t)$ becomes a paraboloid (other choices are relevant for the study of different properties: see [1]). We define the *local energy functions*

$$E(P) := \int_{P(t)} |Du(x,\tau)|^p\, dxd\tau, \quad C(P) := \int_{P(t)} |u(x,\tau)|\, dxd\tau$$

$$b(T) := ess \sup_{s\in(t,T)} \int_{|x-x_0|<\vartheta(s-t)^\upsilon} |u(x,s)|^{\gamma+1}\, dx.$$

Although our results have a local nature (for instance, they are independent of the boundary conditions), we shall need some global information on *the global energy function*

$$D(u(\cdot,\cdot)) := ess \sup_{s\in(0,T)} \int_\Omega |u(x,s)|^{\gamma+1}\, dx + \int_Q (|Du|^p + |u|)\, dxdt. \tag{6}$$

For the sake of the exposition, we shall assume the additional condition $\frac{p-1}{p} \leq \gamma \leq p-1$. Our main assumption deals with the forcing term: we assume that there exists $\Theta > 0$ and $\rho > 0$ such that

$$f(x,t) < -\Theta \text{ on } B_\rho(x_0) \subset \Omega, \text{ a.e. } t \in (0,T). \tag{7}$$

In the presence of the first order term, $B(\cdot,\cdot,u,Du)$, we shall need the extra conditions

$$\left\{ \begin{array}{c} \alpha = \gamma - (1+\gamma)\beta/p, \\ C_3 < \left(\Theta \frac{p}{p-1}\right)^{(p-\beta)/p} \left(C_2 \frac{p}{\beta}\right)^{\beta/p} \text{ if } 0 < \beta < p, \\ C_3 < \Theta \text{ if } \beta = 0 \text{ (respectively } \Theta < C_2 \text{ if } \beta = p). \end{array} \right. \tag{8}$$

The next result shows how the multivalued term causes the formation of the null-set of the solution, even for positive initial data.

Theorem 1 *There exist some positive constants M, t^*, and $\upsilon \in (0,1)$ such that any weak solution of problem* (1), (4) *with $D(u) \leq M$ satisfies that $u(x,t) \equiv 0$ in $P(t^*:1,\upsilon)$.*

In the case of the Stefan problem a definition of weak solution can be given in similar terms but the integral identity reads now as follows:

$$\int_Q \{\psi_u \varphi_t - \mathbf{A} \cdot D\varphi - B\,\varphi - C\,\varphi\}\, dxdt - \int_\Omega \psi(u)\varphi\, dx \Big|_{t=0}^{t=T} = -\int_Q f\,\varphi\, dx\, dt, \tag{9}$$

for some $\psi_u \in L^1((0,T)\times\Omega)$, $\psi_u(x,t) \in \psi(u(t,x))$ a.e.$(t,x) \in (0,T)\times\Omega$. To simplify the exposition we shall assume now that $\mathbf{A}$ and B are independent of u.

Theorem 2 *Assume that $f(x,t) < -\Theta$ (respect. $f(x,t) > \Theta$) on $B_\rho(x_0) \subset \Omega$, a.e. $t \in (0,T)$. Then there exist some positive constants M, t^*, and $\upsilon \in (0,1)$ such that any weak solution of problem* (3), (4) *with $D(u) \leq M$ satisfies that $u(x,t) \leq 0$ (respect. $u(x,t) \geq 0$) in $P(t^*:1,\upsilon)$.*

The proof of Theorem 1 consists of several parts: Step 1. *The integration-by-parts formula:*

$$\begin{array}{c} i_1 + i_2 + i_3 + i_4 = \int_{P\cap\{t=T\}} G(u(x,t))dx \\ + \int_P \mathbf{A}\cdot Du\, dxd\theta + \int_P B\, udxd\theta + (\int_P C\, udxd\theta - \int_P uf\,dxd\theta) \\ \leq \int_{\partial_l P} n_x \cdot \mathbf{A}\, u\, d\Gamma d\theta + \int_{\partial_l P} n_\tau G(u(x,t))d\Gamma d\theta \\ + \int_{P\cap\{t=0\}} G(u(x,t))dx + := j_1 + j_2 + j_3, \end{array}$$

were $\partial_l P$ denotes the lateral boundary of P i.e. $\partial_l P = \{(x,s) : |x - x_0| = \vartheta(s-t)^\upsilon,\ s \in (t,T)\}$, $d\Gamma$ is the differential form on the hypersurface $\partial_l P \cap \{t = const\}$, n_x and n_τ are the components of the unit normal vector to $\partial_l P$. This inequality can be proved by taking the cutting function

$$\zeta(x,\theta) := \psi_\varepsilon(|x-x_0|,\theta)\, \xi_k(\theta) \frac{1}{h} \int_\theta^{\theta+h} T_m(u(x,s))\, ds, \quad h > 0,$$

as test function,where T_m is the truncation at the level m,

$$\xi_k(\theta) := \begin{cases} 1 & \text{if } \theta \in [t, T - \frac{1}{k}], \\ k(T-\theta) & \text{for } \theta \in [T-\frac{1}{k}, T], \\ 0 & \text{otherwise}, \quad k \in \mathbb{N}, \end{cases} \qquad \psi_\varepsilon(|x - x_0|, \theta) := \begin{cases} 1 & \text{if } d > \varepsilon, \\ \frac{1}{\varepsilon} d & \text{if } d < \varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

with $d = dist((x,\theta), \partial_l P(t))$ and $\varepsilon > 0$. So, $supp\zeta(x,\theta) \equiv P(t)$, $\zeta, \frac{\partial \zeta}{\partial t} \in L^\infty((0,T)\times\Omega)$ and $\frac{\partial \zeta}{\partial x_i} \in L^p((0,T)\times\Omega)$. Using the monotonicity of β and passing to the limits we get the inequality.

Step 2. *A differential inequality for some energy function.* We assume choice P such that it does not touch the initial plane $\{t = 0\}$ and $P \subset B_\rho(x_0) \times [0,T]$. Then $i_1 + i_2 + i_3 \le j_1 + j_2$. In order to estimate j_1, let us mention that $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_\tau) = \frac{1}{(\vartheta^2 \upsilon^2 + (\theta - t)^{2(1-\upsilon)})^{1/2}}((\theta - t)^{1-\upsilon}\mathbf{e}_x - \upsilon \mathbf{e}_\tau)$ with $\mathbf{e}_x, \mathbf{e}_\tau$ orthogonal unit vectors to the hyperplane $t = 0$ and the axis t, respectively. Then, if we denotes by (ρ,ω), $\rho \ge 0$ and $\omega \in \partial B_1$ the spherical coordinate system in $\mathbb{R}^N$, if $\Phi(\rho,\omega,\theta)$ is the spherical representation of a general function $F(x,t)$, we have

$$I(t) := \int_P F(x,\theta)dxd\theta \equiv \int_t^T d\theta \int_0^{\rho(\theta,t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J|d\omega,$$

where J is the Jacobi matrix and $\rho(\theta,t) = \vartheta(\theta - t)^\upsilon$. So,

$$\begin{aligned} \frac{dI(t)}{dt} &= -\int_0^{\rho(\theta,t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J|d\omega \Bigg|_{\theta = t} \\ &\quad + \int_t^T \rho_t \rho^{N-1} d\theta \int_{\partial B_1} \Phi(\rho,\omega,t)|J|d\omega = \int_{\partial_l P} \rho_t F(x,\theta) d\Gamma d\theta. \end{aligned} \tag{10}$$

Then, by Hölder's inequality, we get

$$\begin{aligned} \left| \int_{\partial_l P} n_x \cdot \mathbf{A}\, u\, d\Gamma d\theta \right| &\le M_2 \int_{\partial_l P} |\, n_x||\nabla u|^{p-1}|u| d\Gamma d\theta \\ &\le M_2 \left(\int_{\partial_l P} |\rho_t||\nabla u|^p d\Gamma d\theta \right)^{(p-1)/p} \left(\int_{\partial_l P} \frac{|\, n_x|^p}{|\rho_t|^{p-1}} |u|^p d\Gamma d\theta \right)^{1/p} \\ &= M_2 \left(-\frac{dE}{dt} \right)^{(p-1)/p} \left(\int_t^T \frac{|\, n_x|^p}{|\rho_t|^{p-1}} \left(\int_{\partial B_{\rho(\theta,t)}} |u|^p d\Gamma \right) d\theta \right)^{1/p}. \end{aligned} \tag{11}$$

To estimate the right-hand side of (11) we use the interpolation inequality ([5]) : if $0 \le \sigma \le p - 1$, then there exists $L_0 > 0$ such that $\forall v \in W^{1,p}(B_\rho)$

$$\|v\|_{p,S_\rho} \le L_0 \left(\|\nabla v\|_{p,B_\rho} + \rho^\delta \|v\|_{\sigma+1,B_\rho} \right)^{\tilde\theta} \cdot \left(\|v\|_{r,B_\rho} \right)^{1-\tilde\theta} \tag{12}$$

$r \in [1, 1+\gamma]$, $\tilde\theta = \frac{pN - r(N-1)}{(N+1)p - Nr}$, $\delta = -\left(1 + \frac{p-1-\sigma}{p(1+\sigma)} N\right)$. In our case, we shall apply it to the limit case $\sigma = 0$. By Hölder's inequality

$$\left(\int_{B_\rho} |u|^r dx \right)^{1/r} \le \left(\int_{B_\rho} |u| dx \right)^{1/qr} \cdot \left(\int_{B_\rho} |u|^{\gamma+1} dx \right)^{(q-1)/qr},$$

with $q = \frac{\gamma}{\gamma - r + 1}$. Then

$$\int_{\partial B_\rho} |u|^p d\Gamma \le L_0 \left(\left(\int_{B_\rho} |\nabla u|^p + \rho^{\delta p} \left(\int_{B_\rho} |u| \right)^{p/2} \right)^{\tilde\theta} \times \left(\int_{B_\rho} |u|^r \right)^{p(1-\tilde\theta)/r}\right.$$

$$\le L_0 \rho^{\delta\tilde\theta p} \left(\int_{B_\rho} |\nabla u|^p + \int_{B_\rho} |u| \right)^{\tilde\theta} \times \left(\int_{B_\rho} |u| \right)^{p(1-\tilde\theta)/qr} \left(\int_{B_\rho} |u|^{\gamma+1} \right)^{p(q-1)(1-\tilde\theta)/qr}$$

$$\le K\rho^{\delta\tilde\theta p} \left(E_* + C_*\right)^{\tilde\theta} C_*^{(1-\tilde\theta)p/qr} b^{(q-1)(1-\tilde\theta)p/qr}$$

$$\le K\rho^{\delta\tilde\theta p} \left(E_* + C_*\right)^{\tilde\theta + (1-\tilde\theta)p/qr} b^{(q-1)(1-\tilde\theta)p/qr}, \tag{13}$$

where $E_*(t, \rho) := \int_{B_\rho} |\nabla u|^p dx$, $C_*(t, \rho) := \int_{B_\rho} |u| dx$ and K is a suitable positive constant. Taking $r \in \left[\frac{p(\gamma+1)}{p+\gamma}, \gamma + 1 \right]$ we get that $\mu = \tilde\theta + p\frac{1-\tilde\theta}{qr} < 1$. Applying once again Hölder's inequality with the exponent μ, we have from (13)

$$|j_1| \le L \left(-\frac{dE}{dt} \right)^{(p-1)/p} \times \left(\int_t^T \frac{|\vec{n}_x|^p}{|\rho_t|^{p-1}} K\rho^{\delta\tilde\theta p} \left(E_* + C_*\right)^{\mu} b^{(q-1)(1-\tilde\theta)p/qr} d\tau \right)^{1/p}$$

$$\le L \left(-\frac{dE}{dt} \right)^{(p-1)/p} b^{(q-1)(1-\tilde\theta)/qr}$$

$$\times \left(\int_t^T \left(E_* + C_*\right) d\tau \right)^{\frac{\mu}{p}} \left(\int_t^T \left(\frac{|\vec{n}_x|^p}{|\rho_t|^{p-1}} \rho^{\delta\tilde\theta p}(\tau) \right)^{\frac{1}{1-\mu}} d\tau \right)^{\frac{1-\mu}{p}}$$

$$\le L\sigma(t) \left(-\frac{d\left(E + C\right)}{dt} \right)^{(p-1)/p} b^{(q-1)(1-\tilde\theta)/qr} \left(E + C\right)^{\frac{\tilde\theta}{p} + \frac{1-\tilde\theta}{qr}}, \tag{14}$$

for a suitable positive constant L. To obtain (14) we have assumed that

$$\sigma(t) := \left(\int_t^T \left(\frac{1}{|\rho_t|^{p-1}} \rho^{\delta\tilde\theta p}(\tau) \right)^{\frac{1}{1-\mu}} d\tau \right)^{\frac{1-\mu}{p}} < \infty$$

which is fulfilled if we choose $\nu \in (0, 1)$ sufficiently small because the condition of convergence of the integral $\sigma(t)$ has the form $(1-\nu)(p-1) + \nu\delta\tilde\theta p > -(1-\tilde\theta)\left(1 - \frac{p}{qr}\right)$. So, we have obtained an estimate of the following type

$$|j_1| \le L_1 \Lambda(t) D(u)^{(q-1)(1-\tilde\theta)/qr - \lambda} \left(E + C + b\right)^{1-\omega+\lambda} \left(-\frac{d(E + C)}{dt} \right)^{(p-1)/p}, \tag{15}$$

where L_1 is a universal positive constant, $D(u)$ is *the total energy* of the solution under investigation, $\lambda \in [0, (q-1)(1-\tilde{\theta})/qr]$ and $\omega := 1 - \frac{\tilde{\theta}}{p} - \frac{1-\tilde{\theta}}{qr} \in \left(1 - \frac{1}{p}, 1\right)$. This allows us to choose λ so that $\frac{p(\omega-\lambda)}{p-1} \in (0,1)$. Let us estimate j_2. Using the expression for n_τ, we have $|j_2| \le C_5 \int_{\partial_l P} |u|^{1+\gamma} d\Gamma d\theta$. We apply then the interpolation inequality (for the limit case $\sigma = 0$)

$$\|v\|_{\gamma+1,\partial B_\rho} \le L_0 \left(\|\nabla v\|_{p,B_\rho} + \rho^{\delta}\|v\|_{\sigma+1,B_\rho}\right)^s \cdot \|v\|_{r,B_\rho}^{1-s} \quad \forall v \in W^{1,p}(B_\rho) \tag{16}$$

with a universal positive constant $L_0 > 0$ and exponents $s = \frac{(\gamma+1)N - r(N-1)}{(N+r)p - Nr} \frac{p}{\gamma+1}$, $r \in [1+\sigma, 1+\gamma]$. Again

$$\begin{aligned}\int_{\partial B_\rho} |u|^{\gamma+1} dx \le L^{1+\gamma} K^{s(\gamma+1)/\tilde{\theta}p} \left(\int_{B_\rho} |\nabla u|^p dx + \int_{B_\rho} |u|^{\sigma+1} dx\right)^{s(\gamma+1)/p} \\ \times \left[\left(\int_{B_\rho} |u|^{\sigma+1} dx\right)^{1/qr} \left(\int_{B_\rho} |u|^{\gamma+1} dx\right)^{(q-1)/qr}\right]^{(1-s)(\gamma+1)}.\end{aligned} \tag{17}$$

Here K is the same as before. Let $\eta = \frac{s(\gamma+1)}{p} + \frac{(1-s)(\gamma+1)}{qr} < 1$, $\pi = \frac{(q-1)(1-s)(\gamma+1)}{qr}$, $\eta + \pi \ge 1$. Then,

$$\begin{aligned}|j_2| &= \left|\int_t^T d\tau \int_{\partial B_{\rho(\tau)}} |u|^{\gamma+1} d\Gamma\right| \\ &\le L\,(b(T))^{\pi} \left(\int_t^T K^{s(\gamma+1)/\tilde{\theta}p} (E_* + C_*)^{\eta} |n_\tau| d\tau\right) \\ &\le L\,(E + C + b(T,\Omega))\,(b(T,\Omega))^{\kappa} \left(\int_t^T \left(K^{s(\gamma+1)/\tilde{\theta}p}\right)^{\varepsilon} d\tau\right)^{1/\varepsilon},\end{aligned} \tag{18}$$

for some $L = L(C_5, L_0)$ and exponents $\kappa := \eta + \pi - 1$, $\quad \varepsilon = 1/(1-\eta)$. Then. we have

$$C_5 \int_{P \cap \{t=T\}} |u|^{1+\gamma} dx + E + C\Theta \le i_1 + i_2 + i_3, \tag{19}$$

$$|i_4| \le \varepsilon C_3 \frac{p-\beta}{p} C(\rho,t) + \frac{\beta C_3}{pC_2} \varepsilon^{-(p-\beta)/\beta} E(\rho,t), \tag{20}$$

$$K\left(\int_{P \cap \{t=T\}} |u|^{1+\gamma} dx + E + C\right) \le i_1 + i_2 + i_3 + i_4, \tag{21}$$

for different positive constants K. Now, assuming $T - t$ and $D(u)$ so small that

$$L\,(b(T,\Omega))^{\kappa} \left(\int_t^T \left(K^{s(\gamma+1)/\tilde{\theta}p}\right)^{\varepsilon} d\tau\right)^{1/\varepsilon} < \frac{K}{2},$$

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