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Chapter 14

DIFFUSIVE ENERGY BALANCE MODELS IN CLIMATOLOGY

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1. Introduction

This paper contains an expanded and updated version of my lecture at the Collège de France, on May 1997, where I collected several results on a diffusive energy balance model given by a nonlinear parabolic problem formulated in the following terms:

$$(P)\begin{cases} u_t - div(|\nabla u|^{p-2}\nabla u) \in QS(x)\beta(u) - \mathcal{G}(u) + f(x,t) & \text{in } (0,T)\times\mathcal{M}, \\ u(x,0) = u_0(x) & \text{in } \mathcal{M}, \end{cases}$$

where $\mathcal{M}$ is a C^∞ two-dimensional compact connectedoriented Riemannian manifold without boundary. We assume $T > 0$ arbitrarily fixed, $Q > 0$, $S \in L^\infty(\mathcal{M})$ and $p \geq 2$. The function $\mathcal{G}$ is increasing and β represents a bounded maximal monotone graph in $\mathbb{R}^2$ (of Heaviside type). We also consider the associate stationary problem

$$(P_{Q,f}) \qquad -div(|\nabla u|^{p-2}\nabla u) + \mathcal{G}(u) \in QS(x)\beta(u) + f_\infty(x) \ \text{ on } \mathcal{M}.$$

Through the paper we shall use the notation $div(|\nabla u|^{p-2}\nabla u) = \Delta_p u$. Problem (P) arises in the modeling of some problems in Climatology: the so-called *Energy Balance Models* introduced independently, in 1969 by M.I. Budyko [15] and W.D. Sellers [64]. The models have a diagnostic character and intended to understand the evolution of the global climate on a long time scale. Their main characteristic is the high sensitivity to the variation ofsolar and terrestrial parameters. This kind of models has been used inthe study of the Milankovitch theory of the ice-ages (see, e.g. North, Mengel and Short [60]).

The model is obtained from the thermodynamics equation of the *atmosphere primitive equations* via averaging process (see, e.g. Lions, Temam and Wang [53] for a mathematical study of those equations, Kiehl [50] for the application of averaging processes and Remark 1 for some nonlocal variants of (P)). More simply, the model can be formulated by using the energy balance on the Earth's surface: internal energy flux variation = $R_a - R_e + D$, were R_a and R_e represent the absorbed solar and the emitted terrestrial energy flux, respectively and D is the horizontal heat diffusion.

Let us express the components of the above balance in mathematical terms. The distribution of temperature $u(x,t)$ is expressed pointwise after standard average process, where the spatial variable x is in the Earth's surface which may be identified with a compact Riemannian manifold without boundary $\mathcal{M}$ (for instance, the two-sphere S^2), and t is the time variable. The time scale is considered relatively long. Nevertheless, in the so called *seasonal models* a smaller scale of time is introduced in order to analyze the effect of the seasonal cycles in the climate and in particular in the ice caps formation (see Remark 2 for the connection with the associate time periodical problem).

To simplify the presentation we assume that the internal energy flux variation is simply given as the product of the heat capacity c (a given constant which can be assumed equal to one by rescaling) and the partial derivative of the temperature u with respect to the time. For a more general modeling see Remark 1.

The *absorbed energy* R_a depends on the planetary *coalbedo* β. The coalbedo function represents the fraction of the incoming radiation flux which is absorbed by the surface. In ice-covered zones, reflection is greater than over oceans, therefore, the coalbedo is smaller. One observes that there is a sharp transition between zones of high and low coalbedo. In the energy balance climate models, a main change of the coalbedo occurs in a neighborhood of a critical temperature for which ice becomes white, usually taken as $u = -10^0C$.

The different coalbedo is modelled as a discontinuous function of the temperature in the *Budyko model*. Here it will be treated as a maximal monotone graph in $\mathbb{R}^2$

$$\beta(u) = \begin{cases} m & u < -10 \\ [m, M] & u = -10 \\ M & u > -10, \end{cases} \tag{1}$$

where $m = \beta_i$ and $M = \beta_w$ represent the coalbedo in the ice-covered zone and the free-ice zone, respectively and $0 < \beta_i < \beta_w < 1$ (the value of these constants has been estimated by observation from satellites). In the *Sellers model*, β is assumed to be a more regular function (at least Lipschitz continuous), as for instance

$$\beta(u) = \begin{cases} m & u < u_i, \\ m + (\frac{u-u_i}{u_w-u_i})(M-m) & u_i \leq u \leq u_w, \\ M & u > u_w, \end{cases}$$

where u_i and u_w are fixed temperatures closed to -10^0C.In both models, the absorbed energy is given by $R_a = QS(x)\beta(u)$where $S(x)$ is the *insolation function* and Q is the so-called *solar constant.*

The Earth's surface and atmosphere, warmed by the Sun, reemit part of the absorbed solar flux as an infrared long-wave radiation. This energy R_e is represented, in the Budyko model, according to the Newton cooling law, that is,

$$R_e = Bu + C. \tag{2}$$

Here, B and C are positive parameters, which are obtained by observation, and can depend on the *greenhouse effect.* However, in the Sellers model, R_e is expressed according to the Stefan - Boltzman law

$$R_e \ = \ \sigma u^4, \tag{3}$$

where σ is called *emissivity constant* and now u is in Kelvin degrees.

The *heat diffusion* D is given by the divergence of the conduction heat flux F_c and the advection heat flux F_a. Fourier's law expresses $F_c = k_c \nabla u$ where k_c is the *conduction coefficient.* The advection heat flux is given by $F_a = \mathbf{v} \cdot \nabla u$ and it is known (see e.g. Ghil and Childress [35]) that, to the level of the planetary scale, it can be modeled in terms of $k_a \nabla u$ for a suitable diffusion coefficient k_a. So, $D = \div(k\nabla u)$ with $k = k_c + k_a$. In the pioneering models, the diffusion coefficient k was considered as a positive constant. Nevertheless, in 1972, P.H. Stone [68] proposed a coefficient $k = |\nabla u|$, in order to consider negative feedback in the eddy fluxes. So, in that case the heat diffusion is represented by the quasilinear operator $D = div(|\nabla u|\nabla u)$. Our formulation (P) takes into account such a case which corresponds to the special choice $p = 3$ (notice that the case $p = 2$ leads to the linear diffusion). These physical laws lead to problem (P) with $R_e(u) = \mathcal{G}(u) - f$.

In Section 2 we start by presenting some results on the existence and uniqueness of solutions which generalize some previous results in the literature for a one-dimensional simplified formulation. Such simplification considers the averaged temperature over each parallel as the unknown. So, the two-dimensional model (P) is reduced in a one-dimensional model when $\mathcal{M}$ is the two dimensional sphere and considering the spherical coordinates. Therefore, the model becomes

$$(P^1) \begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x \ \in \ QS(x)\beta(u) - R_e(u) & \text{in } (-1,1) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } (-1,1), \end{cases}$$

with $\rho(x) = (1 - x^2)^{\frac{p}{2}}$ where $x = \sin\theta$ and θ is the latitude. Notice that again there is no boundary condition since the meridional heat flux $(1 - x^2)^{\frac{p}{2}}|u_x|^{p-2}u_x$ vanishes at the poles $x = \pm 1$. We also include in this section some comments on the *free boundaries* associated to the Budyko type model (the curves separating the regions $\{x : u(x,t) < -10\}$ and $\{x : u(x,t) > -10\}$). We end the section with a result on the stabilization of solutions as $t \to \infty$. Some references on the question of the approximate controllability for the transient model are given in Remark 4.

Section 3 is devoted to the study of the number of stationary solutions according to the parameter Q, when β is not necessarily Lipschitz continuous and $p \geq 2$. We start by estimating an interval of values for Q where there exist at least *three* stationary solutions and other complementary intervals for Q where the stationary solution is unique. A more precise study of the bifurcation diagram of solutions for different positive values of Q is available once we specialize $f_\infty(x) \equiv C$ with C a prescribed constant. Then problem $(P_{Q,f})$ becomes

$$(P_{Q,C}) \qquad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \mathcal{G}(u) + C \in QS(x)\beta(u) \text{ on } \mathcal{M}.$$

We denote by Σ the set of pairs $(Q,u) \in \mathbb{R}^+ \times V$, where u satisfies the equation $(P_{Q,C})$. We show that, under suitable conditions, Σ contains an unbounded connected component which is S-shaped containing $(0, \mathcal{G}^{-1}(-C))$ with at least one turning point to the right (and so at least another one to the left). We end Section 3 with a remark on a simplified version of problem $(P_{Q,C})$ for which it is still possible to find more precise answers: if $Q_1 < Q < Q_2$, for some suitable positive constants $Q_1 < Q_2$, then we have infinitely many solutions. More precisely, there exists $k_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, $k \geq k_0 \in \mathbb{N}$ there exists at least a solution u_k which crosses the level $u_k = -10$, exactly k times.

2. The transient model

2.1. On the existence of solutions

Motivated by the model background described in the Introduction, we introduce the following structure hypotheses: $p \geq 2$, $Q > 0$,

– $(\mathrm{H}_\mathcal{M})$ $\mathcal{M}$ is a C^∞ two-dimensional compact connected oriented Riemannian manifold of $\mathbb{R}^3$ without boundary,

– (H_β) β is a bounded maximal monotone graph in $\mathbb{R}^2$, i.e. $m \leq z \leq M$, $\forall z \in \beta(s)$, $\forall s \in \mathbb{R}$,

– $(\mathrm{H}_\mathcal{G})$ $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\mathcal{G}(0) = 0$, and $|\mathcal{G}(\sigma)| \geq C|\sigma|^r$ for some $r \geq 1$,

– (H_s) $S : \mathcal{M} \to \mathbb{R}$, $S \in L^\infty(\mathcal{M})$, $S_1 \geq S(x) \geq S_0 > 0$ a.e.$x \in \mathcal{M}$,
– (H_f) $f \in L^\infty(\mathcal{M} \times (0,T))$, (resp.– ($H_f^\infty$) $f \in L^\infty(\mathcal{M} \times (0,\infty))$),
– (H_0) $u_0 \in L^\infty(\mathcal{M})$.

The possible discontinuity in the coalbedo function causes that (P) does not have classical solutions in general, even if the data u_0 and f are smooth. Therefore, we must introduce the notion of weak solution. The natural "energy space" associated to (P) is the one given by

$$V := \{u : \mathcal{M} \to R,\ u \in L^2(\mathcal{M}),\ \nabla_{\mathcal{M}} u \in L^p(T\mathcal{M})\},$$

which is a reflexive Banach space if $1 < p < \infty$. Here $T\mathcal{M}$ denotes the tangent bundle and any differential operator must be understood in terms of the *Riemannian metric* g given on $\mathcal{M}$ (see, e.g. Aubin [8] and Díaz and Tello [26]).

Definition 1 – *We say that $u : \mathcal{M} \to \mathbb{R}$ is a bounded weak solution of (P) if i) $u \in C([0,T]; L^2(\mathcal{M})) \cap L^p(0,T;V) \cap L^\infty(\mathcal{M} \times (0,T))$ and ii) there exists $z \in L^\infty(\mathcal{M} \times (0,T))$ with $z(x,t) \in \beta(u(x,t))$ a.e. $(x,t) \in \mathcal{M} \times (0,T)$ such that*

$$\int_{\mathcal{M}} u(x,T)v(x,T)dA - \int_0^T < v_t(x,t), u(x,t) >_{V' \times V} dt +$$

$$+ \int_0^T \int_{\mathcal{M}} < |\nabla u|^{p-2} \nabla u,\ \nabla v > dAdt + \int_0^T \int_{\mathcal{M}} \mathcal{G}(u) v \, dAdt =$$

$$= \int_0^T \int_{\mathcal{M}} QS(x) z(x,t)\, v dAdt + \int_0^T \int_{\mathcal{M}} f\, v \, dAdt + \int_{\mathcal{M}} u_0(x) v(x,0)\, dA$$

$$\forall v \in L^p(0,T;V) \cap L^\infty(\mathcal{M} \times (0,T)) \text{ such that } v_t \in L^{p'}(0,T;V'),$$

where $<,>_{V' \times V}$ denotes the duality product in $V' \times V$.

We have

Theorem 1 – *There exists at least a bounded weak solution of (P). Moreover, if $T = +\infty$ and f verifies (H_f^∞), the solution u of (P) can be extended to $[0,\infty) \times \mathcal{M}$ in such a way that $u \in C([0,\infty), L^2(\mathcal{M})) \cap L^\infty(\mathcal{M} \times (0,\infty))$ $\cap\ L^p_{loc}(0,\infty;V)$.*

The above result can be proved in different ways. As in the case of the one-dimensional model (Diaz [19]) we can apply the techniques of Diaz and Vrabie [30] based on fixed point arguments which are useful for multivalued non monotone equations. We start by defining the operator A:$D(A) \subset$

$L^2(\mathcal{M}) \longrightarrow L^2(\mathcal{M}), A(u) = -\Delta_p u + \mathcal{G}(u)$ if $u \in D(A) = \{u \in L^2(\mathcal{M}) : -\Delta_p u + \mathcal{G}(u) \in L^2(\mathcal{M})\}$. The Cauchy associated problem

$$(P_h)\begin{cases} \frac{du}{dt}(t) + Au(t) \ni h(t) & t \in (0,T), \text{ in } X = L^2(\mathcal{M}) \\ u(0) = u_0, & u_0 \in L^2(\mathcal{M}), \end{cases}$$

is well posed (it has a unique mild solution in $C([0,T]; L^2(\mathcal{M}))$ for every $h \in L^2(0,T; L^2(\mathcal{M}))$ by the abstract results of Brezis [14]) since we have

Proposition 1 – *Let*

$$\phi(u) = \begin{cases} \frac{1}{p}\int_{\mathcal{M}} |\nabla u|^p dA + \int_{\mathcal{M}} G(u)dA & u \in D(\phi) \\ +\infty & u \notin D(\phi) \end{cases} \tag{4}$$

where $G(u) = \int_0^u \mathcal{G}(\sigma)d\sigma$ *with* $D(\phi) := \{u \in L^2(\mathcal{M}),\ \nabla u \in L^p(T\mathcal{M})$ *and* $\int_{\mathcal{M}} G(u)dA < +\infty\}$. *Then i)* ϕ *is proper, convex and lower semicontinuous in* $L^2(\mathcal{M})$. *ii)* $A = \partial\phi$, *and* $\overline{D(A)} = L^2(\mathcal{M})$, *and iii)* A *generates a compact semigroup of contractions* $S(t)$ *on* $L^2(\mathcal{M})$.

Besides, from Brezis [14] we know that u, solution of (P_h), verifies that $u \in L^p(0,T;V)$, $\sqrt{t}u_t \in L^2(0,T;L^2(\mathcal{M}))$, $u \in W^{1,2}(\delta,T;L^2(\mathcal{M}))$, $0 < \delta < T$. Let us prove the existence of solutions for the problem (P) via a fixed point for a certain operator $\mathcal{L}$. Let $Y = L^p(0,T;L^2(\mathcal{M}))$ and define $\mathcal{L} : K \to 2^{L^p(0,T;L^2(\mathcal{M}))}$ by the following process: Let us define

$$K = \{z \in L^p(0,T;L^\infty(\mathcal{M})) \ : \ \|z(t)\|_{L^\infty(\mathcal{M})} \leq C_0 \text{ a.e.} t \in (0,T)\}$$

with $C_0 = QS_1M + \|f\|_{L^\infty(0,T;L^\infty(\mathcal{M}))}$.

Now, we fix $u_0 \in L^2(\mathcal{M})$ and define the *solution operator* (or generalized Green operator) $I_0 : K \to C([0,T];L^2(\mathcal{M}))$ by $I_0(z) = v$ solution of (P_h) associated to $h \equiv z$. Given $f \in L^p((0,T);L^2(\mathcal{M}))$, we define the *superposition operator* $\mathcal{F}: L^p((0,T);L^2(\mathcal{M})) \to 2^{L^p((0,T);L^2(\mathcal{M}))}$ by $\mathcal{F}(v(t)) = \{h : h(x,t) \in QS(x)\beta(v(x,t)) + f(x,t) \text{ a.e. } x \in \mathcal{M}\}$.

Finally, we define

$$\mathcal{L}(z) = \{h \in L^p(0,T;L^2(\mathcal{M})): \ h(t) \in \mathcal{F}(I_0(z)(t)) \text{ in } L^2(\mathcal{M}) \text{ a.e. } t \in (0,T)\}.$$

It is not difficult to check (see Diaz and Tello [26]) that using the results by Vrabie [71] and Diaz and Vrabie [30], $\mathcal{L}$ verifies the assumptions required to apply a version of the Schauder-Tychonoff Theorem due to Arino et al. [7] and hence the existence of a local solution to (P) is proved.

In order to complete the proof of Theorem 1, we are going to show that the solution u can be continued up to $t = \infty$, when (H_f^∞) is fulfilled. Taking u as test function we get

$$\int_{\mathcal{M}} u_t u dA + \int_{\mathcal{M}} |\nabla u|^p dA + \int_{\mathcal{M}} \mathcal{G}(u) u dA$$
$$= \int_{\mathcal{M}} QSzu dA + \int_{\mathcal{M}} fu dA, \quad z \in \beta(u). \tag{5}$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}} |u|^2 dA + \int_{\mathcal{M}} |\nabla u|^p dA + c\int_{\mathcal{M}} |u|^2 dA \le C + \epsilon \|u\|^2_{L^2(\mathcal{M})}$$

and by Gronwall's inequality $\|u(t)\|_{L^2(\mathcal{M})} \le k \quad \forall t > 0$ with k independent of t. By a well known result (see e.g. Theorem 4.3.4 of Cazenave and Haraux [16]) u can be extended to $(0, \infty)$ and so $u \in C([0,\infty); L^2(\mathcal{M}))$. To see that $u \in L^\infty(\mathcal{M} \times (0,\infty))$ we introduce $\overline{u}(x,t)$ and $\underline{u}(x,t)$, given as the unique solutions of the problem

$$\begin{cases} \overline{u}_t - \Delta_p \overline{u} + \mathcal{G}(\overline{u}) = MQS(x) + f^+(x,t) & \text{on } \mathcal{M} \times (0,\infty) \\ \overline{u}(0,x) = u_0^+(x) = \max\{0, u_0(x)\} & \text{on} \mathcal{M}, \end{cases}$$

and

$$\begin{cases} \underline{u}_t - \Delta_p \underline{u} + \mathcal{G}(\underline{u}) = mQS(x) + f^-(x,t) \\ \underline{u}(0,x) = u_0^-(x) = \min\{0, u_0(x)\}, \end{cases}$$

respectively. Since the operator $-\Delta_p u + \mathcal{G}(u)$ is T-accretive in $L^2(\mathcal{M})$, it is easy to see that $\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t)$ which proves the assertion once that

$$\|\overline{u}\|_{L^\infty((0,\infty)\times\mathcal{M})} \le \max\{\|u_0^+\|_\infty \, , \, \mathcal{G}^{-1}(\|MQS\|_\infty + \|f^+\|_\infty)\}$$

$$\|\underline{u}\|_{L^\infty((0,\infty)\times\mathcal{M})} \le \max\{\|u_0^-\|_\infty \, , \, \mathcal{G}^{-1}(\|mQS\|_\infty + \|f^-\|_\infty)\}.$$

■

Remark 1. More realistic energy balance models are formulated in terms of functional equations adding some non local terms to problem (P) . So, for instance, in linking the albedo to the surface temperature u alone, one neglects the very important long response times the cryosphere exhibits. E.g. the expansion or the retreat of the huge continental ices heets occurs with response times of thousands of years, a feature which Bhattacharya, Ghil and Vulis [13] proposed to incorporate by substituting u by a long term average of u, e.g. $w(x,t) := \int_{-T}^{0} \gamma(x,s) u(x, t+s) ds$ with $T \approx 10^4$ years with

$\gamma(.,-T) \equiv 0$, $\gamma(.,s) > 0$ for $s \in (-T, 0]$ and $\int_{-T}^{0} \gamma(.,s)ds = 1$. Of course, one can refine this procedure by having independent ice- and snow-lines. In that case, one understands ice-lines as the boundaries of regions that are covered by continental ice-sheets or huge glaciers (slow response times in comparison with the ten-year mean), whereas snow-lines refer to boundaries of regions where the variations in ice- or snow cover occur on the time-scale of u. This approach was chosen in Diaz and Hetzer [25] (see also Hetzer and Schmidt [48] and Hetzer [45],[46], [44]).

On the other hand, if we disregard the latent energy stored in continental ice sheets and glaciers, the internal energy flux is given by $e(x,t) = c(x)u(x,t)$ with c the heat capacity which varies considerably with x due to the land-water distribution. However, a more accurate modeling suggests to set $e(x,t) = c(x)u(x,t) + h(w(x,t))$ where c denotes the thermal inertia and $h(w(x,t))$ stands for the latent energy density due to huge ice accumulations. This approach is closely related to the one for the Stefan problem (cf., e.g., Meirmanov [55])with the obvious change that h should depend on the long-term temperature mean w rather than on u in view of the time scales relevant for the latent fluxes. Here h is a nonnegative bounded decreasing function with derivative ξ having compact support. Using that $e_t = [cu + h(w)]_t = cu_t + \xi(w)w_t$ and observing that $w_t = \int_{-T}^{0} \gamma(s,.)u_t(.,t+s)ds$ in case that u is sufficiently smooth, one obtains $cu_t + \xi(w)\gamma(.,0)u - \xi(w)\int_{-T}^{0} \gamma_s(.,s)u(.,t+s)ds$ via integration by parts for e_t. Collecting all terms one is led to a non linear and nonlocal quasilinear parabolic problem. Some results on the existence of solutions for such a model can be found in Diaz and Hetzer [25]. We also mention the treatment made in Bermejo, Diaz and Tello [12] for the study of the general case $c = c(x)$ (but without any nonlocal terms) and study of the multi-layer model made in Hetzer and Tello [49].

Remark 2. A more realistic description of the *incoming solar radiation flux* $QS(x)$ is obtained by replacing it by a time depending function $Q(x,t)$ under the general assumption $Q(x,t)$ is $T-$periodical in time and $Q(x,t) \geq 0$. This last inequality allows to consider the polar night phenomena (time where $Q(x,t) = 0$ for some subsets of the manifold $\mathcal{M}$). The consideration of the periodicity of the forcing term is motivated by the seasonal variation of the incoming solar radiation flux during one natural year. The existence of periodical solutions for the associated model was the object of the paper Badii and Diaz [9]. The existence of periodic solutions for the Sellers type model was obtained by considering the Poincaré map F associated to the Cauchy problem (P), i.e. the operator assigning, to every initial data, the solution of (P) evaluated after T-period. We prove that F is a continuous, compact and pointwise increasing map and so, the Schauder fixed point theorem can be applied. The existence of a periodic solution for the Budyko type model needs some different arguments. The Poincaré map can not be well defined as a univalued operator and so we apply a variant of the Schauder-Tychonoff fixed point theorem for a suitable multivalued operator.

Remark 3. For different purposes it is useful to get existence results via regularization of the multivalued term $\beta(u)$. See, e.g., Xu [72] and Feireisl and Norbury [33] for some special formulations when $p = 2$. In our case it can be obtained as an easy adaptation of the results of Section 3. We also mention some results on the numerical approach due to Lin and North [51], Hetzer, Jarausch and Mackens [42], Bermejo [11] and Diaz, Bermejo and Tello [12].

2.2. On the uniqueness of solutions

The question of uniqueness has different answers for the different coalbedo functions under consideration depending on whether the coalbedo is supposed to be discontinuous or not. For the Sellers model (β locally Lipschitz), the uniqueness is obtained by standard methods (see e.g. Diaz [19]). Nevertheless, in the Budyko model (β multivalued), there are cases of nonuniqueness (in spite of the parabolic nature of(P)). The first nonuniqueness result in this context seems to be the one given in Diaz [19] where infinitely many solutions are found for the one-dimensional model (P^1) for any initial condition u_0 satisfying

$$\left.\begin{array}{l} u_0 \in C^\infty(I),\ u_0(x) = u_0(-x)\ \forall x \in [0,1], \\ u_0(0) = -10,\ u_0^{(k)}(0) = 0,\ k = 1,2 \\ u_0'(1) = 0,\ u_0'(x) < 0,\ x \in (0,1). \end{array}\right\} \qquad (6)$$

Notice that these initial data u_0 are very "flat" at the level-10. A similar non uniqueness result for the Budyko model with a suitable initial datum carries over to the two-dimensional model when $\mathcal{M} = S^2$. Each solution $u_1(x,t)$ of (P^1) generates a solution $u_2(x,y,t)$ of 2D model by rotation about the axis through the poles (notice that the initial datum $u_2(x,y,0)$ is independent of the longitude), i.e. $u_2(x,y,t) = u_1(\sin\theta, t)$ where $(x,y) \in S^2$ with latitude θ. It is not difficult to prove that u_2 is asolution of (P) for the initial datum $u_1 \sin\theta, 0)$. Other non uniqueness results can be found by using selfsimilar special solutions as in Gianni and Hulshof [37].

Since non uniqueness of solutions may arise, it is useful to know (see Diaz and Tello [26]) that in any case problem (P) has a maximal solution u^* and a minimal solution u_* (i.e. u^* and u_* are solutions of (P) such that everysolution u of (P) verifies that $u_* \leq u \leq u^*$ in$(0,T) \times \mathcal{M})$. In order to obtain a criterion for the uniqueness of solutions for Budyko type models we introduce the notion of *nondegeneracy property* for functions defined on $\mathcal{M}$.

Definition 2 – *Let $w \in L^\infty(\mathcal{M})$. We say that w satisfies the strong nondegeneracy property (resp. weak) if there exist $C > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $|\{x \in \mathcal{M} : |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$ (resp. $|\{x \in \mathcal{M} : 0 < |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$), where $|E|$ denotes the Lebesgue measure on the manifold $\mathcal{M}$ for all $E \subset \mathcal{M}$.*

Theorem 2 – *i) Assume that there exists a solution u of (P) suchthat $u(.,t)$ verifies the strong nondegeneracy property for any $t \in [0,T]$. Then u is the unique bounded weak solution of (P). ii) There exists at most one solution of (P) verifying the weak nondegeneracy property.*

The proof is based in the fact (adapted from Feireisl and Norbury [33]) that β generates a continuous operator from $L^\infty(\mathcal{M})$ to $L^q(\mathcal{M})$, $\forall q \in [1, \infty)$, although β is discontinuous, when the domain of such operator is the set of functions verifying the strong nondegeneracy property. More precisely, we have (see Diaz [19], Diaz and Tello [26])

Lemma 1 – *(i) Let $w, \hat{w} \in L^\infty(\mathcal{M})$ and assume that w satisfies the strong nondegeneracy property. Then for any $q \in [1, \infty)$, there exists $\tilde{C} > 0$ such that for any $z, \hat{z} \in L^\infty(\mathcal{M})$ with $z(x) \in \beta(w(x))$ and $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in \mathcal{M}$, we have*

$$\| z - \hat{z} \|_{L^q(\mathcal{M})} \leq (b_w - b_i) \min\{\tilde{C} \| w - \hat{w} \|_{L^\infty(\mathcal{M})}^{(p-1)/q} \ , \ |\mathcal{M}|^{1/q}\}. \quad (7)$$

(ii) If $w, \hat{w} \in L^\infty(\mathcal{M})$ and satisfy the weak nondegeneracy property then

$$\int_{\mathcal{M}} (z(x) - \hat{z}(x))(w(x) - \hat{w}(x))dA \leq (b_w - b_i)C \| w - \hat{w} \|_{L^\infty(\mathcal{M})}^p . \quad (8)$$

Idea of the proof of Theorem 2. Assume that there exist two bounded weak solutions u and $\hat{u}$ of (P), where u verifies the strong nondegeneracy property, i.e. $u_t - \Delta_p u + \mathcal{G}(u) = QSz + f$ and $\hat{u}_t - \Delta_p \hat{u} + \mathcal{G}(\hat{u}) = QS\hat{z} + f$, in $(0,T) \times \mathcal{M}$, for some $z \in \beta(u)$ and $\hat{z} \in \beta(\hat{u})$. Taking $(u - \hat{u})$ as the test function we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}} |u(t) - \hat{u}(t)|^2 dA + \int_{\mathcal{M}} (\mathcal{G}(u) - \mathcal{G}(\hat{u}))(u - \hat{u})dA +$$

$$\int_{\mathcal{M}} < |\nabla u(t)|^{p-2}\nabla u(t) - |\nabla \hat{u}(t)|^{p-2}\nabla \hat{u}(t),\ \nabla u(t) - \nabla \hat{u}(t) > dA =$$

$$= Q\int_{\mathcal{M}} S(x)(z(x,t) - \hat{z}(x,t))(u(x,t) - \hat{u}(x,t))dA. \quad (9)$$

By using the embedding $V \hookrightarrow L^\infty(\mathcal{M})$ if $p > 2$ and $V \subset L^\sigma(\mathcal{M})$ for all $\sigma \in [1, \infty)$ if $p = 2$ (recall that $\mathcal{M}$ is a two-dimensional compact Riemannian manifold: see, e.g. Aubin [8]) we arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} &\leq (C_l Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \frac{C_0}{C_{1,p,\infty}}) \parallel u - \hat{u} \parallel^p_{L^\infty(\mathcal{M})} + \\ &+ \tilde{C}_0 \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})}, \end{aligned} \tag{10}$$

in the case $p > 2$ and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} &\leq (C_l Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \frac{|\mathcal{M}|^{\frac{2}{\sigma}}}{C_{1,2,\sigma}}) \parallel u - \hat{u} \parallel^2_{L^\infty(\mathcal{M})} + \\ &+ \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} + \frac{\epsilon}{C_{1,2,\sigma}}, \end{aligned} \tag{11}$$

for the case $p = 2$ where ϵ and $\sigma = \sigma(\epsilon)$ .Now, we distinguish two cases: CASE 1: if $C_l Q \parallel S \parallel_\infty - \dfrac{C_0}{C_{1,p,\infty}} \leq 0$ and $p > 2$, then

$$\frac{1}{2}\frac{d}{dt} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} \leq \tilde{C}_0 \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} .$$

and the result holds by Gronwall's Lemma. If $p = 2$ and $C_l Q \parallel S \parallel_\infty - \dfrac{1}{C_{1,2,\sigma}} \leq 0$

$$\begin{aligned} \parallel u - \hat{u} \parallel^2_{L^2(\mathcal{M})} &\leq e^{2t} \parallel u_0 - \hat{u}_0 \parallel^2_{L^2(\mathcal{M})} - \frac{2\epsilon}{C_{1,2,\sigma}}(e^{2t} - 1) \leq \\ &\leq -2\epsilon C_l Q \parallel S \parallel_\infty (e^{2t} - 1) \end{aligned}$$

and since the above inequality is true for all ϵ, we conclude the uniqueness.

CASE 2: if $C_l Q \parallel S \parallel_\infty - \frac{1}{C_{1,p,\infty}} > 0$, we consider a suitable rescaling $(\mathcal{M} \mapsto \mathcal{M}_\delta)$ given by the dilatation D of magnitude $\delta > 0$ on the manifold $(\mathcal{M}, \mathrm{g})$, $D : \mathcal{M} \subset \mathbb{R}^3 \to \mathbb{R}^3$, $D(x) = \tilde{x} = \delta x$. So, if u is any function defined on $\mathcal{M}$, its local representation in the new coordinates is $\tilde{u} : \mathcal{M}_\delta \to \mathbb{R}$, $\tilde{u}(\tilde{x}) = u(\frac{\tilde{x}}{\delta})$ and we have

$$\frac{\partial \tilde{u}}{\partial \tilde{x}_i}(\tilde{x}) = \frac{1}{\delta}\frac{\partial u}{\partial x_i}(\frac{\tilde{x}}{\delta}) \qquad i = 1, 2, 3.$$

So problem (P) in the new coordinates becomes

$$(P_\delta) \begin{cases} \tilde{u}_t - \delta^p \mathrm{div}_{\mathcal{M}_\delta} \left(|\nabla_{\mathcal{M}_\delta} \tilde{u}|^{p-2} \nabla_{\mathcal{M}_\delta} \tilde{u} \right) + \mathcal{G}(\tilde{u}) \in QS\beta(\tilde{u}) + f & \text{in } (0,T) \times \mathcal{M}_\delta \\ \tilde{u}(0, \tilde{x}) = u_0(\frac{\tilde{x}}{\delta}) & \text{on } \mathcal{M}_\delta. \end{cases}$$

Clearly, if $\tilde{u}$ is a solution of (P_δ) then $\tilde{u}(\delta x, t)$ is a solution of (P). Moreover, the uniqueness of (P_δ) implies the uniqueness of (P), and conversely.

Let us see that there exists $\delta > 0$ such that the solution of (P_δ) is unique. Let u_δ and $\tilde{u}_\delta$ be two solutions of (P_δ) with u_δ verifying the strong nondegeneracy property. Arguing as before we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}_\delta} |u_\delta(t) - \hat{u}_\delta(t)|^2 dA_\delta$$

$$+\delta^p \int_{\mathcal{M}_\delta} < |\nabla u_\delta|^{p-2}\nabla u_\delta - |\nabla \hat{u}_\delta|^{p-2}\nabla \hat{u}_\delta, \nabla u_\delta - \nabla \hat{u}_\delta > dA_\delta$$

$$+\int_{\mathcal{M}_\delta} (\mathcal{G}(u_\delta) - \mathcal{G}(\hat{u}_\delta))(u_\delta - \hat{u}_\delta) dA_\delta = Q\int_{\mathcal{M}_\delta} S_\delta(z_\delta - \hat{z}_\delta)(u_\delta - \hat{u}_\delta) dA_\delta$$

for some $z_\delta \in \beta(u_\delta)$ and $\hat{z}_\delta \in \beta(\hat{u}_\delta)$. Here, S_δ is defined by $S_\delta : \mathcal{M}_\delta \to \mathbb{R}$, $S_\delta(\tilde{x}) = S(\frac{\tilde{x}}{\delta})$. (10) and (11) allow us toestimate $u_\delta - \hat{u}_\delta$ for u_δ and $\hat{u}_\delta$ solutions of (P_δ)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \| u_\delta - \hat{u}_\delta \|^2_{L^2(\mathcal{M}_\delta)} \\ &\qquad \le (C_{l,\delta}\, Q \| S_\delta \|_{L^\infty(\mathcal{M}_\delta)} - \frac{\delta^2 |\mathcal{M}_\delta|^{\frac{1}{\sigma}}}{C_{1,2,\sigma,\delta}}) \| u_\delta - \hat{u}_\delta \|^2_{L^\infty(\mathcal{M}_\delta)} \\ &\qquad + \| u_\delta - \hat{u}_\delta \|^2_{L^2(\mathcal{M}_\delta)} + \frac{\epsilon}{C_{1,2,\sigma,\delta}}. \end{aligned} \tag{12}$$

A careful study of the dependence on δ of the involved constants (see Diaz and Tello [26]) allows us to see that if we define the constant

$$K_{p,\delta} = C_{l,\delta} Q \quad \| S_\delta \|_{L^\infty(\mathcal{M}_\delta)} - \frac{\delta^2 |\mathcal{M}|^{\frac{2}{\sigma}}}{C_{1,2,\sigma,\delta}},$$

we have that $\lim_{\delta \to 0} K_{p,\delta} = 0$. This reduces the proof to Case 1 and the proof of (i) follows. For the proof of (ii) we use the second part of Lemma 1 and so

$$\frac{1}{2}\frac{d}{dt} \| u - \hat{u} \|_2^2 \le (C_d Q \| S \|_{L^\infty(\mathcal{M})} - \frac{C_0}{C_{1,p,q}}) \| u - \hat{u} \|^p_\infty + \tilde{C}_0 \| u - \hat{u} \|_2^2$$

where C_d is the constant of the weak nondegeneracy property (Lemma1). The uniqueness follows as in (i), by studying the sign of the constant $C_d Q \| S \|_{L^\infty(\mathcal{M})} - \dfrac{1}{C_{1,p,q}}$ and by rescaling when it is negative. ∎

Remark 4. It is possible to give several sufficient criteria for the nondegeneracy property. For instance, in the one-dimensional case and $p = 2$, if $u_0 \in C^1((-1,1))$ is such that there exists $\epsilon_0 > 0$ such that the set $\{x \in (-1,1) : \ |u_0(x) + 10| \leq \epsilon_0\}$ has a finite number of connected components I_j with $j = 1,..,N$ and for any j there exists $x_j \in I_j$ such that $u_0(x_j) = -10$,and $|u_{0x}(x)| \geq \delta_0$ for some $\delta_0 > 0$ and any $x \in I_j$ close to x_j, then there exists a solution $u(x,t)$ satisfying the strong nondegeneracy property on $(0, T^*)$ for some T^* (see Diaz and Tello [26]). Some results on solutions with $|\nabla u| \neq 0$ on the level where β becomes multivalued for a similar bidimensional problem are given in Gianni [36].

2.3. On the free boundary for Budyko type models

The discontinuity of the albedo function assumed in the Budyko model (β multivalued) generates a natural *free boundary* or interface $\zeta(t)$ between the ice-covered area ($\{x \in \mathcal{M} : u(x,t) < -10\}$) and the ice-free area ($\{x \in \mathcal{M} : u(x,t) > -10\}$). The free boundary is then given as $\zeta(t) = \{x \in \mathcal{M} : u(x,t) = -10\}$. In Xu [72], the Budyko model for $p = 2$ is considered in the one-dimensional case. He shows that if the initial datum u_0 satisfies

$$\begin{aligned}&u_0(x) = u_0(-x),\ u_0 \in C^3([-1,1]),\ u_0'(x) < 0 \text{ for any } x \in (0,1)\\ &\text{and there exists } \zeta(0) \in (0,1) \text{ such that } (u_0(x)+10)(x-\zeta(0)) < 0\\ &\text{for any } x \in [0,\zeta(0)) \cup (\zeta(0),1],\end{aligned}$$

then there exists a bounded weak solution u of (P) for which one has $\zeta(t) = \{\zeta_+(t)\} \cup \{\zeta_-(t)\}$ with $x = \zeta_+(t)$ a smooth curve, $\zeta_-(t) = \zeta_+(t)$ and $\zeta_+(\cdot) \in C^\infty([0, T^*))$, where T^* is characterized as the first time t for which $\zeta_+(t) = 1$. He also gives an expression for the derivative $\zeta_+'(t)$ (some related results for a model corresponding to $\rho(x) = 1$ can be found in Feireisl and Norbury [33], Gianni and Hulshof [37] and Stakgold [67]). We point out that the uniqueness result can be applied for such an initial datum. For the study of the free boundary in the bidimensional case see Diaz [22] and Gianni [36].

The interpretation of the size of the separating zone $\zeta(t)$ for other models is in fact a controversial question. So, some satellite pictures (Image of the Weddell sea taken by the satellite Spot on December 10, 1987) show that the separating region between the ice-free and the ice-covered zones is not a simple line on the Earth but a narrow zone where ice and water are mixed. Mathematically it could correspond to say that the set

$$M(t) = \{x \in \mathcal{M}\colon\ u(x,t) = -10\}$$

is a positively measured set. In the following we shall denote this set as the *mushy region* (since it plays the same role than in changing phase problems, see e.g. Diaz , Fasano and Meirmanov [23]).

Using the strong maximum principle, it is possible to show that if $p = 2$ the interior set of the mushy region $M(t)$ is empty even if the interior of

$M(0)$ is a nonempty open set (see Gianni and Hulshof [37]). As we shall see, this is not the case when $p > 2$ (recall that $p = 3$ in Stone [68]). A necessary condition for the Budyko model (with $R_e = Bu + C$) for $\overset{\circ}{M}(t) \neq \emptyset$ is that

$$C - 10B \in [\beta_i QS(x), \beta_w QS(x)] \quad \text{for a.e. } x \in \mathcal{M}. \tag{13}$$

It is possible to show that if $p > 2$, this condition is also sufficient. Here we merely present a result for the one-dimensional case (see Diaz [22] for the bidimensional case):

Theorem 3 – *Let $p > 2$. Assume (13) and $u_0 \in L^\infty(I)$ such that there exist $x_0 \in I$ and $R_0 > 0$ satisfying*

$$M(0) = \{x \in I : u_0(x) = -10\} \supset B(x_0, R_0)(= \{x \in I : |x - x_0| < R_0\}).$$

If u is the bounded weak solution of (P) satisfying the weak nondegeneracy property, then there exist $T^* \in (0, T]$ and a nonincreasing function $R(t)$ with $R(0) = R_0$ such that

$$M(t) = \{x \in I : u(x,t) = -10\} \supset B(x_0, R(t))$$

for any $t \in [0, T^*)$.

Proof. We shall use an energy method as developed in Diaz and Veron [29]. Given u bounded weak solution of (P), we define $v = u + 10$. As in Lemma 3.1 of the above reference, by multiplying the equation by v we obtain that for a.e. $R \in (0, R_0)$ and $t \in (0, T)$, we have

$$\frac{1}{2}\int_{B(x_0,R)} |v(x,t)|^2 dx + \int_0^t \int_{B(x_0,R)} \rho(x)|v_x|^p dx d\tau$$

$$+B\int_0^t \int_{B(x_0,R)} |v(x,\tau)|^2 dx d\tau \leq$$

$$\leq \int_0^t \int_{S(x_0,R)} \rho |v_x|^{p-2} v_x \cdot \vec{n} v ds d\tau + \int_0^t \int_{B(x_0,R)} \{QSz - C + 10B\} v dx d\tau$$

$$= I_1 + I_2,$$

where $\rho(x) = (1 - x^2)^{\frac{p}{2}}$, $S(x_0, R) = \partial B(x_0, R) = \{x_0 - R\} \cup \{x_0 + R\}$ and $z(x,t) \in \beta(u(x,t))$ for a.e. $x \in B(x_0, R)$ and $t \in (0, T]$. We introduce the energy functions

$$E(R,t) = \int_0^t \int_{B(x_0,R)} \rho(x)|v_x|^p dx d\tau$$

$$b(R,t) = \sup_{0 \leq \tau \leq t} \text{ess} \int_{B(x_0,R)} |v(x,\tau)|^2 dx.$$

Using Holder's inequality and the interpolation-trace Lemma of Diaz -Veron [29], we get

$$I_1 \leq \left(\frac{\partial E}{\partial R}(R,t)\right)^{(p-1)/p} \left(\int_0^t \int_{S(x_0,R)} |v|^p dx d\tau\right)^{1/p} \leq$$

$$\leq Ct^{(1-\theta)/p} \left(\frac{\partial E}{\partial R}(R,t)\right)^{(p-1)/p}$$

$$\times \left(E(R,t)^{1/p} + R^{\delta} t^{1/p} b(R,t)^{1/2}\right)^{\theta} b(R,t)^{(1-\theta)/2},$$

where

$$\theta = p/(3p-2) \text{ and } \delta = -(3p-2)/2p.$$

Using the assumption (13), we have $\hat{z}(\cdot) = [(C-10B)/QS(\cdot)] \in \beta(-10)$. Then applying Lemma 3 we get

$$I_2 \leq (M-m)Q \parallel S \parallel_{L^\infty(I)} C' \int_0^t \parallel v(\tau) \parallel^p_{L^\infty(B(x_0,R)} d\tau.$$

From Theorem 4 of Rakotoson and Simon [62], we have the estimate

$$\parallel v \parallel_{L^\infty(J)} \leq C_1 \parallel v_x \parallel_{L^p(J:\rho)} + C_2 \parallel v \parallel_{L^1(J:\rho)}, \quad \forall v \in V, \tag{14}$$

for some positive constants independent on the interval J. Then we obtain

$$I_2 \leq (M-m)Q \parallel S \parallel_{L^\infty(I)} C'(C_1 E(R,t) + tC_3(R) b(R,t)),$$

where now

$$C_3(R) = \frac{\left(\int_{B(x_0,R)} \rho(x)^2 dx\right)^{p-2}}{C_1^p \left(\int_{B(x_0,R)} \rho(x) dx\right)^p} \parallel u + 10 \parallel^p_{L^\infty((0,T):L^2(I))}.$$

As in the proof of the uniqueness, we can assume C_1 small enough without loss of generality. Then, there exists $T^* \in (0,T]$ and $\lambda \in (0,1]$ such that

$$\lambda(E(R,t) + b(R,t)) \leq I_1$$

which implies that

$$\lambda E^{\mu} \leq t^{(1-\theta)/p} \frac{\partial E}{\partial R}$$

for some $\mu \in (0,1)$ and for any $t \in [0,T^*)$. The proof ends as in Diaz and Veron [29](see also and Antonsev and Diaz [4]). ■

Remark 5. The existence of the mushy region (for anyvalue of $p \in (1,\infty)$) can be proved for a different class of models by taking into account a discontinuous diffusivity (see Held, Linder and Suárez [40]). In that case the problem is a variant of the Stefan problem (see, e.g., Meirmanov [55]). It would be interesting to find sufficient conditions implying the persistence of a mushy region for any time $t > 0$. The fact that a mushy region may exist for the stationary problem can be found from the results of Diaz [?] (see Theorem 1.14).

2.4. Stabilization of solutions when $t \to +\infty$

In order to analyze the stabilization of the solutions of (P), following Diaz, Hernández and Tello [24], we assume the additional condition

- $(\mathrm{H}_\infty)]$ $f \in L^\infty((0,\infty)\times\mathcal{M})$ and there exists $f_\infty \in V'$ such that

$$\int_{t-1}^{t+1} \|f(\tau,\cdot) - f_\infty(\cdot)\|_{V'} d\tau \to 0 \qquad \text{as } t \to \infty.$$

We start by recalling a global regularity of the solutions on $(0,\infty)$.

Lemma 2 — *Assume we are given* $u_0 \in V \cap L^\infty(\mathcal{M})$, $f \in L^\infty(\mathcal{M}\times(0,\infty))$ $\cap W^{1,1}_{loc}((0,\infty); L^1(\mathcal{M}))$ *and* $\int_t^{t+1} \|f_t(s,\cdot)\|_{L^1(\mathcal{M})} ds \le C_0$, $\forall t > 0$ *where* C_0*is a time independent constant. Then there exists a weak solution of* (P) *such that*

$$u \in L^\infty(0,\infty; V) \quad \text{and} \quad u_t \in L^2(0,\infty; L^2(\mathcal{M})). \tag{15}$$

A key point in the proof is to check that $\varphi(t) = \frac{1}{p}\int_{\mathcal{M}} |\nabla u(x,t)|^p dA$ satisfies

$$\varphi(t+1) \le C[\varphi(t) - \varphi(t+1)] + \theta(t) \qquad t > 0, \tag{16}$$

where C is a positive constant and $\theta(t) > 0$ when t is large enough with $\theta(t) = O(1)$ when $t \to \infty$. Then, thanks to a technical lemma due to Nakao [56], we conclude that $\varphi(t) = O(1)$ which is equivalent to $u \in L^\infty(0,\infty;V)$.

The following theorem proves the stabilization of the solutions u satisfying (15). As usual, given u bounded weak solution of(P), we define the ω-limit set of u by

$$\omega(u) = \{u_\infty \in V \cap L^\infty(\mathcal{M}) : \exists t_n \to +\infty \text{ such that } u(t_n,\cdot) \to u_\infty \text{ in } L^2(\mathcal{M})\}.$$

Theorem 4 — *Let* $u_0 \in L^\infty(\mathcal{M}) \cap V$ *and let* u *be any bounded weak solution satisfying (15). Then, i)* $\omega(u) \ne \emptyset$ *and if* $u_\infty \in \omega(u)$, $\exists t_n \to +\infty$ *such that* $u(\cdot, t_n + s) \to u_\infty$ *in* $L^2(-1,1;L^2(\mathcal{M}))$ *and* $u_\infty \in V$ *is a weak solution of the stationary problem associated to* f_∞ *; ii) in fact, if* $u_\infty \in \omega(u)$, *then* $\exists\{\hat{t}_n\} \to +\infty$ *such that* $u(\cdot,\hat{t}_n) \to u_\infty$ *strongly in* V.

Proof. Let u_∞ be an element of $\omega(u)$. Then,

$$\|u(t_n+s)-u(t_n)\|^2_{L^2(\mathcal{M})} \leq 2\ \|u_t\|^2_{L^2((t_n-1,t_n+1);L^2(\mathcal{M}))}.$$

Since $u_t \in L^2(0,\infty;L^2(\mathcal{M}))$, $\|u_t\|^2_{L^2((t_n-1,t_n+1);L^2(\mathcal{M}))} \to 0$ when $t_n \to \infty$ and by the Lebesgue convergence theorem we conclude that $u(\cdot,t_n+\cdot) \to u_\infty$ in $L^2((-1,1);L^2(\mathcal{M}))$.

To prove that u_∞ is a solution of (P_Q), we consider the test functions $v(x,t) = \xi(x)\varphi(t-t_n)$ with $\xi \in V \cap L^\infty(\mathcal{M})$ and $\varphi \in \mathcal{D}(-1,1)$, $\varphi \geq 0$, $\int_{-1}^1 \varphi = 1$. Then

$$\int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} u_t\xi\varphi(t-t_n) + \int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} |\nabla u|^{p-2}\nabla u\cdot\nabla\xi\varphi(t-t_n)$$
$$+\int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} \mathcal{G}(u)\xi\varphi(t-t_n)$$
$$=\int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} Qz\xi\varphi(t-t_n) - \int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} f(x,t)\xi\varphi(t-t_n) \quad z\in\beta(u(x,t)).$$

Changing variables, namely $s = t-t_n$ and defining $U_n(x,s) = u(x,t_n+s)$, we obtain that

$$\begin{array}{llll} U_n \rightharpoonup u_\infty & \text{weakly in} & L^\sigma((-1,1);V) & \forall\sigma>1 \\ |\nabla U_n|^{p-2}\nabla U_n \rightharpoonup Y & \text{weakly in} & L^\sigma((-1,1);L^p(T\mathcal{M})) & \forall\sigma>1. \end{array}$$

Applying Aubin's compactness result (see e.g. Simon [66]), a well known property of the maximal monotone graphs (see Brezis [14]) and Lebesgue's theorem, we get that $z_n \rightharpoonup z_\infty \in \beta(u_\infty)$ weakly in $L^\sigma(\mathcal{M}\times(-1,1))$ $\forall\sigma>1$ and $\mathcal{G}(U_n)\to\mathcal{G}(u_\infty)$ in $L^1(\mathcal{M}\times(-1,1))$. Letting $n\to\infty$, we arrive to

$$\int_{-1}^1\int_{\mathcal{M}} Y\cdot\nabla\xi\varphi + \int_{\mathcal{M}}\mathcal{G}(u_\infty)\xi = \int_{\mathcal{M}} QSz_\infty\xi + \int_{\mathcal{M}} f_\infty\xi \quad \forall\xi\in V\cap L^\infty(\mathcal{M}).$$

Now, the main difficulty is to prove that

$$\int_{-1}^1 Y(s,\cdot)\varphi(s) = |\nabla u_\infty|^{p-2}\nabla u_\infty.$$

Due to the coercivity of the p-Laplacian operator we obtain the following inequality :

$$\lim_{n\to\infty}\int_{-1}^1\int_{\mathcal{M}}(|\nabla U_n|^{p-2}\nabla U_n - |\nabla\chi|^{p-2}\nabla\chi)\cdot(\nabla u_\infty - \nabla\chi)\varphi(s)\, dAds \geq 0, \tag{17}$$

which holds for $\chi \in V$. We arrive to the desired convergence by taking $\chi = u_\infty + \lambda\xi$ and applying a Minty type argument to(17) as in Diaz and de Thélin [28].

The proof of (ii) uses the coerciveness of the operator and the fact that

$$\int_{-1}^{1}\int_{\mathcal{M}}(|\nabla U_n|^{p-2}\nabla U_n - |\nabla u_\infty|^{p-2}\nabla u_\infty)\cdot(\nabla U_n - \nabla u_\infty)\varphi(s)\, dAds \to 0.$$

The inequality $|\zeta-\hat{\zeta}|^p \leq (|\zeta|^{p-2}\zeta - |\hat{\zeta}|^{p-2}\hat{\zeta})\cdot(\zeta-\hat{\zeta})$, $\forall\zeta,\hat{\zeta} \in \mathbb{R}^N$ allows us to obtain

$$\lim_{n\to\infty}\int_{-1}^{1}\int_{\mathcal{M}}|\nabla U_n - \nabla u_\infty|^p\varphi(s)\, dAds \;=\; 0,\ \forall\varphi.$$

This implies that there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$, where $s_n \in (-1,1)$ such that

$$\lim_{n\to\infty}\int_{\mathcal{M}}|\nabla u(t_n+s_n,\cdot) - \nabla u_\infty|^p dA = 0$$

and so we prove the assertion. ∎

Remark 6. If u_∞ is an isolated point of $\omega(u)$, it is easy to see that in fact the above convergences hold as $t\to\infty$ (and not merely for a sequence $t_n \to \infty$). The proof of this convergence is an open problem in the remaining cases. In fact, in some cases the set of stationary points is a continuum (see Remark 11) and the convergence when $t\to\infty$ is far from trivial (for some results in this direction see Feireisl and Simondon [34]).

Remark 7. A result on the convergence (in a suitable sense) of the free boundaries to the free boundary of the solution of the stationary problem is given in Diaz [22] (see also Gianni [36]).

Remark 8. The question of the approximate controllability was considered in Diaz [?] and [21]. To avoid technical difficulties, in these articles the manifold $\mathcal{M}$ is replaced by an open regular bounded set Ω of $\mathbb{R}^2$ (here $\mathbb{R}^2$ can be also substituted by $\mathbb{R}^N$ with $N \in \mathbb{N}$) and p is taken as $p=2$. As a boundary condition on $(0,T)\times\partial\Omega$, it is chosen the one of Neumann type since it leads to a set of test functions for the weak formulation very similar to the one corresponding to the case of a Riemannian manifold without boundary. The case of an internal control is considered by taking $f(x,t){=}v(x,t)\chi_\omega$ with v the control to be searched, and χ_ω, the characteristic function of ω, a given open bounded subset of Ω. Thus, the new formulation is now the following: given y_0, $y_d : \Omega \to \mathbb{R}$ and $\varepsilon > 0$, find $v_\varepsilon : \omega\times(0,T)\to\mathbb{R}$ such that $d(y(T:v_\varepsilon),y_d) \leq \varepsilon$ where, in general, $y(T:v)$ represents the solution of problem

$$(\mathcal{P}_\omega)\begin{cases} y_t - \Delta y + \mathcal{G}(y) \in QS(x)\beta(y) + v\chi_\omega & \text{in } \Omega \times (0,T) \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ y(0,\cdot) = y_0(\cdot) & \text{on } \Omega, \end{cases}$$

where n is the outer unit vector to $\partial\Omega$. It is shown that the answer to the approximate controllability question depends on the asymptotic behaviour of the nonlinearity $\mathcal{G}(y)$ of the equation. If, for instance $\mathcal{G}(y) = |y|^{p-2}y$, then the approximate controllability property holds when $p \in (0,1]$ but if $p > 1$, an *obstruction phenomenon* appears, implying the impossibility of the controllability for general data. Some results concerning a special class of data for the superlinear case $p > 1$ are presented in Diaz [21]. We point out that in 1955, John von Neumann [57] proposed to control the climate by acting on the albedo and that this still remains a mathematical open question. Finally, we mention the "rain making" (see Dennis [17]) as a practical example of the application of control problems in environment.

3. On the stationary problem

We consider the problem $(P_{Q,f})$ obtained in the last subsection. Following Diaz, Hernández and Tello [24], we made in this section the additional assumptions

$-(\mathrm{H}^*_{\mathcal{G}})$ $\mathcal{G}$ satisfies $(\mathrm{H}_{\mathcal{G}})$ and $\lim_{|s|\to\infty} |\mathcal{G}(s)| = +\infty$.

$-(\mathrm{H}_{f_\infty})$ $f_\infty \in L^\infty(\mathcal{M})$ and there exists $C_f > 0$ such that $-\|f_\infty\|_{L^\infty(\mathcal{M})} \leq f(x) \leq -C_f$ *a.e.* $x \in \mathcal{M}$

$-(\mathrm{H}^*_\beta)$ there exist two real numbers $0 < m < M$ and $\epsilon > 0$ such that $\beta(r) = \{m\}$ for any $r \in (-\infty, -10-\epsilon)$ and $\beta(r) = \{M\}$ for any $r \in (-10+\epsilon, +\infty)$.

$-(\mathrm{H}_{C_f})$ $\mathcal{G}(-10-\epsilon) + C_f > 0$ and $\dfrac{\mathcal{G}(-10+\epsilon) + \|f_\infty\|_{L^\infty(\mathcal{M})}}{\mathcal{G}(-10-\epsilon) + C_f} \leq \dfrac{S_0 M}{S_1 m}$.

A function $u \in V \cap L^\infty(\mathcal{M})$ is called a *bounded weak solution* of $(P_{Q,f})$ if there exists $z \in L^\infty(\mathcal{M})$, $z(x) \in \beta(u(x))$ a.e. $x \in \mathcal{M}$ such that

$$\int_\mathcal{M} (|\nabla u|^{p-2}\nabla u) \cdot \nabla v \, dA + \int_\mathcal{M} \mathcal{G}(u) v \, dA = \int_\mathcal{M} QS(x) z v \, dA + \int_\mathcal{M} f_\infty v \, dA,$$

for any $v \in V$.

3.1. Existence of at least three solutions for suitable Q

We start with a multiplicity result given in Diaz, Hernández and Tello [24]

Theorem 5 – *Let u_m, u_M be the (unique) solutions of the problems*

$$(P_m) \quad -\Delta_p u + \mathcal{G}(u) = QS(x)m + f_\infty(x) \text{ on } \mathcal{M},$$

$$(P_M) \quad -\Delta_p u + \mathcal{G}(u) = QS(x)M + f_\infty(x) \text{ on } \mathcal{M},$$

respectively. Then: i) for any $Q > 0$, there is a minimal solution $\underline{u}$ (resp. a maximal solution $\overline{u}$) of problem($P_{Q,f}$). Moreover, any other solution u must satisfy

$$u_m \leq \underline{u} \leq u \leq \overline{u} \leq u_M \tag{18}$$

$$\mathcal{G}^{-1}(QS_0 m - \|f_\infty\|_{L^\infty(\mathcal{M})}) \tag{19}$$
$$lequ_m \leq \mathcal{G}^{-1}(QS_1 m - C_f),$$
$$\mathcal{G}^{-1}(QS_0 M - \|f_\infty\|_{L^\infty(\mathcal{M})})$$
$$lequ_M \leq \mathcal{G}^{-1}(QS_1 M - C_f). \tag{20}$$

ii) for any Q there is, at least, a solution u of $(P_{Q,f})$ which is a global minimum of the functional

$$J(w) = \frac{1}{p}\int_{\mathcal{M}} |\nabla w|^p dA + \int_{\mathcal{M}} G(w)dA - \int_{\mathcal{M}} f_\infty w dA - \int_{\mathcal{M}} QS(x)\jmath(w)dA,$$

on the set $K = \{w \in V,\ G(w) \in L^1(\mathcal{M})\}$, where $\beta = \partial\jmath$. Moreover, if (H_{C_f}) holds, then: iii) if $0 < Q < Q_1$, then ($P_{Q,f}$) has a unique solution $u = u_m$, $u < -10$, u is the minimum of J on K, and

$$\mathcal{G}^{-1}(-\|f_\infty\|_{L^\infty(\mathcal{M})}) \leq \lim_{Q\searrow 0}\ inf\, \|u\|_{L^\infty(\mathcal{M})} \leq \lim_{Q\searrow 0}\ sup\, \|u\|_{L^\infty(\mathcal{M})} u$$

$$\leq \mathcal{G}^{-1}(-C_f),$$

iv) if $Q_2 < Q < Q_3$, then $(P_{Q,f})$ has at least three solutions,u_i, $i = 1,2,3$ with $u_1 = u_M$, $u_1 > -10$, $u_2 = u_m$, $u_2 < -10$and $u_1 \geq u_3 \geq u_2$ on $\mathcal{M}$. Moreover, u_1 and u_2 are local minima of J on $K \cap L^\infty(\mathcal{M})$ and, if $p > 2$, and v) if $Q_4 < Q$, then $(P_{Q,f})$ has a unique solution $u = u_M$, $u > -10$, u is the minimum of J on K and $\|u\|_{L^\infty(\mathcal{M})} \to +\infty$ when $Q \to +\infty$, where

$$Q_1 = \frac{\mathcal{G}(-10-\epsilon) + C_f}{S_1 M} \qquad Q_2 = \frac{\mathcal{G}(-10+\epsilon) + \|f_\infty\|_{L^\infty(\mathcal{M})}}{S_0 M} \tag{21}$$

$$Q_3 = \frac{\mathcal{G}(-10-\epsilon) + C_f}{S_1 m} \qquad Q_4 = \frac{\mathcal{G}(-10+\epsilon) + \|f_\infty\|_{L^\infty(\mathcal{M})}}{S_0 m}. \tag{22}$$

Proof. i) It is a consequence of the fact that the comparison principle holds for problems (P_m), (P_M) and then the method of sub and supersolutions can be applied (see e.g. Amann [1]). ii) The conclusion follows from the Weierstrass argument by using Lebesgue convergence theorem. iii) From assumption (H_{C_f}) and since $0 < Q < Q_1$, we obtain that $\overline{u}_1 < -10 - \epsilon$ and $overline u_2 < -10 - \epsilon$. The proof of v) is analogous to i).

The proof of iv) is divided into several steps. First, we construct two constant subsolutions V_i and two constant supersolutions U_i such that

$$V_2 \;<\; U_2 \;<\; -10-\epsilon \;<\; -10+\epsilon \;<\; V_1 \;<\; U_1, \tag{23}$$

proving the existence of, at least two solutions of $(P_{Q,f})$. Later, we introduce the approximate model (P_Q^λ)

$$(P_Q^\lambda) \qquad -\Delta_p u + \mathcal{G}(u) \;=\; QS(x)\beta_\lambda(u) + f_\infty(x) \quad \text{ on } \mathcal{M},$$

where β_λ is the Lipschitz function $\beta_\lambda = \frac{1}{\lambda}(I - (I - \lambda\beta)^{-1})$, $\lambda > 0$ (the Yosida approximation of β). Since β verifies (H_β^*), we get that β_λ is a bounded and nondecreasing function $\forall\lambda > 0$, $\beta_\lambda(s) = \beta(s)$ for any $s \notin [-10-\epsilon, -10+\epsilon+\lambda M]$, $\beta_\lambda(s) \to \beta(s)$ in the sense of maximal monotone graphs when $\lambda \to 0$. If β is a Lipschitz function, we take simply $\beta_\lambda = \beta$. The existence of a solution of (P_Q^λ) is obtained by a topological fixed point argument.

Let us show the convergence of the mentioned solution of (P_Q^λ) to a third solution of $(P_{Q,f})$. For $\lambda < \lambda_0$ (a certain positive parameter) U_1, U_2 are supersolutions of(P_Q^λ) and V_1, V_2 are subsolutions of(P_Q^λ). So, arguing as in ii), we obtain two families of solutions $\{u_1^\lambda\}$ and $\{u_2^\lambda\}$ of (P_Q^λ) such that

$$\begin{aligned} -10+\epsilon+\lambda_0 M \;<\; V_1 \quad &\le \quad u_1^\lambda \le\; U_1 \\ V_2 \;\le u_2^\lambda \quad &\le \quad U_2 \;<\; -10-\epsilon. \end{aligned}$$

Moreover, since $\beta_\lambda(u_1^\lambda) = \beta(u_1^\lambda)$, we deduce that $u_1^\lambda = u_1$. Analogously, we conclude that $u_2^\lambda = u_2$. In order to prove that (P_Q^λ) has a third solution u_3^λ, different from u_1^λ and u_2^λ, we show the applicability of a result due to Amann [1] to the function $F(v) := (-\Delta_p + \mathcal{G})^{-1}(QS(\cdot)\beta_\lambda(v) + f_\infty(\cdot))$ on the space $E = L^\infty(\mathcal{M})$. Finally, we get the a priori estimates

$$\int_{\mathcal{M}} |\nabla u_\lambda|^p dA + \int_{\mathcal{M}} \mathcal{G}(u_\lambda)u_\lambda dA = \int_{\mathcal{M}} QS(x)\beta_\lambda(u_\lambda)u_\lambda dA + \int_{\mathcal{M}} f_\infty u_\lambda dA.$$

$$\int_{\mathcal{M}} QS(x)\beta_\lambda(u_\lambda)u_\lambda dA + \int_{\mathcal{M}} f_\infty u_\lambda dA \le C_1$$

which allows to conclude that $u_\lambda \to u$ strongly in $L^2(\mathcal{M})$, $\beta_\lambda(u_\lambda) \rightharpoonup$ weakly in $L^2(\mathcal{M})$ and that $\beta_\lambda \to \beta$ in the sense of maximal monotone graphs (so, $z \in \beta(u)$ see, e.g.,Benilan, Crandall and Saks [10]). Moreover, we get that $\lim_{\lambda\to 0} \| \nabla u_\lambda - \nabla u \|_{L^p(T\mathcal{M})} = 0$. Since $u_3^\lambda \to u_3$ uniformly and $u_1 > -10+\epsilon_0$, there exists ϵ_1 such that $\forall\epsilon < \epsilon_1$, $u_3^\epsilon > -10+\epsilon_0$, which is a contradiction (u_3 necessarily crosses the level-10). ■

Corollary 1 – *Let $R_e(u) = Bu + C$ with β given by (1), $-10B + C > 0$ and $\frac{S_1}{S_0} \leq \frac{M}{m}$. Then we have i) if $0 < Q < \frac{-10B+C}{S_1 M}$, then $(P_{Q,f})$ has a unique solution, ii) if $\frac{-10B+C}{S_0 M} < Q < \frac{-10B+C}{S_1 m}$, then $(P_{Q,f})$ has at least three solutions, iii) if $\frac{-10B+C}{S_0 m} < Q$, then $(P_{Q,f})$ has a unique solution.*

Remark 9. As pointed out in Hetzer [44], the uniqueness of solutions for Q small and Q large still holds if conditions (H_β^*) and ($H_\mathcal{G}$) are replaced by $\mathcal{G} \in C^1(\mathbb{R})$, $\beta \in C^1(\mathbb{R}-\{-10\})$, $m \leq \beta(r) \leq M$, $\forall r \in \mathbb{R}-\{-10\}$, $\inf\{\frac{\mathcal{G}'(r)}{\beta'(r)}, r \in [\underline{U}, -10-\epsilon]\} > 0$, where $\underline{U} := \mathcal{G}^{-1}(-\|f_\infty\|_{L^\infty(\mathcal{M})})$ and $\inf\{\frac{\mathcal{G}'(r)}{\beta'(r)}, r \in [-10+\epsilon, +\infty)\} > 0$. Indeed, if Q is small enough, we can construct a supersolution showing that any possible solution u satisfies that $\underline{U} \leq u \leq -10-\epsilon$ on $\mathcal{M}$. Then, any solution u must satisfy $-\Delta_p u + \mathcal{F}(x,u) = f_\infty(x)$ with $\mathcal{F}(x,u) := \mathcal{G}(u) - QS(x)\beta(u)$. Since $\mathcal{F}(x,u)$is a strictly increasing function on $[\underline{U}, -10-\epsilon]$, for a.e. $x \in \mathcal{M}$ we have the uniqueness of solutions. The assumption on $\mathcal{G}$ leads to a similar conclusion when Q is large enough.

3.2. S-shaped bifurcation diagram

As a continuation of the previous results we can improve the answer for the special formulation

$$(P_{Q,C}) \qquad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \mathcal{G}(u) + C \in QS(x)\beta(u) \text{ on } \mathcal{M}.$$

Following Arcoya, Diaz and Tello [6], we shall describe more precisely the bifurcation diagram and in particular, we shall prove that the principal branch (emanating from $(0, \mathcal{G}^{-1}(-C)) \in \mathbb{R}^+ \times L^\infty(\mathcal{M})$) is S-shaped, i.e. it contains at least one turning point to the left and another one to the right. By a turning point to the left (respectively, to the right), we understand a point (Q^*, u^*) in the principal branch such that in a neighborhood in $\mathbb{R}^+ \times L^\infty(\mathcal{M})$ of it, the principal branch is contained in $\{(Q,u) \in \mathbb{R}^+ \times L^\infty(\mathcal{M}) \,/\, Q \leq Q^*\}$(respectively, $\{(Q,u) \in \mathbb{R}^+ \times L^\infty(\mathcal{M}) \,/\, Q \geq Q^*\}$). A previous result is due to Hetzer [43], for the special case of $p = 2$ and β a C^1 function. He proves that the principal branch of the bifurcation diagram has an even number(including zero) of turning points. Our main result already improves this information showing that indeed, this number of turning points is greater than or equal to two.

Semilinear problems with discontinuous forcing terms on an open bounded set and with Dirichlet boundary conditions have been considered in Ambrosetti [2], Ambrosetti, Calahorrano and Dobarro [3], Arcoya and Calahorrano [5] (see also Drazin and Griffel [31], North [59] and Schmidt [65] in the context of energy balance models).

We make the additional assumption

$-(\mathrm{H}_C)$ $\mathcal{G}(-10-\epsilon)+C>0$ and $\dfrac{\mathcal{G}(-10+\epsilon)+C}{\mathcal{G}(-10-\epsilon)+C}\leq\dfrac{S_2M}{S_1m}$.

We start by considering the problem with β a Lipschitz function (as in the Sellers model).

Theorem 6 – *Let β be a Lipschitz continuous function verifying (H_β^*). Then Σ contains an unbounded connected component which is S-shaped containing $(0,\mathcal{G}^{-1}(-C))$ with at least one turning point to the right contained in the region $(Q_1,Q_2)\times L^\infty(\mathcal{M})$, and another one to the left in $(Q_3,Q_4)\times L^\infty(\mathcal{M})$.*

Proof. Step 1. Σ has an unbounded component containing $(0,\mathcal{G}^{-1}(-C))$: We claim that the following result, due to Rabinowitz [61], can be applied to our case: "Let E a Banach space. If $F:\mathbb{R}\times E\to E$ is compact and $F(0,u)\equiv 0$, then Σ contains a pair of unbounded components C^+ and C^- in $\mathbb{R}^+\times E$, $\mathbb{R}^-\times E$, respectively and $C^+\cap C^-=\{(0,0)\}$". To do so, we consider the translation of u given by $v:=u-\mathcal{G}^{-1}(-C)$. Obviously, v is a solution of

$$-\Delta_p v+\hat{\mathcal{G}}(v)=QS(x)\hat{\beta}(v)\qquad\text{on }\mathcal{M}\tag{24}$$

where $\hat{\mathcal{G}}(\sigma)=\mathcal{G}(\sigma+\mathcal{G}^{-1}(-C))+C$ and $\hat{\beta}(\sigma)=\beta(\sigma+\mathcal{G}^{-1}(-C))$. We define $\hat{\Sigma}$ in an analogous way to Σ. Let $E=L^\infty(\mathcal{M})$ and define $F(Q,v)=(-\Delta_p+\hat{\mathcal{G}})^{-1}(QS(x)\hat{\beta}(v))$. Then F is the composition of a continuous operator and a compact one (recall that $p\geq 2$), so F is also compact. On the other hand, if $Q=0$ problem (24) has a unique solution $v=0$, so $F(0,0)=0$. In conclusion, $\hat{\Sigma}$ contains two unbounded components $\hat{C}^+$and $\hat{C}^-$ on $\mathbb{R}^+\times L^\infty(\mathcal{M})$ and $\mathbb{R}^-\times L^\infty(\mathcal{M})$ respectively and $\hat{C}^+\cap\hat{C}^-=\{(0,0)\}$. Since Σ is a translation of $\hat{\Sigma}$, Σ contains two unbounded components C^+ and C^- on $\mathbb{R}^+\times L^\infty(\mathcal{M})$ and $\mathbb{R}^-\times L^\infty(\mathcal{M})$ respectively, and that $C^+\cap C^-=\{(0,\mathcal{G}^{-1}(-C))\}$. Since $Q\geq 0$ in the studied model, we are interested inC^+. In order to establish the behaviour of C^+, we also recall that for every $q>0$, there exists a constant $L=L(q)$ such that, if $0\leq Q\leq q$, then every solution u_Q of $(P_{Q,C})$ verifies $\|u_Q\|_{L^\infty(\mathcal{M})}\leq L(q)$. Since the principal component is unbounded, its projection over the Q-axis is $[0,\infty)$. On the other hand, if Q is large enough, $(P_{Q,C})$ has a unique solution u_Q, and this solution

is greater than $\mathcal{G}^{-1}(QS_0M - C)$. Since $\lim_{|s|\to\infty} |\mathcal{G}(s)| = +\infty$, then the unbounded branch containing $(0, \mathcal{G}^{-1}(-C))C^+$ should go to (∞, ∞).
Step 2. Bifurcation diagram for two auxiliary problems: We consider the auxiliary zero-dimensional models

$$\begin{array}{llcll} (P_1) & \mathcal{G}(u) + C & = & QS_1\beta(u) & u \in \mathbb{R}, \\ (P_2) & \mathcal{G}(u) + C & = & QS_2\beta(u) & u \in \mathbb{R}. \end{array}$$

The number of solutions to these problems depends clearly on the values of Q. Let us call Σ_1 and Σ_2 the bifurcation diagrams of (P_1) and (P_2), respectively. By assumptions $(H_\mathcal{G})$, (H_β^*) and (H_C), the principal components of Σ_1 and Σ_2 are S-shaped. We also remark that the sets

$$K_1^i := \{(Q, u_Q) \in \mathbb{R}^2 : 0 \leq Q \leq \frac{\mathcal{G}(-10 - \epsilon) + C}{S_i m},\ u_Q = \mathcal{G}^{-1}(QS_i m - C)\},$$

$$K_2^i := \{(Q, u_Q) \in \mathbb{R}^2 : Q \geq \frac{\mathcal{G}(-10 + \epsilon) + C}{S_i M},\ u_Q = \mathcal{G}^{-1}(QS_i M - C)\}$$

are contained in Σ_i, $i = 1, 2$.
Step 3. A comparison argument: If $Q < Q_3$, there exists u_Q solution of $(P_{Q,C})$ such that $u_Q < 10 - \epsilon$. Thus u_Q satisfies

$$QS_2m \leq -\Delta_p u + \mathcal{G}(u) + C \leq QS_1m \text{ on } \mathcal{M}.$$

Let u_Q^1 and u_Q^2 be the solutions of the problems

$$\begin{array}{lcl} \mathcal{G}(u) + C & = & QS_1m \text{ on } \mathcal{M} \\ \mathcal{G}(u) + C & = & QS_2m \text{ on } \mathcal{M}, \end{array}$$

respectively. That is, (Q, u_Q^1) and (Q, u_Q^2) live in Σ_1 and Σ_2, respectively. Now, if $Q < Q_3$,

$$-\Delta_p u_Q^2 + \mathcal{G}(u_Q^2) \leq -\Delta_p u_Q + \mathcal{G}(u_Q) \leq -\Delta_p u_Q^1 + \mathcal{G}(u_Q^1),$$

and so by the comparison principle for the monotone problem $-\Delta_p u + \mathcal{G}(u) = f \in L^2(\mathcal{M})$ on $\mathcal{M}$, we have that u_Q^2
$leq\ u_Q \leq u_Q^1$. Therefore, the component of Σ starting in $(0, \mathcal{G}^{-1}(-C))$, lives between Σ_1 andΣ_2 to arrive at (Q_3, u_{Q_3}), where u_{Q_3} is the minimal solution of (P_{Q_3}). Analogously, if we denote by u_{Q_2} the maximal solution of (P_{Q_2}), we can prove that the component ofΣ which connects (Q_2, u_Q^2) with (∞, ∞) lives between Σ_1 and Σ_2. From $Q_2 < Q_3$, the branch containing $(0, \mathcal{G}^{-1}(-C))$ is un bounded and by the uniqueness of solution for $(P_{Q,C})$ when $Q > Q_4$, we get that this branch is necessarily S-shaped. ∎

Our next result avoids the Lipschitz assumption made in Theorem 6.

Theorem 7 – *Let β a general maximal monotone graph satisfying (H^*_β) and assume (H_C). Then Σ has an unbounded S-shaped component containing $(0, \mathcal{G}^{-1}(-C))$ with at least one turning point to the right contained in the region $(Q_1, Q_2) \times L^\infty(\mathcal{M})$ and another one to the left in $(Q_3, Q_4) \times L^\infty(\mathcal{M})$, respectively.*

To prove Theorem 7, we approximate problem $(P_{Q,C})$ when β is not Lipschitz continuous. We only need to show the *convergence* of the principal branches C_n of these approximating problems to a S-shaped unbounded connected set C of solutions of $(P_{Q,C})$. For this reason, let us recall the notions of *liminf* and *limsup* of a sequence of subsets C_n of a metric space X:

$$\begin{aligned}\liminf_{n\to\infty} C_n := \{p \in\ & X: \text{ for any neighbourhood } U(p) \text{ of } p \text{ in } X \\ & \exists n_0 \in \mathbb{N} : \mathrm{U(p)} \cap \mathrm{C_n} \neq \emptyset\ \forall \mathrm{n} \geq \mathrm{n_0}\}, \\ \limsup_{n\to\infty} C_n := \{p \in\ & X: \text{ for any neighbourhood } U(p) \text{ of } p \text{ in } X \\ & U(p) \cap C_n \neq \emptyset \text{ for infinitely many } n\}.\end{aligned}$$

A lemma due to Whyburn [70] shows that if i) $\lim_{n\to\infty}\ inf\ C_n \neq \emptyset$ and ii) $\cup_{n=1}^\infty C_n$ is precompact, then $\lim_{n\to\infty}\ sup\ C_n$ is a nonempty, precompact, closed and connected set. *Proof of Theorem 7.* The method of super and sub solutions proves that if $Q > Q_2$, then there exists a solution of $(P_{Q,C})$ greater than $-10+\epsilon$. Analogously, we know that if $0 \leq Q < Q_3$, then (P_Q) has a solution smaller than $-10-\epsilon$. It is clear that these functions are not the unique solutions of $(P_{Q,C})$ in those intervals and that the uniqueness holds at least in the Q-intervals $[0, Q_1)$ and (Q_4, ∞). Since we can not apply directly Rabinowitz theorem to our problem, we consider the family $\beta_n = n(I - (I - \frac{1}{n}\beta)^{-1})$, $n \in \mathbb{N}$ to approximate β in the sense of maximal monotone graphs when $n \to \infty$. Notice that since βverifies (H^*_β), then β_n is a Lipschitz bounded nondecreasing function (see Brezis [14]) and that $\beta_n(s) = \beta(s)$ for any $s \notin [-10-\epsilon, -10+\epsilon+\frac{M}{n}]$, $\forall n$.

Let u_n be the solutions of the approximated problem

$$(P^n_Q) \quad -\Delta_p u_n + \mathcal{G}(u_n) + C \;=\; QS(x)\beta_n(u_n) \text{ on } \mathcal{M}$$

and let Σ_n the bifurcation diagrams for (P^n_Q). Let us denote by S_n the component of Σ_n containing $(0, \mathcal{G}^{-1}(-C))$. By Theorem 6, every S_n is an unbounded, connected and S-shaped set. First of all, we are going to prove that lim sup S_n is a connected and closed set of solutions to problem (P_Q). In order to apply Whyburn' result, we consider the sets C^j_n $(j > Q_4)$ defined as $S_n \cap ([0, j] \times L^\infty(\Omega))$, $\forall n \in \mathbb{N}$ containing $(0, \mathcal{G}^{-1}(-C))$. It is easy to see that these sets are connected and that i) is verified. Let us check (ii),$\cup_{n=1}^{\infty} C_n^j$ is precompact. Since X is a Banach space, it suffices to prove that every sequence $\{(Q_l, u_l)\}_{l\in \mathbb{N}} \subset \cup_{n=1}^{\infty} C_n^j$ contains a subsequence $\{(Q_{l_k}, u_{l_k})\}$ converging in X. From $Q_l \in [0,j]$, there exists $Q \in [0,j]$ and a subsequence of $\{Q_l\}$which we still call $\{Q_l\}$, such that $Q_l \to Q$. On the other hand,u_l is a solution of the problem

$$-\Delta_p u_l + \mathcal{G}(u_l) + C \;=\; QS(x)\beta_l(u_l) \text{ on } \mathcal{M}.$$

Taking u_l as a test function in this equation, we obtain the estimate

$$\int_{\mathcal{M}} |\nabla u_l|^p dA + \int_{\mathcal{M}} \frac{B}{2}|u_l|^2 dA \leq \frac{(j\|S\|_\infty M + C)^2|\mathcal{M}|}{2B} \tag{25}$$

where $|\mathcal{M}|$ is the Hausdorf measure of $\mathcal{M}$. Then u_l is a bounded sequence in V. From the compact embedding $V \subset L^\infty(\mathcal{M})$ when $p > 2$, there exist $u \in L^\infty(\mathcal{M})$ and asubsequence $\{u_{l_k}\}$ of $\{u_l\}$ such that $u_{lk} \to u$in $L^\infty(\mathcal{M})$. If $p = 2$, then $\{u_l\}$ is a bounded sequence in the Sobolev space $H^2(\mathcal{M})$. From the compact embedding $H^2(\mathcal{M}) \subset C(\mathcal{M})$, we deduce the existence of a subsequence $\{u_{lk}\}$ and $u \in C(\mathcal{M})$, such that $u_{lk} \to u$ in $L^\infty(\mathcal{M})$. Thus $\cup_{n=1}^{\infty} C_n^j$ is precompact. Then by Whyburn's result $C^j \equiv \lim_{n\to\infty} sup\, C_n^j$ is a connected and compact set in X. Moreover, since every S_n is unbounded and fixed $\overline{Q}$, the solutions u_Q are uniformly bounded in $L^\infty(\mathcal{M})$, for $Q < \overline{Q}$, we have that $C_k^j \cap (\{j\} \times L^\infty(\mathcal{M})) \neq \emptyset$, for all $j \in \mathbb{N}$.

Now, we prove that the set C^j is contained in Σ. Let us see that for every $Q \in [Q_1, Q_4]$, we have that every $(Q, u) \in C^j$ verifies that u is a solution of (P_Q) (notice that it is true for every $Q \in (0, Q_1] \cup [Q_4, +\infty)$ from $C_n^j = C^j$ in these intervals). Let $(Q, u) \in C^j = \lim_{n\to\infty} \sup C_n^j$, that is, there exists a subsequence of $(Q_n, u_n) \in C_n^j$ such that $(Q_{n_k}, u_{n_k}) \to (Q, u)$ in $\mathbb{R}\times L^\infty(\mathcal{M})$. From estimate (25) and the compact embedding $H^2(\mathcal{M}) \subset L^\infty(\mathcal{M})$ (for $p = 2$) and $V \subset L^\infty(\mathcal{M})$ (for $p > 2$), we deduce the existence of $u \in L^\infty(\mathcal{M})$ and a subsequence of $\{(Q_{n_k}, u_{n_k})\}$ which we call $\{(Q_{n_k}, u_{n_k})\}$, such that

$$(Q_{n_k}, u_{n_k}) \;\to\; (Q, u) \text{ in } \mathbb{R} \times L^\infty(\mathcal{M}),$$

Since $\beta_n \to \beta$ in the sense of maximal monotone graphs of $\mathbb{R}^2$, we have that $\beta_{n_k}(u_{n_k}) \rightharpoonup z \in \beta(u)$ weakly in $L^2(\mathcal{M})$. Using a Minty's type argument we deduce that u is a solution of the problem $(P_{Q,C})$. Thus $(Q, u) \in \Sigma$ and $C^j \subset \Sigma$. Since for all n and j, $C_n^j \cap (\{j\} \times L^\infty(\mathcal{M})) \neq \emptyset$, there exists $\{(j, u_n)\}_{n\in\mathbb{N}}$ such that $(j, u_n) \in C_n^j$, that is,

$$-\Delta_p u_n + \mathcal{G}(u_n) = jS(x)\beta_n(u_n) - C \quad \text{in } \mathcal{M}.$$

Using that the operator $(\Delta_p + \mathcal{G})^{-1}$ is compact in $L^\infty(\mathcal{M})$, there exists a subsequence $u_{n_k} \to u$ in $L^\infty(\mathcal{M})$. Thus $(j, u) \in C^j$ and $C^j \cap (\{j\} \times$

$L^\infty(\mathcal{M})) \neq \emptyset$. Since $j > Q_4$, u_j is the unique solution of $(P_{Q,C})$. On the other hand, we know that $\Sigma \cap (j,\infty) \times L^\infty(\mathcal{M}) = \Sigma_M \cap (j,\infty) \times L^\infty(\mathcal{M})$. So, we have obtained a connected unbounded set which starts in $(0, \mathcal{G}^{-1}(-C))$. The proof ends with the argument used in the proof of Theorem 6 for $Q_2 < Q_3$. ∎

Remark 10. We point out that our results remain true for the more general equation

$$-\text{div}(k(x)|\nabla u|^{p-2}\nabla u) + \mathcal{G}(u) + C \in QS(x)\beta(u) \text{ on } \mathcal{M},$$

where $k(x)$ is a given bounded function with $k(x) \geq k_0 > 0$ a.e.$x \in \mathcal{M}$, representing the *eddy diffusion coefficient.* When $\mathcal{M} = \mathcal{S}^1$, it is usually assumed that $S(x) = S(\lambda)$ and $k(x) = k(\lambda, \phi)$ with λ the latitude and ϕ the longitude. So, in that case, the corresponding solutions are not ϕ-invariant.

Remark 11. By using a shooting method, it is possible to show (see Diaz and Tello [27] that there exist infinitely many equilibrium solutions for some values of Q when we study the one-dimensional problem

$$(P_{1,Q,C}) \begin{cases} -(|u'|^{p-2}u')' + Bu + C \in Q\beta(u) & x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases}$$

If $Q_1 < Q < Q_2$ then $(P_{1,Q,C})$ has infinitely many solutions. Moreover, there exists $K_0 \in \mathbb{N}$ such that for every $K \in \mathbb{N}$, $K \geq K_0 \in \mathbb{N}$ there exists at least a solution which crosses the level $u_K = -10$, exactly K times.

Remark 12. After my lecture at the Collège de France, Professor J.L. Lions pointed out to me the reference Rahmstorf [63] where a S-shaped diagram bifurcation curve arises in the context of the Atlantic Thermohaline Circulation in reponse to changes in the hydrological cycle.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Review, 18, 4 (1976), 620–709.

[2] A. Ambrosetti, Critical points and non linear variational problems. Supplement au Bulletin de la Societé Mathématique de France. Mémoire 49 (1992).

[3] A. Ambrosseti, M. Calahorrano and F.Dobarro, Global branching for discontinuous problems, Comment. Math. Univ. Carolinae, 31 (1990), 213–222.

[4] S.N. Antontsev and J.I. Diaz, New results on localization of solutions of nonlinear elliptic and parabolic equations obtained by energy methods, Soviet Math. Dokl., 38 (1989), 535–539.

[5] D. Arcoya, and M. Calahorrano, Multivalued non-positone problems, Rend. Mat. Acc. Lincei, 9 (1990), 117–123.
[6] D. Arcoya, J.I. Diaz and L. Tello, S-shaped bifurcation branch in a quasilinear multivalued model arising in climatology, Journal of Differential Equations, 150 (1998), 215–225.
[7] O. Arino, S. Gautier, and J.P. Penot, A fixed point theorem for sequentially continuous mappings with applications to ordinary differential equations, Funkcialaj Ekvacioj, 27 (1984), 273–279.
[8] T. Aubin, Nonlinear analysis on manifolds. Monge-Ampere equations. Springer-Verlag, New York, 1982.
[9] M. Badii and J.I.Diaz, Time periodic solutions for a diffusive energy balance model in climatology, J. Mathematical Analysis and Applications, 233 (1999), 713–729.
[10] Ph. Benilan, M.G. Crandall and P.Saks, Some L^1 existence and dependence results for semilinear elliptic equations under nonlinear boundary conditions, Appl. Math. Optimization, 17 (1988), 203–224.
[11] R. Bermejo, Numerical solution to a two-dimensional diffusive climate model, in Modelado de sistemas en oceanografia, climatologia y ciencias medio-ambientales: Aspectos matemáticos y numéricos, (A. Valle and C. Parés eds.), Universidad de Málaga, (1994), 15–30.
[12] R. Bermejo, J.I. Diaz and L. Tello, Article in preparation.
[13] K. Bhattacharya, M. Ghil and I.L. Vulis, Internal variability of an energy balance climate model, J. Atmosph. Sci., 39(1982), 1747–1773.
[14] H. Brezis, Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert, North Holland, Amsterdam, 1973.
[15] M.I. Budyko, The effects of solar radiation variations on the climate of the Earth, Tellus, 21 (1969), 611–619.
[16] T. Cazenave and A. Haraux, Introduction aux problèmes d'évolution semi-linéaires. Mathématiques et Applications, Ellipses, Paris, 1990.
[17] A.S. Dennis, Weather modifications by cloud seeding. Academic Press, 1980.
[18] J.I. Diaz , Nonlinear partial differential equations and free boundaries. Pitman, Londres, 1984.
[19] J.I. Diaz, Mathematical analysis of some diffusive energy balance climate models, in Mathematics, climate and environment, (J.I. Diaz and J.-L.Lions, eds.), Masson, Paris, (1993), 28–56.
[20] J.I. Diaz, On the controllability of some simple climate models, in Environment, economics and their mathematical models, (J.I.Diaz and J.-L. Lions eds.), Masson, (1994), 29–44.
[21] J.I. Diaz, On the mathematical treatment of energy balance climate models, in The mathematics of models for climatology and environment, NATO ASI Series, Serie I: Global Environmental Change, 48, (J.I. Diaz ed.), Springer, Berlin, (1996), 217–252.

[22] J.I. Diaz, On a free boundary problem arising in climatology, to appear.

[23] J.I. Diaz, A. Fasano and A.Meirmanov, On the disappearence of the mushy region in multidimensional Stefan problems, in Free boundary problems: theory and applications, Vol. VII. Pitman. London (1993), 124–129.

[24] J.I. Diaz, J. Hernandez and L. Tello, On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in climatology, J. Math. An. Appl., 216 (1997), 593–613.

[25] J.I. Diaz and G. Hetzer, A quasilinear functional reaction-diffusion equation arising in climatology, in Équations aux dérivées partielles et applications: Articles dédiés à Jacques-Louis Lions, Gauthier-Villars, Paris (1998), 461–480.

[26] J.I. Diaz and L. Tello, A nonlinear parabolic problem on a Riemannian manifold without boundary arising in climatology, Collect. Math., 50 (1999), 19–51.

[27] J.I. Diaz and L. Tello, Infinidad de soluciones estacionarias para un modelo climatológico simple via un método de tiro, Actas XVI CEDYA, VI CMA, (R. Montenegro, G. Montero and G. Winter eds.), Servicio de Publicaciones de la Univ. de Las Plamas, 1999, 1469–1476.

[28] J.I. Diaz and F. de Thélin, On a nonlinear parabolic problem arising in some models related to turbulent flows, SIAM J. Math. An., 25, 4 (1994), 1085–1111.

[29] J.I. Diaz and L. Veron, Local vanishing properties of solutions of elliptic and parabolic problems quasilinear equations, Trans. A.M.S., 290 (1985), 787–814.

[30] J.I. Diaz and I.I. Vrabie, Existence for reaction-diffusion systems, J. Math. Analysis and Appl., 188 (1994), 521–528.

[31] P.G. Drazin and D.H. Griffel, On the branching structure of diffusive climatological models, J. Atmos. Sci., 34 (1977), 1696–1706.

[32] E. Fereisel, A note on uniqueness for parabolic problems with discontinuous nonlinearities, Nonlinear Analysis, 16(1991), 1053–1056.

[33] E. Fereisel and J. Norbury, Some existence, uniqueness, and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities, Proc. Royal Soc. Edinburgh, 119A (1991), 1–17.

[34] E. Fereisel and F. Simondon, Convergence for degenerate parabolic equations in one dimension, C.R. Acad. Sci. Paris, Sér. I, 323 (1996), 251–255.

[35] M. Ghill and S. Childress, Topics in geophysical fluid dynamics, Springer-Verlag, 1987.

[36] R. Gianni, Existence of the free boundary in a multi-dimensional combustion problem, Proc. Royal Soc. Edinburgh, 125A, (1995), 525–543.

[37] R. Gianni and J. Hulshof, Thesemilinear heat equation with a Heaviside source term. Euro. J. of Applied Mathematics, 3 (1992), 367–379.

[38] R. Gianni and P. Mannucci, Some existence theorems for an N-dimensional parabolic equation with a discontinuous source term, SIAM J. Math. Anal., (1993), 618–633.
[39] I.M. Held and M.J. Suarez, Simple albedo feedback models of the ice-caps, Tellus, 36 (1974).
[40] I.M. Held, D.I. Linder and M.J. Suarez, Albedo feedback, the meridional structure of the effective heat diffusivity, and climatic sensitivity: Results from dynamic and diffusive models, American Meteorological Society (1981), 1911–1927.
[41] K. Henderson-Sellers and McGufiie, A climate modelling primer, John Wiley and Sons, Chichester, 1987.
[42] G. Hetzer, H. Jarausch and W. Mackens, A multiparameter sensitivity analysis of a 2D diffusive climate model, Impact and computing in science and engineering, 1 (1989),327–393.
[43] G. Hetzer, The structure of the principal component for semilinear diffusion equations from energy balanceclimate models, Houston Journal of Math., 16 (1990), 203–216.
[44] G. Hetzer, S-shapedness for energy balance climate models of Sellers-type, in The Mathematics of models for climatology and environment, NATO ASI Series, Serie I: global environmental change, 48, (J.I. Diaz ed.), Springer, Berlin,(1996), 25–288.
[45] G. Hetzer, A functional reaction-diffusion equation from climate modeling: S-shapedness of the principal branch. Diff. and Integral Eq. 8 (1995), 1047–1059.
[46] G. Hetzer, Global existence, uniqueness, and continuous dependence for a reaction-diffusion equation with memory, EJDE, (1996), 1–16.
[47] G. Hetzer and P.G. Schmidt, A global attractor and stationary solutions for a reaction-diffusion system arising from climate modeling, Nonlinear analysis. TMA, 14 (1990), 915–926.
[48] G. Hetzer and P.G. Schmidt, Analysis of energy balance models, in World congress of nonlinear analysts'92, Tampa, FL, August 19-26, 1992, (V. Laksmikantham ed.), Walter de Gruyter, Berlin-New York (1996),1609–1618.
[49] G. Hetzer and L. Tello, Sobre un sistema parabólico no lineal "multicapa" en limatologia, in Actas del XV Congreso de ecuaciones diferenciales y aplicaciones, Serv. Publ. Univ. de Vigo (1998), 471–476.
[50] J. T. Kiehl, Atmoshperic general circulation modeling, In Climate system modeling, (K.E. Trenberth ed.), Cambridge University Press, (1992), 319–370.
[51] R.Q. Lin and G.R. North, A study of abrupt climate change in a simple nonlinear climate model. Climate dynamics, 4 (1990), 253–261.
[52] J.-L. Lions, El Planeta Tierra, Espasa-Calpe. Serie Instituto de España. Madrid, 1990.

[53] J.-L. Lions, R. Temam and S. Wang, New formulations of the primitive equations of atmosphere and applications, Nonlinearity, 5 (1992), 237–288.

[54] J.G. Mengel, D.A. Short and G.R. North, Seasonal snowline instability in an energy balance model, Climate Dynamics, 2 (1988), 127–131.

[55] A.M. Meirmanov, The Stefan problem, Walter de Gruyter, Berlin-New York, 1992.

[56] M. Nakao, A difference inequality and its application to nonlinear evolution equations, J. Math. Soc. Japan, 30 (1978), 747–762.

[57] J. von Neumann, Can we survive technology?, Nature, 1955. (Also in Collected works. Vol VI, Pergamon, 1966.)

[58] G.R. North, Multiple solutions in energy balance climate models, in Paleogeography, paleoclimatology, paleoecology, 82, Elsevier Science Publishers, B.V. Amsterdam (1990), 225–235.

[59] G.R. North, Introduction to simple climate model, in Mathematics, climate and environment, (J.I. Diaz and J.-L.Lions eds.), Masson, Paris (1993), 139–159.

[60] G.R. North, J.G. Mengel and D.A. Short, Simple energy balance model resolving the season and continents: Applications to astronomical theory of ice ages. J. Geophys. Res., 88 (1983), 6576–6586.

[61] P.H. Rabinowitz, A global theorem for nonlinear eigenvalue problems and applications, in Contributions to nonlinear functional analysis, (E.H. Zarantonello ed.), Academic Press, New York (1971), 11–36.

[62] J.M. Rakotoson and B. Simon, Relative rearrangement on a measure space. Application to the regularity of weighted monotone rearrangement. Part. II. Appl. Math. Lett., 6 (1993), 79–82.

[63] S. Rahmstorf, Bifurcations of the atlantic thermohaline circulation in reponse to changes in the hydrological cycle, Nature, 378 (1995), 145–149.

[64] W.D. Sellers, A global climatic model based on the energy balance of the earth-atmosphere system, J. Appl. Meteorol., 8 (1969), 392–400.

[65] B.E. Schmidt, Bifurcation of stationary solutions for Legendre-type boundary value problems arising from climate modeling. Ph. D. Thesis. Auburn Univ. 1994.

[66] J. Simon, Compact sets in the space $L^p(0,T;B)$, Annali Mat. Pura et Appl., CXLVI (1987), 65–96.

[67] I. Stakgold, Free boundary problems in climate modeling, in Mathematics, climate and environment, (J.I.Diaz and J.-L.Lions eds.), Masson (1993), 177–188.

[68] P.H. Stone, A simplified radiative-dynamical model for the static stability of rotating atmospheres, J. Atmos. Sci., 29, 3 (1972), 405–418.

[69] L. Tello, Tratamiento matemático de algunos modelos no lineales que aparecen en climatologia. Ph. D. Thesis. Univ. Complutense de Madrid, 1996.
[70] G.T. Whyburn, Topological analysis, Princeton Univ. Press, Princeton, 1955.
[71] I.I. Vrabie, Compactness methods for non linear evolutions, Pitman, London, 1987.
[72] X. Xu, Existence and regularity theorems for a free boundary problem governing a simple climate model, Aplicable Anal., 42 (1991), 33–59.

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