Infinitely many stationary solutions for a simple climate model via a shooting method

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Abstract

In this paper we study the number of steady solutions of a nonlinear model arising in Climatology. By applying a shooting method we show the existence of infinitely many steady solutions for some values of a parameter (the solar constant). This method allows us to determine how many times a solution attains the critical temperature (-10^oC) at which the coalbedo is assumed to be discontinuous.

1 Introduction

The model is based on a global energy balance over the Earth surface. The unknown is the mean temperature over each parallel and the time scale is considered relatively large. These models were introduced independently by M. Budyko and W. Sellers on 1969.

The energy balance is obtained when one expresses the *heat variation* in terms of $R_a - R_e + D$, where R_a, R_e represent the absorbed and the emitted energies by the Earth and D the heat diffusion which is given by a second order elliptic operator. By expressing each component of the above balance in mathematical terms as function on the temperature u, we obtain a nonlinear parabolic partial differential equation. The general spatial domain for this kind of models is a compact two-dimensional Riemannian manifold $\mathcal{M}$ simulating the Earth. Often, the two-dimensional model is reduced in a one-dimensional model by considering the averaged temperature over each parallel. Such simplification considers the temperature $u(x,t)$ only dependent on the latitude λ and $x = \sin\lambda$. By assuming North South symmetry, the obtained model is

$$(P_0)\begin{cases} u_t - (|u_x|^{p-2}u_x)_x + \mathcal{G}(u) + C \in Q\beta(u), & (t,x) \in (0,T)\times(0,1), \\ u_x(t,0) = u_x(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x) & x \in (0,1). \end{cases}$$

In the pioneer models, the diffusion operator was linear ($p = 2$). Later, Stone [1972] proposed a nonlinear diffusion operator for this kind of models ($p = 3$) in order to consider the negative feedback in the eddy flux. The formulation in (P) include both cases. In this equation it appears a multivalued term $\beta(u)$ which represents the planetary coalbedo (the fraction of the incoming radiation flux which is absorbed by the surface). This term is multiplying by the Solar constant Q. From the physical point of view, such a constant can have small variations depending on the obliquity of the terrestrial axis, eccentricity of the Earth orbit, etc.

These climatological models have been studied by different authors. These models are sensitive with respect to variations of the parameter Q. By using a shooting method we show that there exist infinitely many equilibrium solutions for some values of Q. This result give more precision than the obtained for two-dimensional climatological models (see Díaz – Hernández – Tello [1997] and Arcoya – Díaz – Tello [1998]), where it was proven that there exist at least three stationary solutions for values of Q in a bounded interval, as well as the uniqueness of solution for Q big or small enough. As a consequence of such results we can affirm

(i) if $Q < Q_1$ or $Q > Q_2$ then the stationary problem associated to (P_0) has a unique solution;

(ii) if $Q_1 < Q < Q_2$ then the stationary problem associated to (P_0) has at least three solutions,

where

$$Q_1 := \frac{\mathcal{G}(-10) + C}{M} \qquad \text{and} \qquad Q_2 := \frac{\mathcal{G}(-10) + C}{m}, \tag{1}$$

with m and M the infimum and the supremum of β, respectively.

The results of this paper improve also a part of Tello [1996], where we assume $p = 2$. We also mention the works Hetzer [1992], Drazin-Griffel [1977], North [1993], B. Schmidt [1994] for other analysis of multiplicity of solutions.

2 Multiplicity of stationary solutions.

We are concerning with the stationary boundary value problem associated to the model (P_0),

$$(P) \begin{cases} -(|u'|^{p-2}u')' + \mathcal{G}(u) + C \ \in \ Q\beta(u) & x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases}$$

where $p \geq 2$ and $Q > 0$. We assume the following conditions:

(H_1) β is a bounded maximal monotone graph of the Heaviside type defined by

$$\beta(u) \ = \ \begin{cases} m & \text{if } u < -10, \\ [m, M] & \text{if } u = -10, \\ M & \text{if } u > -10, \end{cases}$$

with $0 < m < M$,

(H_2) $\mathcal{G}$ is continuous increasing function with $\mathcal{G}(0) = 0$ and $\lim_{s\to\infty} |\mathcal{G}(s)| = +\infty$.

(H_3) $\mathcal{G}(-10) + C > 0$.

We will say that u is a solution of (P) if $u \in C^1([0,1])$ and there exists $z \in L^\infty(0,1)$, $z(x) \in \beta(u(x))$ a.e. $x \in (0,1)$ such that u verifies the equation $-(|u'|^{p-2}u')'+\mathcal{G}(u)+C = Qz$ in the weak sense.
The main goal of this paper is to prove that there exists an interval of Q where the problem has infinitely many solutions.

Theorem 1 *If $Q_1 < Q < Q_2$ then (P) has infinitely many solutions. Moreover, there exists $N_0 \in \mathbb{N}$ such that for every $K \geq N_0$ there exists at least a solution u_K which cross its level $u_K = -10$, exactly K times.*

Proof.
We start by computing the intersections between the graphs $\mathcal{G}(u) + C$ and $Q\beta(u)$. These are constant solutions of problem (P) and the number of intersections depends on the value of the parameter $Q > 0$. We compute without difficulty two significative values, Q_1 and Q_2. If $Q = Q_1$ or $Q = Q_2$ then the graphs has two points in common. Moreover, if $Q < Q_1$ or $Q > Q_2$ then the intersection is only one point. If $Q_1 < Q < Q_2$ then the intersection points are the following:

$$u_1 = \mathcal{G}^{-1}(Qm - C) < -10, \qquad u_2 = -10 \text{ and } u_3 = \mathcal{G}^{-1}(QM - C) > -10. \tag{2}$$

Step 1. We study the phase portrait (u, u') for an auxiliar Cauchy problem. Since the equation (P) is conservative, we get the conservation of the total energy

$$\frac{|u'(x)|^p}{p} + V(u(x)) = E, \qquad \forall x \in \mathbb{R}, \tag{3}$$

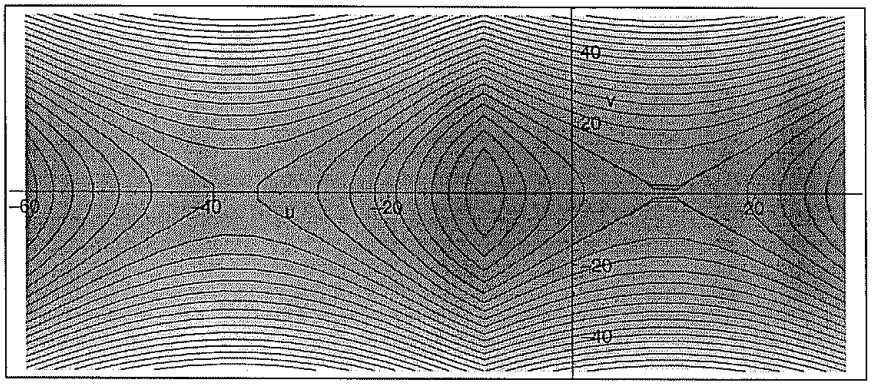
for some constant E and for the following potential function

$$V(u) = \begin{cases} (QM - C)u - G(u), & u \geq -10, \\ (Qm - C)u - G(u) - 10Q(M - m), & u < -10, \end{cases} \tag{4}$$

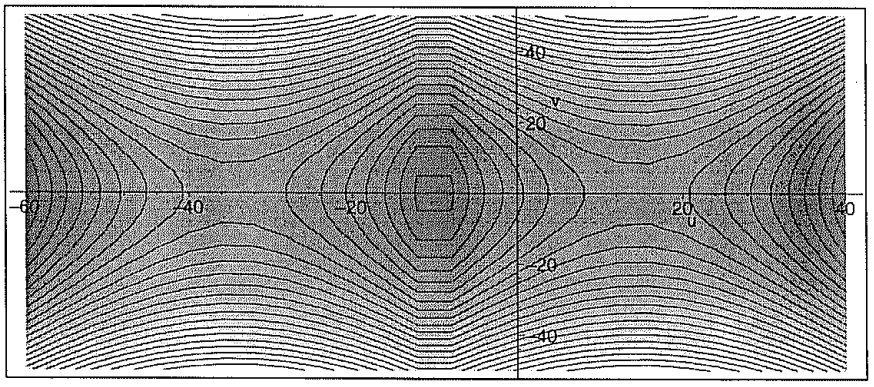
where $G(u) = \int_0^u \mathcal{G}(s)ds$. This function V allows us to get the trajectories (u, u') corresponding to each energy level E, from the equation (3) (notice that V is continuous but it is not C^1). From the restrictions of $-V$ on the sets $x \leq -10$ and $x \geq -10$ are convex, we get that V has three relative extrema: two of them are maxima, u_1 and u_3, and the other u_2 is a minimum. So, we get three constant stationary solutions: $(u_1, 0)$ and $(u_3, 0)$ are saddle points and $(u_2, 0)$ is a centre.
If $Q_1 < Q < Q_2$ then $V(u_2) < V(u_3)$ and $V(u_2) < V(u_1)$. We obtain trajectories which does not cross the axis $u' = 0$: they correspond to the energy levels $E > \max\{V(u_1), V(u_3)\}$. The trajectories corresponding to the energy levels $E < V(u_2)$ and $\min\{V(u_1), V(u_3)\} < E < \max\{V(u_1), V(u_3)\}$ cross the axis $u' = 0$ exactly in one point. Finally, if $V(u_2) < E < \min\{V(u_1), V(u_3)\}$ we find periodic trajectories (which cut in two different points $(a, 0)$ and $(b, 0)$ the axis $u' = 0$, where $u_1 < a < -10 < b < u_3$) and the others only in one point.

When we try to compare $V(u_1)$ and $V(u_3)$ we find a significative value of Q, which we call $Q_3 \in (Q_1, Q_2)$, verifying $V(u_1) = V(u_3)$. We obtain three different cases for the phases portrait (u, u').

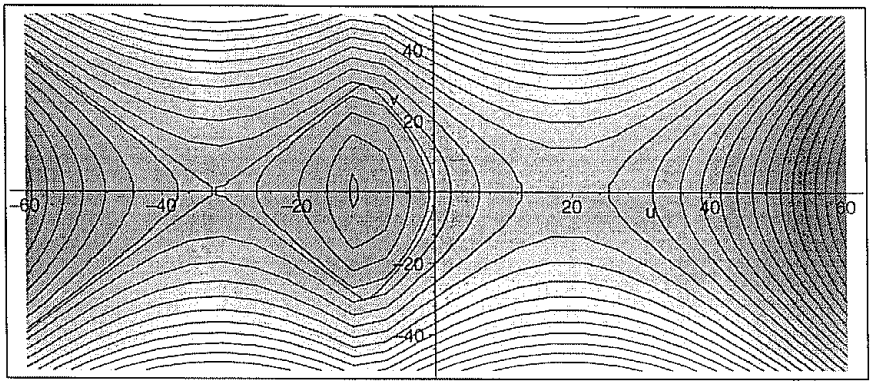
(a) If $Q_1 < Q < Q_3$ then $V(u_3) < V(u_1)$. There exists a homoclinic orbit with ω-limit equal to u_3, which separates a region of the periodic orbits of the others.



(b) If $Q = Q_3$ then $V(u_1) < V(u_3)$. There exists two heteroclinic orbits with ω-limit equal to u_1 and u_3, respectively, which separate a region of the periodic orbits of the others.



(c) If $Q_3 < Q < Q_2$ then $V(u_3) < V(u_1)$. There exists a homoclinic orbit with ω-limit equal to u_1, which separates a region of the periodic orbits of the others.



In order to solve the boundary value (P) we shall use a shooting method, which is described in the following step.

Step 2: Shooting method. We consider the following Cauchy problem depending of the parameter μ,

$$(P_\mu)\begin{cases} -(|u'|^{p-2}u')' + \mathcal{G}(u) + C \in Q\beta(u), & x \in I\!R^+, \\ u'(0) = 0, \\ u(0) = \mu. \end{cases}$$

Our purpose is to determine the values μ such that the solution of (P_μ) verifies $u'(1) = 0$.

From the phases portrait studied in the first step we deduce that the solutions which attain at least two times value $u' = 0$ are the solutions given by the periodic trajectories, that is, the associated to energy level $V(u_2) \leq E \leq \min\{V(u_1), V(u_3)\}$. The idea is to choose the periodic trajectories which starts in $(\mu, 0)$ and arrives to $(\lambda, 0)$ at the time $x = 1$. That is, integrating the conservation law equation (3), we obtain

$$\int_{u(0)}^{u(1)} \frac{ds}{\pm\sqrt[p]{p(E - V(s))}} = \int_0^1 d\sigma,$$

where the sign of $\sqrt[p]{p(E - V(s))}$ is the same of u'. The period of the periodic orbit of the phases portrait is given by the expression:

$$\tau = 2\int_a^{-10} \frac{ds}{\sqrt[p]{p(E - V(s))}} + 2\int_{-10}^{b} \frac{ds}{\sqrt[p]{p(E - V(s))}}. \tag{5}$$

where $(a, 0)$, $(b, 0)$ are the two different points in which the trajectories pass by the axis $u' = 0$. This is equivalent to say that a and b verify: $u_1 < a < -10 < b < u_3$ and $V(b) = V(a) < \min\{V(u_1), V(u_3)\}$. Consequently, there exists $b^* > -10$ such that $V(b^*) = \min\{V(u_1), V(u_3)\}$, then, the below condition can be written as $-10 < b < b^*$.

If $p = 2$ and $\mathcal{G}(u) = Bu$ where B is a positive constant, it is posible to obtain the explicit expression for τ,

$$\tau = \frac{2}{\sqrt{B}} \operatorname{Ln}\left(\frac{\left(\frac{QM-C}{\sqrt{B}} + 10\sqrt{B} + \sqrt{2(E - V(-10))}\right)\left(\frac{C-Qm}{\sqrt{B}} - 10\sqrt{B} + \sqrt{2(E - V(-10))}\right)}{\left(\frac{QM-C}{\sqrt{B}} - b\sqrt{B}\right)\left(\frac{C-Qm}{\sqrt{B}} + a\sqrt{B}\right)}\right), \tag{6}$$

where $(a, 0)$ and $(b, 0)$ are two different points of the periodic orbit.

If $p \geq 2$, under the hypothesis (H_3), we have obtained the following estimates for the period τ of a periodic trajectory which contains the points $(a, 0)$ and $(b, 0)$ with $a < -10 < b$,

$$\tau_1 \leq \tau \leq \tau_2,$$

where

$$\tau_1 = \frac{2p^{1-\frac{1}{p}}}{p-1}((b+10)^{1-\frac{1}{p}}(\mathcal{G}(u_3) - \mathcal{G}(-10))^{-\frac{1}{p}} + (-10-a)^{1-\frac{1}{p}}(\mathcal{G}(-10) - \mathcal{G}(a))^{-\frac{1}{p}}) \tag{7}$$

$$\tau_2 = \frac{2p^{1-\frac{1}{p}}}{p-1}((b+10)^{1-\frac{1}{p}}(\mathcal{G}(u_3) - \mathcal{G}(b))^{-\frac{1}{p}} + (-10-a)^{1-\frac{1}{p}}(\mathcal{G}(a) - \mathcal{G}(u_1))^{-\frac{1}{p}}). \tag{8}$$

We have used the Mean Value Theorem for $\mathcal{G}$ to get these estimates.
The solution u of (P) satisfies one of theses four equalities:

$$\begin{array}{lllllll} I) & u(0) & = & \mu & = & b, & u(1) = a, \\ II) & u(0) & = & \mu & = & a, & u(1) = b, \\ III) & u(0) & = & \mu & = & b = u(1), & \\ IV) & u(0) & = & \mu & = & a = u(1). & \end{array}$$

The solutions of type I and II cross the level -10 an odd number of times, while for the solutions of type III and IV, it is in an even number of times. We only analyze the case I, the others can be analyzed in an analogous way. Moreover, case I is the unique case verifying that $u(0) > u(1)$, and this makes that this is the more realistic because $x = 0$ represents the Equator and $x = 1$ the North pole. So, we assume

$$\begin{array}{rcl} u(0) = \mu = b & > & -10, \\ u(1) = a & < & -10. \end{array}$$

We notice that every μ determines a_μ as the unique solution of $V(s) = V(\mu)$ on $(u_1, -10)$. Now, the problem is to study whether exist or not μ such that the time to arrive from μ to a_μ is exactly $x = 1$, after N complete turns and a half, that is,

$$N\tau + \int_{-10}^{u(0)} \frac{ds}{\sqrt[p]{p(E - V(s))}} + \int_{u(1)}^{-10} \frac{ds}{\sqrt[p]{p(E - V(s))}} = 1. \tag{9}$$

Then, for every $\mu \in (-10, b^*)$, we have $\tau(\mu)$ and we are interested in the number of solutions $(N, \mu) \in I\!N \times (-10, b^*)$ of the equation $(N + \frac{1}{2})\tau(\mu) = 1$, that is,

$$\tau(\mu) = \frac{2}{2N+1}.$$

In order to see that, we study the functions τ_1 and τ_2. We observe that $\tau_1(\mu)$ and $\tau_1(\mu)$ are continuous and increasing functions on the interval $(-10, b^*)$, where

$$\begin{array}{rcl} b^* & = & u_3 \quad \text{if } Q \in (Q_1, Q_3], \\ b^* & < & u_3, \ V(u_1) = V(b^*) \ \text{ if } Q \in (Q_3, Q_2). \end{array}$$

Moreover, the function has a vertical asymptote $\mu = b^*$. From $\tau_1(-10) = \tau_2(-10) = 0$ and the properties of τ_1 and τ_2 we get that there exists N_0 such that for all $N \geq N_0$ there exist μ_1 and μ_2 such that

$$\tau_1(\mu_1) = \frac{2}{2N+1} = \tau_2(\mu_2). \tag{10}$$

Thus, we can conclude that there exists $\mu \in (\mu_1, \mu_2)$ such that $(N + \frac{1}{2})\tau(\mu) = 1$. So, we obtain that for each $N \geq N_0$ there exist a solution of (P) which cross $2N + 1$ times the level -10.
Thus, we have proved the existence of infinitely many solutions of (P) for $Q \in (Q_1, Q_2)$. On the other hand, the construction allows us to deduce the family of solutions obtained is uniformly bounded because every solution u of (P) verifies $u_1 \leq u(x) \leq u_3$ (in fact u' is also bounded since (u, u') is a periodic trajectory).

■

3 Final Comments and Open Problems.

Theorem 1 shows that the studied simple model has a complex behaviour under small and critical variations of Q. The results of Theorem 1 open some new problems:

Problem 1. We know that the solutions $u(t,x)$ of the evolution problem (1-D and 2-D models) go to a stationary solution when $t \to \infty$ in the following sense: We define the ω - limit set as

$$\omega(u) = \{u_\infty \in W^{1,p}(\Omega) \cap L^\infty(\Omega) : \exists t_n \to +\infty \text{ such that } u(t_n, \cdot) \to u_\infty \text{ in } L^2(\Omega)\}.$$

We have proved:

Let $u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ the initial data and let u be the solution. Then (i) $\omega(u) \neq \emptyset$, (ii) if $u_\infty \in \omega(u)$ then $\exists t_n \to +\infty$ such that $u(t_n + s, \cdot) \to u_\infty$ in $L^2(-1,1; L^2(\Omega))$ and $u_\infty \in W^{1,p}(\Omega)$ is a weak solution of the stationary problem. (iii) Actually if $u_\infty \in \omega(u)$ then $\exists \{\hat{t}_n\} \to +\infty$ such that $u(\hat{t}_n, \cdot) \to u_\infty$ strongly in $W^{1,p}(\Omega)$.
Now we know that the model has infinitely many stationary solutions. Which is the limit of $u(t,x)$ when we consider $\forall t$? How can we distinguish such a limit u_∞ in terms of the initial datum and $f(t,\cdot)$ among the infinitely many stationary solutions? Is it true that $\omega(u)$ is formed by a single element u_∞ or it is formed by a multiple set of stationary solutions.

Problem 2. The model studied in Diaz - Hernandez - Tello [1997] includes the *insolation function* $S(x)$ and the multiplicity of at least three solutions for $Q \in (Q_1, Q_2)$ was proved. It would be interesting to extend the conclusion of Theorem 1 to the 1-D model including the insolation effect. A related work for a 1-D EBM, with linear diffusion ($p = 2$), is due to G. Hetzer [2000]. So, second open problem is to analyze the number of solutions for the 1-D model with $R_a(x,u) = QS(x)\beta(u)$, $p \geq 2$ and β multivalued.

Problem 3. For the 2-D EBM we know that there exist *at least three* stationary solutions for $Q \in (Q_1, Q_2)$. Is it possible the extension of the results of Theorem 1 to 2-D EBM?

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