On the stabilization of uniform oscillations for the complex Ginzburg-Landau equation by means of a global delayed mechanism.

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Introduction

This work deals with the stabilization of the uniform oscillations for the *complex Ginzburg-Landau equation* on the one-dimensional domain $\Omega = (0, L)$

$$(P_1) \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - (1 + i\epsilon)\frac{\partial^2 \mathbf{u}}{\partial x^2} = (1 - i\omega)\mathbf{u} - (1 + i\beta)\left|\mathbf{u}\right|^2 \mathbf{u} + \mu e^{i\chi_0}\mathbf{F}(\mathbf{u}, t, \tau) & \Omega \times (0, +\infty), \\ \frac{\partial \mathbf{u}}{\partial x} = 0 & \partial\Omega \times (0, +\infty), \\ \mathbf{u}(x, s) = \mathbf{u}_0(x, s) & \Omega \times [-\tau, 0], \end{cases}$$

by means of some effects including some global feedback delayed terms where

$$\mathbf{F}(\mathbf{u}, t, \tau) = [m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + m_3\mathbf{u}(t - \tau, x) + m_4\overline{\mathbf{u}}(t - \tau)] \text{ with } \overline{\mathbf{u}}(s) = \frac{1}{L}\int_0^L \mathbf{u}(s, x)dx.$$

Here the parameters $\epsilon, \beta, \omega, \mu, \chi_0, m_i$ and τ are real numbers, in contrast with the solution $\mathbf{u}(x, t) = u_1(x, t) + iu_2(x, t)$. We point out that most of our results remain true for N-dimensional domains and other types of boundary conditions.

This type of equations (called as of Stuart-Landau in absence of the diffusion term) arise in the study of the stability of reaction diffusion equations as $\frac{\partial \mathbf{X}}{\partial t} - \mathbf{D}\frac{\partial^2 \mathbf{X}}{\partial x^2} = \mathbf{f}(\mathbf{X} : \eta)$ where $\mathbf{X} : \Omega \times (0, +\infty) \to I\!\!R^n$ and η is a real scalar parameter when the deviation $\mathbf{v}$ from the uniform state solution $\mathbf{X}_\infty$ is developed asymptotically in terms of some multiple scales (see [5]). Coefficient ε measures the degree to which the diffusion matrix $\mathbf{D}$ deviates from a scalar.

Notice that the presence of complex coefficients introduces important differences with the classical Ginzburg-Landau equations arising in superconductivity ([2]).

Our main goal is to carry out a rigorous analysis to some recent studies of a more descriptive nature, but of a great originally and interest, in which the delay term $\mathbf{F}(\mathbf{u}, t, \tau)$ is taken corresponding to $m_4 = 1, m_i = 0$ for $i = 1, 2, 3$ and it was introduced as a control mechanism (see [1], [6]). Here we also want to investigate the possibility of controlling the turbulence by using other terms (see Remark 4).

We concentrate our attention on the so called *slowly varying complex amplitudes* defined by $\mathbf{u}(x, t) = \mathbf{v}(x, t)e^{-i\omega t}$. Thus, $\mathbf{v}$ satisfy

$$(P_2) \begin{cases} \frac{\partial \mathbf{v}}{\partial t} - (1 + i\epsilon)\frac{\partial^2 \mathbf{v}}{\partial x^2} = \mathbf{v} - (1 + i\beta)|\mathbf{v}|^2 \mathbf{v} + \\ \quad + \mu e^{i\chi_0}[m_1\mathbf{v} + m_2\overline{\mathbf{v}} + e^{i\omega\tau}(m_3\mathbf{v}(t - \tau, x) + m_4\overline{\mathbf{v}}(t - \tau))] & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \mathbf{v}}{\partial x} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ \mathbf{v}(x, s) = \mathbf{u}_0(x, s)e^{i\omega s} & \text{on } \Omega \times [-\tau, 0]. \end{cases}$$

We study the stability of *uniform oscillations*, i.e., special solutions of (P_2) of the form $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ which determines completely ρ_0 and θ. As we shall see, the only effect of the delay τ is that it controls the effective phase shift $\chi(\tau)$.

In absence of delay ($\tau = 0$), and for $L = +\infty$ and $\mu = 0$, it is known (see [5] and [6]) that the *Benjamin-Feir condition* $\beta < -\frac{1}{\epsilon}$ implies the instability of such uniform oscillations. Here we shall assume merely that

$$\beta \leq 0 \text{ and } \varepsilon \geq 0 \tag{1}$$

and we shall prove that this instability holds, in absence of delay, for $L < +\infty$ once $\chi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\mu > \frac{1}{|\cos\chi_0|}$. Moreover, we shall also prove that when $\tau > 0$ is suitably chosen then the uniform oscillation becomes linearly stable. We point out that the above stabilization phenomenon requires a non zero complex component perturbation (notice that χ_0 can not be zero) and that it applies to the case of $\mu > 0$ and $\epsilon = \beta = \omega = 0$.

On the linearization principle

We start by pointing out that the existence and uniqueness of a solution of (P_1) can be proven once we assume that $\mathbf{u}_0 \in \mathbf{C}([-\tau,0] : \mathbf{L}^2(\Omega))$ (see [3]).

We are interested in the stability analysis of the time-periodical function $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$. In order to avoid the application of techniques for the study of the stability of periodical solutions we can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t) = \mathbf{v}(x,t)e^{i\theta t}$ where $\mathbf{v}(x,t)$ is a solution of (P_2). Thus $\mathbf{z}(x,t)$ satisfies

$$(P_3) \begin{cases} \frac{\partial \mathbf{z}}{\partial t} - (1+i\epsilon)\frac{\partial^2 \mathbf{z}}{\partial x^2} = (1+i\theta)\mathbf{z} - (1+i\beta)|\mathbf{z}|^2\mathbf{z} + \\ \quad + \mu e^{i\chi_0}\left[m_1\mathbf{z} + m_2\bar{\mathbf{z}} + e^{i(\omega+\theta)\tau}(m_3\mathbf{z}(t-\tau,x) + m_4\bar{\mathbf{z}}(t-\tau))\right] & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial \mathbf{z}}{\partial x} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x,s)e^{i(\omega-\theta)s} & \text{on } \Omega \times [-\tau,0]. \end{cases}$$

Notice that now, $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_{uosc}(x,t)e^{i\theta t} = \mathbf{z}_\infty = \rho_0$ is an stationary solution of (P_3): i.e.

$$0 = (1+i\theta)\mathbf{z}_\infty - (1+i\beta)|\mathbf{z}_\infty|^2\mathbf{z}_\infty + \mu e^{i\chi_0}\left[m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3+m_4)\right]\mathbf{z}_\infty. \tag{2}$$

In order to keep some resemblance with [1] we shall assume that

$$m_1 + m_2 = 0 \text{ and } m_3 + m_4 = 1 \tag{3}$$

Then we get the expressions $\rho_0(\tau) = (1 + \mu\cos\chi(\tau))^{1/2}$, where $\chi(\tau) = \chi_0 + (\omega + \theta(\tau))\tau$ and with $\theta(\tau)$ given as the solution of the implicit equation

$$\theta = \beta - \mu(\sin(\chi_0 + (\omega+\theta)\tau) - \beta\cos(\chi_0 + (\omega+\theta)\tau)). \tag{4}$$

Notice that if $\mu = 0$ we deduce that $\rho_0(\tau) = 1$ and that $\theta(\tau) = \beta$ for any τ and that $\rho_0(0) = (1 + \mu\cos\chi_0)^{1/2}, \theta(0) = \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)$. It is not difficult to prove (see

Proposition 1) the existence and uniqueness of such a function $\theta(\tau)$ and that $\theta \in C^1$. Our main result is the following:

Theorem 1. *Assume (1), (3), $\chi_0 \in (\pi, \frac{3\pi}{2})$,*

$$3 - m_1 - 2m_3 \geq 0, \quad m_1 + m_3 \geq 0, \quad 3 + 2m_3 > 0, \tag{5}$$

$$\mu > \max\Big\{ \frac{1}{|\cos \chi_0|}, \frac{3\beta - \omega + 3(\omega + \beta)\sin \chi_0 + \cos \chi_0}{5(-\beta)\sin \chi_0 \cos \chi_0 + 1},$$
$$\frac{m_3(3\beta - \omega - \varepsilon\frac{\pi^2}{L^2}) + 3(\omega + \beta)\sin \chi_0 + (m_1 + m_3)\cos \chi_0}{(3 - m_1 - 2m_3)\sin^2 \chi_0 + (m_1 + m_3)\cos^2 \chi_0 + (-\beta)(3 + 2m_3)\sin \chi_0 \cos \chi_0} \Big\}.$$

Then there exists some $\tau_0 \in (0,1)$ such that if we assume $\tau \in (\tau_0, 1)$ we get that

$$|\mathbf{v}(x,t) - \rho_0| \leq M e^{-\alpha t} \left\| \mathbf{u}_0(\cdot, \cdot)e^{i\omega\cdot} - \rho_0 \right\|.$$

The proof will be divided in two parts: first we shall show the applicability of some abstract result on the linearized stability principle for the delayed problem

$$(ADP) \begin{cases} \frac{du}{dt}(t) = Au(t) + G(u_t) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0]. \end{cases} \tag{6}$$

In a second part (Section 3) we shall check that the above conditions on the data of the problem allows to prove that any eigenvalue λ of the associate linearized problem has $Re(\lambda) < 0$ which implies the result. As in [8], X is a Banach space (of norm $|\cdot|$),

$$\begin{cases} A \text{ is the generator of a compact semigroup } \{T(t)\}_{t \geq 0} \text{ satisfying} \\ \|T(t)\| \leq M e^{\gamma t} \text{ for some constants } M \text{ and } \gamma, \end{cases} \tag{7}$$

function $G : C \to X$, with $C = C([-\tau, 0] : Y)$ of norm $\|\cdot\|$, $Y \subset D(A)$ with continuous embedding $Y \subset X$, satisfies a local Lipschitz condition, i.e.

$$\begin{cases} \text{for any } R > 0 \text{ there exists } L(R) > 0 \text{ such that} \\ |G(\phi) - G(\psi)| \leq L(R) \|\phi - \psi\| \text{ if } \phi, \psi \in C \text{ and } \|\phi\|, \|\psi\| \leq R. \end{cases} \tag{8}$$

The notation u_t means that $u_t \in C$ and that $u_t(s) = u(t + s)$ for any $s \in [-\tau, 0]$. In this abstract context, the stationary states (or equilibria) are given by the elements $u_\infty \in D(A) \subset X$ such that $0 = Au_\infty + G(\widehat{u}_\infty)$, where $\widehat{u}_\infty \in C$ is the function which takes constant values equal to u_∞. Further assumptions are needed:

$$\begin{cases} \exists\, \delta > 0 \text{ such that } G : B_\delta(\widehat{u}_\infty) \to X \text{ is Frechet differentiable} \\ (B_\delta(\widehat{u}_\infty) = \{\phi \in C : \|\phi - \widehat{u}_\infty\| < \delta\}) \text{ of Frechet derivative} \\ DG(\widehat{u}_\infty)\phi = \int_{-\tau}^0 d\eta(s)\phi(s), \ \phi \in C \text{ for some } \eta : [-\tau, 0] \to B(Y, X) \text{ of} \\ \text{bounded variation and the Frechet derivative is locally Lipschitz continuous} \end{cases} \tag{9}$$

and, which is crucial,

$$\begin{cases} \text{for any } \lambda \in \mathcal{C} \text{ such that } \exists\, v \in D(A), \ v \neq 0, \text{ such that} \\ 0 = Av - \lambda v + DG(\widehat{u}_\infty)(e^{\lambda\cdot}v), \text{ then } Re\lambda < 0, \end{cases} \tag{10}$$

Proof. Notice that, since $Y = \mathbf{L}^6(\Omega)$, G is well defined (i.e. $G(C) \subset X = \mathbf{L}^2(\Omega)$). Given $R > 0, \phi, \psi \in C$ with $\|\phi\|, \|\psi\| \leq R$ we have

$$|G(\phi) - G(\psi)| \leq [(1+\theta^2)^{1/2}L^{1/4}+(1+\beta^2)^{1/2}K(R)]\,|\phi(0) - \psi(0)|_Y +\mu L^{1/4}\,|\phi(-\tau) - \psi(-\tau)|_Y$$

where $K(R)$ is the supremum, on the ball $B_R(0)$ of Y, of the norm of the Frechet derivative of the function $H : Y \to X$ given by $\mathbf{v} \to |\mathbf{v}|^2\,\mathbf{v}$. Thus it suffices to take $L(R) = \max\{(1 + \theta^2)^{1/2}L^{1/4} + (1 + \beta^2)^{1/2}K(R), \mu L^{1/4}\}$. On the other hand, for any $\phi \in C$, since the non-local operator $\phi \to \frac{1}{L} \int_0^L \phi(s)dx$ is linear, we can write $DG(\widehat{\mathbf{z}}_\infty)\phi = \int_{-\tau}^0 d\eta(s)\phi(s)$, with

$$d\eta(s)\mathbf{v}(s) \quad = \quad \delta_0(s)[(1 + i\theta)-3(1 + i\beta)\,|\widehat{\mathbf{z}}_\infty|^2]\mathbf{v}(s)+ \tag{16}$$

$$+\mu e^{i\chi_0}\left[\delta_0(s)(m_1\mathbf{v}(s)+m_2\overline{\mathbf{v}}(s))+e^{i(\omega+\theta)\tau}\delta_{-\tau}(s)(m_3\mathbf{v}(s)+m_4\overline{\mathbf{v}}(s))\right] \tag{17}$$

for any $\mathbf{v} \in C([-\tau, 0]: \mathbf{L}^6(\Omega))$ and any $s \in [-\tau, 0]$, where $\delta_0(s), \delta_{-\tau}(s)$ denote the Dirac delta at the points $s = 0$ and $s = -\tau$ respectively. By well-known results, we have that $\eta : [-\tau, 0] \to B(Y, Y)$ has a bounded variation. Finally, it is easy to check that $DG(\widehat{\mathbf{z}}_\infty)$ is locally Lipschitz continuous as function of $\widehat{\mathbf{z}}_\infty$ (and of the rest of its arguments).

Remark 3. We point out that the above linearization process uses, in a fundamental way, the linearity of operator A. Other linearization principles can be introduced but its rigorous justification can be harder than the above arguments. For instance, very often it is used the representation for the unknown as $\mathbf{z}(x,t) = \rho(x,t)e^{i\phi(x,t)}$. In this way, the delayed nonlinear equation (P_3) leads to a coupled system of delayed equations for ρ and ϕ. This is the procedure followed in [1] and [6]. In spite of the possibility of to state a linearized principle for such nonlinear system we want to mention what is the main difficulty added to the process by using this representation. Let us denote $\mathbf{P} : I\!\!R^2 \to C$ to the representation $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$. Notice that $\mathbf{P}$ is nonlinear and that if $\mathbf{q} = (\rho,\phi)$ then $\mathbf{z}(x,t) = \mathbf{P}(\mathbf{q}(x,t))$ and the (P_3) can be formulated as $\frac{d\mathbf{P}(\mathbf{q}(\cdot,t))}{dt} + A\mathbf{P}(\mathbf{q}(\cdot,t)) = G(\mathbf{P}(\mathbf{q}(\cdot,t))_t)$. By using that the matrix $\mathbf{B}(\mathbf{q}(\cdot,t)) = grad\mathbf{P}(\mathbf{q}(\cdot,t))$ is not singular, we can arrive to the simpler formulation $\frac{d\mathbf{q}}{dt}(\cdot,t) + \mathbf{B}(\mathbf{q}(\cdot,t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot,t)) = \mathbf{B}(\mathbf{q}(\cdot,t))^{-1}G(\mathbf{P}(\mathbf{q}(\cdot,t))_t)$. This delayed system can be also (formally) linearized but notice that then the diffusion operator $\mathbf{B}(\mathbf{q}(\cdot,t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot,t))$ becomes now quasilinear on $\mathbf{q}$ and thus the mathematical justification is much more delicate.

Study of the eigenvalues of the linearized problem

In this section we shall study the eigenvalues $\lambda \in C$, $\lambda = a + ib$ of the linearized problem (10). We start by proving the existence and uniqueness of $\theta(\tau)$

Proposition 1. *There exists a unique function $\theta(\tau)$ such that*

$$\theta(\tau) - \beta + \mu(\sin(\chi_0 + (\omega + \theta(\tau))\,\tau) - \beta\cos(\chi_0 + (\omega + \theta(\tau))\,\tau)) = 0$$

for any $\tau \in [0,1]$. Moreover $\theta \in C^1$.

Proof. It is enough to see that, by the implicit function theorem, $\theta(\tau)$ is characterized as the (unique) solution of the Cauchy problem associated to the ODE

$$\frac{d\theta}{d\tau}(\tau) = \frac{-[\mu(\cos(\chi_0 + (\omega + \theta(\tau))\,\tau)\,(\omega + \theta) + \beta\sin(\chi_0 + (\omega + \theta(\tau))\,\tau))]\,(\omega + \theta(\tau))}{1 + \mu(\cos(\chi_0 + (\omega + \theta(\tau))\,\tau)\,\tau + \beta\sin(\chi_0 + (\omega + \theta(\tau))\,\tau))\tau}.$$

We recall that in our case, $z_\infty = \rho_0$ and so, by using (9), we arrive to the linear problem

$$(P_4) \begin{cases} -(1+i\epsilon)\frac{\partial^2 z}{\partial x^2} = -(a+ib)z + [(1+i\theta)-3(1+i\beta)\rho_0^2)]z & \\ \quad +\mu e^{i\chi_0}\left[m_1 z + m_2\bar{z} + e^{-a\tau+i(\omega+\theta-b)\tau}(m_3 z + m_4\bar{z})\right] & \text{in } \Omega, \\ \frac{\partial z}{\partial x} = 0 & \text{on } \partial\Omega. \end{cases}$$

As usual, the linear structure of the equation leads to the research of nontrivial solutions of the form $z(x) = A_n cos(\pi n x/L)$ (recall that the eigenvalues for the usual Laplacian operator $\frac{\partial^2}{\partial x^2}$ with homogeneous boundary conditions on $\Omega = (0, L)$ are given by $k(n) = \pi n/L$ with $n = 0, 1, 2, \ldots$ with the associate eigenfunctions by $\{cos(kx)\}_{n\geq 1}$). In order to keep a coherent notation with the one used in [1] we introduce the notation $\lambda_k = a_k + ib_k$ for the real and imaginary parts of the eigenvalues. Notice that $\int_0^L cos(\pi n x/L)dx = 0$ for any $n = 1, 2, \ldots$ Then we get that

$$(a_k + ib_k) - (1+i\epsilon)\left(-k^2\right) = (1+i\theta)-3(1+i\beta)\rho_0^2$$
$$+\mu e^{i\chi_0}\left[m_1 + m_2\delta_{0k} + e^{-a\tau+i(\omega+\theta-b)\tau}(m_3 + m_4\delta_{0k})\right]$$

where δ_{0k} denotes the Kronecker delta function. We arrive to

$$\begin{cases} a_k = -k^2 - 2 - 3\mu\cos\chi(\tau) + \mu(m_1 + m_2\delta_{0k})\cos\chi_0 + \\ \quad +\mu e^{-a_k\tau}(m_3 + m_4\delta_{0k})\cos(\chi_0 + (\omega+\theta-b_k)\tau), \\ b_k = \theta - \epsilon k^2 - 3\beta(1 + \mu\cos\chi) + \mu(m_1 + m_2\delta_{0k})\sin\chi_0 + \\ \quad +\mu e^{-a_k\tau}(m_3 + m_4\delta_{0k})\sin(\chi_0 + (\omega+\theta-b_k)\tau). \end{cases} \quad (18)$$

The previous equations are transcendent and we cannot get an explicit expression for the real and imaginary part of the eigenvalues (for some similar transcendent equations arising in delayed ODE's see [4]).

Now, we focus our attention in the dependence of a_k and b_k with respect to τ. So, by the regularity of the involved functions we can assume

$$a_k = a_{k0} + a_{k1}\tau + o(\tau), \quad b_k = b_{k0} + b_{k1}\tau + o(\tau),$$

as we get, for instance, by a "formal" series development in powers of τ argument. Here we used the Landau notation $(f(\tau) = o(\tau)$ means that $\frac{f(\tau)}{\tau} \to 0$ when $\tau \to 0$).

The terms of order zero in τ are obtained by making $\tau = 0$ in (18)

$$\begin{cases} a_{k0} = -(2 + k^2) + \mu\cos\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k}) \\ b_{k0} = 4\beta - \epsilon k^2 + 3\mu\beta\cos\chi_0 + \mu\sin\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k}). \end{cases} \quad (19)$$

So, we can state a first result concerning the case without any delay

Proposition 2. *Assume* $\tau = 0$, $\chi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, *and* $\mu > \frac{1}{|\cos\chi_0|}$. *Then the uniform oscillation* $v_{uosc}(x, t) = \rho_0 e^{-i\theta t}$ *is linearly unstable.*

Proof. From (19) we see that $a_{00} > 0$ and since $\tau = 0$ we get the existence of at least one eigenvalue λ of the linearized problem with $Re(\lambda) > 0$ which implies the result.

The first order terms in τ are calculated below

Lemma 3 *We have*

$$a_{k1} = \left[\frac{da_k}{d\tau}\right]_{\tau=0} = (2 + k^2) + \mu\left[3\,(\omega + \beta)\sin\chi_0 + (m_3 + m_4\delta_{0k})\,(3\beta - \epsilon k^2 - \omega)\right]$$
$$+\mu^2\{-3\sin^2\chi_0 + 3\beta\sin\chi_0\cos\chi_0 + \qquad\qquad\qquad (20)$$
$$+(m_3 + m_4\delta_{0k})\,[\sin^2\chi_0 + 2\beta\sin\chi_0\cos\chi_0 +$$
$$+(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k})]\,(\sin^2\chi_0 - \cos^2\chi_0)\}.$$

Proof. Differentiating in (18) we get that

$$a_{k1} = \left[\frac{da_k}{d\tau}\right]_{\tau=0} = \left[3\mu\sin\chi(\tau)\frac{d\chi}{d\tau}\right]_{\tau=0} + [(-a_k)\,\mu e^{-a_k\tau}(m_3 + m_4\delta_{0k})\cos(\chi_0 + (\omega + \theta - b_k)\tau)]_{\tau=0}$$
$$-[\mu e^{-a_k\tau}(m_3 + m_4\delta_{0k})\sin(\chi_0 + (\omega + \theta - b_k)\tau)]_{\tau=0}\left[\frac{d(\omega + \theta - b_k)\tau}{d\tau}\right]_{\tau=0} =$$
$$= (3\mu\sin\chi_0)\,(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)) -$$
$$- (-(2 + k^2) + \mu\cos\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k}))\,\mu(m_3 + m_4\delta_{0k})\cos\chi_0 -$$
$$-\mu(m_3 + m_4\delta_{0k})(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0) - b_{k0})\sin\chi_0.$$

Thus, by using the expression for b_{k0} (see (18)) we obtain that

$$a_{k1} = (3\mu\sin\chi_0)\,(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)) -$$
$$- (-(2 + k^2) + \mu\cos\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k}))\,\mu(m_3 + m_4\delta_{0k})\cos\chi_0 -$$
$$-\mu(m_3 + m_4\delta_{0k})(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0))\sin\chi_0$$
$$(3\mu\sin\chi_0)\,(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)) -$$
$$- (-(2 + k^2) + \mu\cos\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k}))\,\mu(m_3 + m_4\delta_{0k})\cos\chi_0 -$$
$$-\mu(m_3 + m_4\delta_{0k})(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0))\sin\chi_0$$
$$+\mu(m_3 + m_4\delta_{0k})(4\beta - \epsilon k^2 + 3\mu\beta\cos\chi_0 + \mu\sin\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k}))\sin\chi_0.$$

In consequence

$$a_{k1} = (2 + k^2) + \mu\,(3\,(\omega + \beta)\sin\chi_0 - (m_3 + m_4\delta_{0k})\,(\omega + \beta) + (4\beta - \epsilon k^2)\,(m_3 + m_4\delta_{0k}))$$
$$-\mu^2\,(3\sin\chi_0(\sin\chi_0 - \beta\cos\chi_0) + \cos^2\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k})(m_3 + m_4\delta_{0k})) -$$
$$+\mu^2(m_3 + m_4\delta_{0k})\,[(\sin\chi_0 - \beta\cos\chi_0)\sin\chi_0 + (3\beta\cos\chi_0 + \sin\chi_0(m_1 + m_2\delta_{0k} + m_3 + m_4\delta_{0k}))\sin\chi_0]$$

which proves the result.

Proposition 3. *Assume (1), $\chi_0 \in (\pi, \frac{3\pi}{2})$, (3) and*

$$\mu > \max\{0, \frac{3\beta - \omega + 3(\omega + \beta)\sin\chi_0 + \cos\chi_0}{5(-\beta)\sin\chi_0\cos\chi_0 + 1}\}.$$

Then $a_{00} + a_{01} < 0$.

Proof. By using (19), (20), and (3) we get

$$a_{00} + a_{01} = \mu[(3\beta - \omega + 3(\omega + \beta)\sin\chi_0 + \cos\chi_0) - \mu(5(-\beta)\sin\chi_0\cos\chi_0 + 1)].$$

Then, the assumptions imply the positivity of the coefficient of μ^2 and the result holds.

Proposition 4. *Assume (1), $\chi_0 \in (\pi, \frac{3\pi}{2})$, (5) and*

$$\mu > \max\{0, \frac{m_3(3\beta - \omega - \varepsilon\frac{\pi^2}{L^2}) + 3(\omega + \beta)\sin\chi_0 + (m_1 + m_3)\cos\chi_0}{(3 - m_1 - 2m_3)\sin^2\chi_0 + (m_1 + m_3)\cos^2\chi_0 + (-\beta)(3 + 2m_3)\sin\chi_0\cos\chi_0}\}.$$

Then, for any $n > 0$, $a_{k0} + a_{k1} < 0$. Moreover, for any $n > 1$ and any $\tau \in (0, 1]$,

$$a_{k(n)0} + a_{k(n)1}\tau < a_{k(1)0} + a_{k(1)1}\tau.$$

Proof. By using (19), (20) we obtain that

$$a_{k0} + a_{k1} = \mu[(m_3(3\beta - \omega - \varepsilon\tfrac{\pi^2}{L^2}) + 3(\omega + \beta)\sin\chi_0 + (m_1 + m_3)\cos\chi_0)$$
$$-\mu((3 - m_1 - 2m_3)\sin^2\chi_0 + (m_1 + m_3)\cos^2\chi_0 + (-\beta)(3 + 2m_3)\sin\chi_0\cos\chi_0)].$$

Again, the assumptions made on the parameters imply the positivity of the coefficient of μ^2 and the result holds. Moreover

$$a_{k(n)0} - a_{k(1)0} + (a_{k(n)1} - a_{k(1)1})\tau = -k(n)^2 + k(1)^2 - (m_3\epsilon k(n)^2 - m_3\epsilon k(1)^2)\tau < 0.$$

The proof of Theorem 1 is now complete since from Propositions 3 and 4 we deduce the existence of some $\tau_0 \in (0, 1)$ (independent of $n \in I\!N$) such that for any $n \geq 0$ we have $a_{k0} + a_{k1}\tau < 0$ for any $\tau \in (\tau_0, 1)$. This implies the hypothesis of the abstract result and the conclusion follows.

Remark 4. Notice that Theorem 1 applies to the case $m_1 = m_2 = m_3 = 0$ which corresponds to a formulation similar to the one of [1]. Moreover, it also applies to the choice $m_1 = \kappa, m_2 = -1 - \kappa, m_3 = 0$ and $m_4 = 1$, for any $\kappa \in (0, 1)$ which corresponds to a formulation quite close to the pioneering paper [7] (concerning chaotic ODEs).

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