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ON SOME OF THE MANY NONLINEAR MATHEMATICAL PROBLEMS IN THE OEUVRE OF AMABLE LIÑÁN

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Abstract. The contributions of Amable Liñán to combustion theory and fluid mechanics are important enough to place him among the most brilliant specialists in these fields and, of course, among the leading Spanish scientists of the last decades. Without intention of being exhaustive, I would like to point out in this paper the great impact of his dilated oeuvre in the field of applied mathematics, specially in the education of several Spanish mathematicians, among which I have the good fortune of being included.

Key words: Catalysis, homogenization, finite extinction time, lubrication, Coulomb type friction problems

1. INTRODUCTION

The outstanding contributions of Amable Liñán to combustion theory and fluid mechanics always involve many mathematical arguments which are, in a sense, the core of his ideas. In these pages, and without any intention of being exhaustive, I would like to illustrate the great impact of Liñán's dilated oeuvre in the field of applied mathematics, specially in the education of several Spanish mathematicians, among which I include myself. The selection of topics is motivated by my own experience, but the reader may find many other illustrations in other papers of this volume. I would like to point out also to Liñán's generous collaboration in the organization of mathematical meetings. Besides his active participation in many of those occasions, it seems to me relevant to remember here the large international meeting which we organized in collaboration with M. A. Herrero and J. L. Vázquez; the 8th International Colloquium on Free Boundary Problems: Theory and Applications, Toledo, June 1993 ([30]).

2. CATALYSIS: FRONTS AND HOMOGENIZATION

Perhaps one of the earliest contacts of the Spanish mathematical community with Amable Liñán took place in the first Spanish meeting on Differential Equations and its Applications (*I Congreso de Ecuaciones Diferenciales y Aplicaciones CEDYA. El Escorial, May 29th-31th, 1978*).

In his lecture [51], Liñán presented a cascade of mathematical models, obtained via singular perturbation methods in the study of the temperature and concentration in the interior of a catalyst particle. In my opinion, one of the main lessons of his lecture notes was at the time rather new for most of the mathematicians attending the meeting, educated perhaps in excessively abstract mathematics. It dealt with an initial model of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \alpha u^p e^{\gamma(v-1)/v} & \text{in } \Omega \times (0,\infty), \\[2mm] \dfrac{\partial v}{\partial t} = \Delta v + \beta u^p e^{\gamma(v-1)/v} & \text{in } \Omega \times (0,\infty), \\[2mm] \dfrac{\partial u}{\partial n} = \sigma(1-u), \ \dfrac{\partial v}{\partial n} = v(1-v) & \text{on } \partial\Omega \times (0,\infty). \\[2mm] u(x,0) = u_i(x), \ v(x,0) = v_i(x) & \text{on } \Omega, \end{cases} \tag{1}$$

which can be simplified in several ways, depending on scaling. Some additional conversations with Liñán and his (then) Ph.D. student J. M. Vega, allowed several of us to know their deep results (Liñán and Vega [62]) on the "formation of the dead core" typical of reactions of low order ($p \in [0,1)$)[1]. The study of this phenomenon without any assumption of symmetry of the domain or the solution was the main goal of a series of papers by J. Hernández and this author ([25], [26], [27]) and of the monographs [20] and [21].

A second type of problems suggested by A. Liñán concerned the *homogenization process* related to the overall modeling in the presence of two spatial scales. In fact, it is not difficult to track the scientific connections between Liñán and one of the pioneers in this area, E. Sánchez-Palencia, before the latter moved to France at the end of the sixties. Connections also exist with J. L. Lions; see [54] and [14], and with collaborators of Sánchez-Palencia, specially M. Lobo Hidalgo.

Starting in 1985, on the occasion of the visit of C. Conca to the Universidad Complutense de Madrid, we considered the homogenization of chemical reactive flows through the exterior of a domain containing periodically distributed reactive solid grains (or reactive obstacles). A partial account of our results can be seen in Díaz [24]. The final version was published as a joint paper [19], incorporating also C. Timofte, which we summarize in the rest of this section.

We focus our attention on two nonlinear problems that describe the motion of a reactive fluid having different chemical properties. For a nice presentation of the chemical aspects involved in our first model (and also for mathematical and historical background) we refer the reader to Antontsev et al. [8], Bear [13], Díaz [23], Hornung [44] and Norman [55] and the references therein.

Let Ω be an open bounded set in $\mathbb{R}^n$ and let us introduce a set of periodically distributed reactive obstacles. As a result, we obtain an open set Ω^ε which will be

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[1]The mathematical consideration of the case of singular chemical kinetics was considered in Díaz, Morel and Oswald [36] and Hernández, Mancebo and Vega [42], among other papers. The limit case $\gamma \to +\infty$, relevant in combustion, was studied in Urrutia and Liñán [60] and then in Kapila, Matkowsky and Vega [47]

referred to as being the *exterior domain*; ε represents a small parameter related to the characteristic size of the reactive obstacles.

The first nonlinear problem studied concerns the stationary reactive flow of a fluid confined in Ω^ε, of concentration u^ε, reacting on the boundary of the obstacles. A simplified version of this problem can be written as follows:

$$\begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) & \text{on } S^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \qquad (2)$$

Here, ν is the exterior unit normal to Ω^ε, $a > 0$, $f \in L^2(\Omega)$ and S^ε is the boundary of our exterior medium $\Omega \setminus \overline{\Omega^\varepsilon}$. Moreover, the fluid is assumed to be homogeneous and isotropic, with a constant diffusion coefficient $D_f > 0$.

The semilinear boundary condition on S^ε describes the chemical reactions which take place locally at the interface between the reactive fluid and the grains. From strictly chemical point of view, this situation represents, equivalently, the effective reaction on the walls of the chemical reactor between the fluid filling Ω^ε and a chemical reactant located in the rigid solid grains.

The function g is assumed to be given. Two model situations will be considered; the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$ and the case of a maximal monotone graph with $g(0) = 0$, i.e. the case in which g is the subdifferential of a convex lower semicontinuous function G. These two general situations are well illustrated by the following important practical examples:

$$g(v) = \frac{\alpha v}{1 + \beta v}, \quad \alpha, \beta > 0 \quad \text{(Langmuir kinetics)} \qquad (3)$$

and

$$b) \quad g(v) = |v|^{p-1} v, \quad 0 < p < 1 \quad \text{(Freundlich kinetics)}. \qquad (4)$$

The exponent p is called *the order of the reaction*. In some applications the limit case $(p = 0)$ is of great relevance. It is worth remarking that if we assume $f \geq 0$, one can prove (see, e.g. [21]) that $u^\varepsilon \geq 0$ in $\Omega \setminus \overline{\Omega^\varepsilon}$ and $u^\varepsilon > 0$ in Ω^ε, although u^ε is not uniformly positive, except in the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$, as, for instance, in example (3).

The existence and uniqueness of a weak solution can be settled by using the classical theory of semilinear monotone problems (see, for instance, [17], [23] and [53]). As a result, we know that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \cap H^2(\Omega^\varepsilon)$, where

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}.$$

Moreover, if in the second model situation we associate the following nonempty convex subset of V^ε:

$$K^\varepsilon = \{v \in V^\varepsilon \mid G(v)|_{S^\varepsilon} \in L^1(S^\varepsilon)\}, \qquad (5)$$

then u^ε is also known to be characterized as being the unique solution of the following variational problem: Find $u^\varepsilon \in K^\varepsilon$ such that

$$D_f \int_{\Omega^\varepsilon} Du^\varepsilon D(v^\varepsilon - u^\varepsilon)dx - \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx + a\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\rangle \geq 0 \qquad (6)$$

for all $v^\varepsilon \in K^\varepsilon$, where μ^ε is the linear form on $W_0^{1,1}(\Omega)$ defined by

$$\langle \mu^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} \varphi d\sigma \quad \forall \varphi \in W_0^{1,1}(\Omega).$$

From a geometrical point of view, we shall just consider periodic structures obtained by removing periodically from Ω, with period εY (where Y is a given hyper-rectangle in $\mathbb{R}^n$), an elementary reactive obstacle T which has been appropriated rescaled and which is strictly included in Y, i.e. $\overline{T} \subset Y$.

As usual in homogenization, we shall be interested in obtaining a suitable description of the asymptotic behavior, as ε tends to zero, of the solution u^ε in such domains. We wonder, for example, whether the solution u^ε converges to a limit u as $\varepsilon \to 0$. And if this limit exists, can it be characterized?

We proved in [19] that, in the second model situation (in absence of any additional regularity on g), the solution u^ε, properly extended to the whole of Ω, converges to the unique solution of the following variational inequality: $u \in H_0^1(\Omega)$

$$\int_\Omega QDuD(v - u)dx \geq \int_\Omega f(v - u)dx - a\frac{|\partial T|}{|Y \setminus T|}\int_\Omega (G(v) - G(u))dx, \qquad (7)$$

for all $v \in H_0^1(\Omega)$. Here, $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries are defined as follows:

$$q_{ij} = D_f \left(\delta_{ij} + \frac{1}{|Y \setminus T|} \int_{Y \setminus T} \frac{\partial \chi_j}{\partial y_i} dy \right) \qquad (8)$$

in terms of the functions χ_i, $i = 1, ..., n$, solutions of the so-called cell problems

$$\begin{cases} -\Delta \chi_i = 0 & \text{in } Y \setminus T, \\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 & \text{on } \partial T, \\ \chi_i \; Y- \text{ periodic.} \end{cases} \qquad (9)$$

Notice that if g is smooth, then g is the classical derivative of G.

The chemical situation behind the second nonlinear problem we treat in [19] is slightly different that the previous one; it also involves a chemical reactor containing reactive grains, but we assume that now there is an internal reaction inside the grains, instead just on their boundaries. In fact, it is therefore a transmission problem with an unknown flux on the boundary of each grain.

To simplify matters, we shall just focus on the case of a function g which is continuous, monotone increasing and such that $g(0) = 0$; examples (3) and (4) are both covered by this class of functions $g's$ and, of course, both are still our main practical examples.

A simplified setting of this kind of models is as follows:

$$\begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_p \Delta v^\varepsilon + ag(v^\varepsilon) = 0, & \text{in } \Omega \setminus \overline{\Omega^\varepsilon} \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = D_p \dfrac{\partial v^\varepsilon}{\partial \nu} & \text{on } S^\varepsilon, \\ u^\varepsilon = v^\varepsilon & \text{on } S^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \qquad (10)$$

We assumed the initial and boundary conditions corresponding to an static initially full pipeline with one closed boundary point and other in which the discharge make take place at the pressure p_0 ($p_0 > p_a$) and temperature T_0 (for any time)

$$\begin{cases} u(x,0) = 0, \ T(0,x) = 1, \ p(x,0) = 1 & x \in (0,1), \\ u(0,t) = 0 & t > 0, \\ \dfrac{u(1,t)}{\sqrt{\gamma T(1,t)}} < 1 & \text{if } p(1,t) = p_a, \\ \dfrac{u(1,t)}{\sqrt{\gamma T(1,t)}} = 1 & \text{if } p(1,t) > p_a. \end{cases}$$

The system has the same characteristic lines that the Euler system

$$\frac{dx}{dt} = u, \quad \frac{dx}{dt} = u \pm a$$

with $a = \sqrt{\gamma T}$ the sound speed. We show that, asymptotically, there are two different steps in the discharge: in the first one (very short, of the order $1/f$) all the terms at the equation (15) are of the same order but the auxiliary conditions can be simplified allowing a local study made by using the Riemann invariants

$$2\frac{a}{(\gamma - 1)} + u \text{ and } 2\frac{a}{(\gamma - 1)} - u.$$

In the second step, when $t >> 1/f$ we show that the second and fourth equation can be simplified, by neglecting lower order terms and using some suitable variable scales), to

$$0 = -\frac{\partial p}{\partial x} - \frac{1}{2}\rho |u| u \text{ and } \frac{p}{\rho} = T = 1.$$

Then, from the first equation we deduce that p satisfies that

$$\begin{cases} \dfrac{\partial p}{\partial t} - \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial p^2}{\partial x}\right|^{1/2} sign\left(\dfrac{\partial p^2}{\partial x}\right)\right) = 0 & t > 0, \ x \in (0,1), \\ \dfrac{\partial p}{\partial x}(0,t) = 0, \ p(1,t) = p_a & t > 0, \\ p(x,0) = 1 & x \in (0,1). \end{cases}$$

Notice that since $u \geq 0$, making $p^2 - p_a^2 := w$ we arrive to the doubly nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial \psi(w)}{\partial t} - \Delta_q w = 0 & t > 0, \ x \in (0,1), \\ \dfrac{\partial w}{\partial x}(0,t) = 0, \ w(1,t) = 0 & t > 0, \\ w(x,0) = w_0 & x \in (0,1), \end{cases} \tag{16}$$

with $\psi(w) = (w + p_a^m)^{1/m}$ which is a nondecreasing function of w. The correct exponents are $m = 2$ and $q = 3/2$ nevertheless other interesting cases are $m = 7/4$ and $q = 11/7$ (case of very polished pipes) and $m = 1$ and $q = 3/2$ (laminar regime). The existence and uniqueness of solutions of a larger class of problems of this type was the main motivation of the paper [37]. In the paper [31] we study the *finite*

extinction property: there exists a finite time t_0 such that $w(x,t) \equiv p_a$ for any $t \geq t_0$ and $x \in (0,1)$.

By using an integral energy method (in the spirit of [7]) we show that the property holds if $p_a > 0$ and $q < 2$ (for any $m > 0$ arbitrary) either $p_a = 0$ and $m(q-1) < 1$. Those assumptions are, in some sense, optimal since, if by the contrary, we assume $p_a \geq 0$ and $m(q-1) = 1$ we show the existence of two positive constants τ_1, τ_2 such that

$$e^{-\lambda_1(t+\tau_1)}v(x) \leq w(x,t) \leq e^{-\lambda_1(t+\tau_2)}v(x), \quad \text{for any } t \geq 0 \text{ and } x \in (0,1),$$

where λ_1 and v are the first eigenvalue and eigenfunction of the problem

$$(NEP) \begin{cases} -\Delta_q v = \lambda v^{1/m} & x \in (0,1), \\ \dfrac{\partial v}{\partial x}(0) = 0, \ v(1) = 0. \end{cases}$$

(notice that $v(x) > 0$ for any $x \in (0,1)$). Moreover we prove that if $p_a = 0$ and $m(q-1) < 1$ then the discharge is global at time $t = t_0$ since we prove that there exists an increasing sequence $t_n \to t_0$ and a solution $v > 0$ of the stationary problem (NEP) such that the function

$$z(t,x) := \begin{cases} \dfrac{w(x,t)}{g(t)} & \text{if } 0 \leq t < t_0 \\ 0 & \text{if } t \geq t_0, \end{cases}$$

with $g(t) = [\lambda(1-m(q-1))(t_0-t)]^{\frac{1}{1-m(q-1)}}$, verifies that $z(t_n,x) \to v^m(x)$, as $n \to \infty$, in $L^p(0,1)$, *for all* $1 \leq p < \infty$. This result extended some previous theorem due to Berryman and Holland [15] for the case of linear diffusions $q = 2$. More recently, the limit case $q = 1$ (and $m = 1$) was considered in Andreu, Caselles, Díaz, Mazón [4]. This corresponds to the so called *total variation flow* equation

$$\frac{\partial u}{\partial t} = \text{div}\left(\frac{Du}{|Du|}\right) \tag{17}$$

arising in many questions related to differential geometry, image processing and microgranular materials (see, for instance, Kobayashi and Giga [49] and its references).

4. A SOURCE OF PROBLEMS IN LUBRICATION

The lecture by A. Liñán "Problemas matemáticos de lubricación hidrodinámica" given at the Seminario de Matemática Aplicada of the Universidad Complutense de Madrid (UCM) on April 14th 1986 was the origin of a long production by many Spanish mathematicians. In this lecture, he presented the cavitation phenomena as one of the harder free boundary problems formulated in fluid mechanics. The correct mathematical condition satisfied at the free boundary attracted the attention of many mathematicians. Moreover the question of the uniqueness of the associated weak solutions of the stationary problem was the main goal of the thesis by S.J. Álvarez at the UCM which appeared later coauthored with his thesis adviser (Álvarez and Carrillo [1]). The evolution problem was considered in Álvarez, Carrillo and Díaz [2] and Díaz [22].

The mathematical interest for this type of problems propagated very fast to specialists of the universities of Santiago de Compostela (Bermúdez de Castro) and

Vigo (Durany). The doctoral dissertation of C. Vázquez (who maintain today a great activity in this domain with collaborations with different authors) was one of the consequences, as well as the celebration of the international meeting *Mathematical Modelling Lubrication* (G.Bayada, M.Chambat y J.Durany ed.) Universidad de Vigo in 1991.

Another set of results, originated this time in the postgraduate course "Introducción a la Mecánica de Fluidos", given by A. Liñán and this author since 1996 (the first two years also in collaboration with M. García Velarde) concerned the question of the regularity of solutions of some very simple problems arising in lubrication. In Díaz and Tello [38] (see also Tello [57], [58]), we present the mathematical treatment of a problem of hydrodynamic lubrication, relevant in the applications, which leads to a formulation lacking a classical solution. So, the solvability must be necessarily boarded in terms of weak solutions. This type of arguments, justifying the needed of weak solutions, is typical of nonlinear hyperbolic equations. What we underline in that paper is that this situation also arises with some linear elliptic equations which are relevant in the applications, and not merely a mathematical exercise searched as a sophisticated counterexample.

Consider, for instance, the problem of the lubricating the friction between a fixed undeformable solid presenting some abrupt edges and a regular surface in movement by using an incompressible fluid in the separating region. This kind of problem frequently appears in different engineering applications, as in "feedbox" or "shaft-bearing" systems. We assume, for simplicity that the surface reduces to the one given by $z = 0$ and that it moves with a given velocity $(U_0, V_0, 0)$, (i.e. parallel to the own surface). Let $h(t, x, y)$ be the distance between the surface and the solid. That we want to describe is the fluid velocity $\mathbf{u} = (u, v, w)$ and pressure P. We suppose the fluid incompressible of density ρ (a positive know constant). Starting from the usual conservation principles

$$\text{mass conservation} \quad \frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0,$$
$$\text{momentum conservation} \quad \rho \mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + \mu \Delta \mathbf{u},$$

using dimensional analysis and supposing h *small with respect the solid size*, we can simplify the momentum equation leading to the system

$$-\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{in the } x \text{ component},$$
$$-\frac{\partial P}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2} = 0 \quad \text{in the } y \text{ component},$$
$$-\frac{\partial P}{\partial z} = 0 \quad \text{in the } z \text{ component}.$$

The boundary conditions are
$$u = v = 0, \quad w = \tfrac{\partial h}{\partial t} \quad \text{on} \quad z = h,$$
$$u - U_0 = v - V_0 = w = 0 \quad \text{on} \quad z = 0.$$
Therefore, we have that

$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x}z(z - h) + U_0\left(1 - \frac{z}{h}\right),$$

$$v = \frac{1}{2\mu}\frac{\partial P}{\partial y}z(z - h) + V_0\left(1 - \frac{z}{h}\right).$$

The flow is given by

$$q_x = \int_0^h u\,dz = \frac{U_0 h}{2} - \frac{h^3}{12\mu}\frac{\partial P}{\partial x},$$

$$q_y = \int_0^h v\,dz = \frac{V_0 h}{2} - \frac{h^3}{12\mu}\frac{\partial P}{\partial y}.$$

Integrating in the mass equation, we get that

$$\begin{cases} \dfrac{\partial \rho h}{\partial t} + \dfrac{\partial}{\partial x}[\rho q_x] + \dfrac{\partial}{\partial y}[\rho q_y] = 0 & \text{in } \Omega, \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

If, for simplicity, we suppose that $h(t, \cdot) = h(\cdot)$ we arrive to the, so called *Reynolds equation*

$$(\mathcal{P}) \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{U_0 h}{2} - \dfrac{h^3}{12\mu}\dfrac{\partial P}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{V_0 h}{2} - \dfrac{h^3}{12\mu y}\dfrac{\partial P}{\partial y}\right) = 0 & \text{in } \Omega, \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

In fact, in what follows, we shall always assume that

$$h \in L^\infty(\Omega), 0 < h_0 \leq h(x, y) \leq h_1, a.e. \text{ on } \Omega. \tag{18}$$

We point out that more general situations, in which the surface is more complicated, can be considered by expressing the pde in terms of a general coordinates system (α, β, z) associated to the surface, getting formulations of the type

$$\begin{cases} \dfrac{\partial}{\partial \alpha}\left(g_\beta\dfrac{U_0 h}{2} - g_\beta\dfrac{h^3}{12\mu g_\alpha}\dfrac{\partial P}{\partial \alpha}\right) + \dfrac{\partial}{\partial \beta}\left(g_\alpha\dfrac{V_0 h}{2} - g_\alpha\dfrac{h^3}{12\mu g_\beta}\dfrac{\partial P}{\partial \beta}\right) = 0 & \text{in } \Omega, \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

In order to present an example where no classical solution of $(\mathcal{P})$ may exits we consider the case in which $h(x, y)$ is discontinuous (case of solids with abrupt edges). This is specially easy to present in the unidimensional case (i.e. an uniform solid which is understood as unbounded). We start by recalling the notion of weak solution:

Definition We say that P is a weak solution of $(\mathcal{P})$ if $P = u + P_0$, with $u \in H_0^1(\Omega)$ satisfying that

$$\int_\Omega \frac{h^3}{12\mu}\nabla u \cdot \nabla \psi \, d\sigma = \int_\Omega \frac{h}{2}(U_0, V_0) \cdot \nabla\psi d\sigma, \; \forall\psi \in H_0^1(\Omega). \tag{19}$$

A standard application of the Lax-Milgram theorem allows to prove the existence and uniqueness of a weak solution P. In the special discontinuous unidimensional case, if $\Omega = (0, L)$ and

$$h(x) = \begin{cases} h_0 & \text{if } x \in (0, \frac{L}{2}) \\ h_1 & \text{if } x \in (\frac{L}{2}, L). \end{cases} \tag{20}$$

then the weak solution P is non of class C^2 and so is not a classical solution since the one dimensional Reynolds equation becomes

$$\begin{cases} \dfrac{\partial}{\partial x}[\dfrac{U_0 h}{2} - \dfrac{h^3}{12\mu}\dfrac{\partial P(x)}{\partial x}] = 0 & \text{in } \Omega, \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{cases} \qquad (21)$$

and the (unique) weak solution is explicitly given by:

$$P(x) = \begin{cases} -6\mu\dfrac{2k + U_0 h_0}{h_0^3}x + P_0 & \text{if } 0 < x < \dfrac{L}{2} \\ 6\mu\dfrac{-2k + U_0 h_1}{h_1^3}(L - x) + P_0 & \text{if } \dfrac{L}{2} < x < L \end{cases} \qquad (22)$$

where

$$K = \dfrac{LU_0}{4}\left[\dfrac{1}{h_1^2} - \dfrac{1}{h_0^2}\right]\left[\dfrac{1}{h_0^3} - \dfrac{1}{h_1^3}\right]^{-1}.$$

Then, obviously, $P \notin C^1(\Omega)$ (nevertheless, it is easy to see that function given by (22) satisfies that $P \in W^{1,\infty}(0,L)$).

The study of the regularity of the weak solution of $(\mathcal{P})$ associated to $\Omega = (0,L) \times (0,B)$ and eventually discontinuous separation functions $h(x,y)$ as

$$h(x,y) = \begin{cases} h_0 & \text{if } x \in (0, \frac{L}{2}) \\ h_1 & \text{if } x \in (\frac{L}{2}, L) \end{cases} \qquad (23)$$

where $0 < h_0 < h_1$, is far to be trivial. For instance, the regularity $C^{0,\alpha}(\Omega)$, $\forall \alpha \in [0,1)$, of the weak solution of $(\mathcal{P})$ is a direct consequence of the regularity theory (see, e.g. Kinderlehrer-Stampacchia [48]). The $W^{1,p}(\Omega)$ regularity is a more delicate question due to the lack of continuity of h. As far as we know, there is not any general result in the literature that could be applied directly to this case. The main result of Díaz and Tello ([38]) (see also the generalization made in [58]) shows that, in fact, $P \in W^{1,\infty}(\Omega)$.

We end this section by making reference to some inverse problems, also suggested by A. Liñán in the mentioned postgraduate course (see also his notes [52]). After the pioneering work by O. Reynolds, in 1886, it is well-known that the pressure of a lubricating fluid filling the gap between two solid surfaces satisfies the, so called, Reynolds equation involving the distance function, h, between both planes, as a crucial coefficient. Nevertheless, in most of the applications function h is not known a priori. The hard disc of computers or the compact disc player are two examples of the many real situations where this kind of problems appear.

Although several works have been devoted to the study of this problem when some extra information is added to the formulation (see, e.g. the articles Bayada [10], Bayada and El Alaoui Talibi [11] in which the total load supported by the surfaces is prescribed), it seems not well observed the necessity of to impose suitable conditions on the additional information in order to get a well posed formulation.

In Díaz and Tello ([39]) we consider the simple case in which the surfaces are two parallel planes and so the unknown distance between both planes is merely a time function $h = h(t)$, for $t \in (0,T)$.with $T > 0$ given. So, the unknown are the functions $(h(t), P(x,y,t))$, where P denotes the pressure and $(x,y) \in \Omega$, an open and bounded set of $\mathbb{R}^2$. We assume given the initial distance between the planes

$$h(0) = h_0, \qquad (24)$$

the external pressure P_a (a positive constant), the initial pressure distribution $P(x,0) = P_0(x)$ (only for the case of a compressible fluid) and the relative velocity (U,V) of the superior plane (in fact here assumed to be a constant vector). In this note we also assume to be known the total force applied upon the superior plane and that it has only a nonzero component, $F(t)$, in the z-direction (orthogonal to the planes).

The main goal of ([39]) was to give some sufficient conditions on $F(t)$ in order to solve this inverse problem. Moreover, in the incompressible case, we shall show that our sufficient condition on $F(t)$ is also necessary for the existence of a solution (h, P). We recall that in the case of an incompressible fluid, under the above conditions, the Reynolds equation deals to the linear elliptic inverse problem: assumed known $F(t)$ find (h, P) such that

$$\begin{cases} -div(h(t)^3 \nabla P) = -h'(t), & \text{in } \Omega \times (0,T), \\ P = P_a, & \text{on } \partial\Omega \times (0,T) \\ F(t) = \int_\Omega (P(x,y,t) - P_a)dxdy & \text{for } t \in (0,T). \end{cases} \qquad (25)$$

In spite of the simplicity of the above formulation, it seems that the study of necessary and sufficient conditions on $F(t)$ was not clearly indicated before in the literature. The incompressible case (Problem (24) and (25)) can be solved by using the auxiliary problem

$$\begin{cases} -\Delta w = 1, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \qquad (26)$$

Consider

$$K(\Omega) = \int_\Omega w(x,y)dxdy.$$

and assume that

$$F(t) > -\dfrac{P_a K(\Omega)}{\max_{(x,y)\in\Omega}\{w(x,y)\}}, \quad t \in (0,\infty). \qquad (27)$$

then we show that if

$$\int_0^t F(s)ds > -\dfrac{K(\Omega)}{2h_0^2} \; \forall \, t \in (0,\infty) \qquad (28)$$

then there exists a unique solution $(h(t), P(x,y,t))$ of the problem (24), (25) such that

$$\dfrac{h'(t)}{h^3(t)} = -\dfrac{F(t)}{K(\Omega)}, \text{ (and therefore, } \text{sign}(h') = -\text{sign}(F(t))). \qquad (29)$$

In particular $h(t) = \left[\dfrac{1}{h_0^2} + \dfrac{2}{K(\Omega)}\int_0^t F(s)ds\right]^{-\frac{1}{2}}$ and

$$P(x,y,t) = \dfrac{F(t)}{K(\Omega)}w(x,y) + P_a. \qquad (30)$$

Moreover, if there exists $t_0 > 0$ such that

$$\int_0^{t_0} F(s)ds = -\dfrac{K(\Omega)}{2h_0^2} \text{ and } \int_0^t F(s)ds < -\dfrac{K(\Omega)}{2h_0^2} \; \forall t \in (0,t_0), \qquad (31)$$

then $h(t) \to \infty$ when $t \nearrow t_0$ and $P(x, y, t_0) = \frac{F(t_0)}{K(\Omega)} w(x, y) + P_a$.

The compressible case is more delicate since the associate problem becomes parabolic and of quasilinear type. To simplify the formulation, we consider the simpler case in which the spatial domain is reduced to a one dimensional interval $I = (0, L)$ (so, there in no dependence of data and unknowns with respect to the y-variable). Then, under some conditions on the degree of compressibility of the gas (see, e.g., Friedman and Tello [40]) we arrive to the following inverse problem for the Reynolds equation: assumed known $F(t)$ find (h, P) such that

$$\begin{cases} \dfrac{\partial (Ph(t))}{\partial t} + Uh(t)\dfrac{\partial P}{\partial x} - \epsilon\dfrac{\partial}{\partial x}((\alpha h(t)^2 + \beta h(t)^3 P)\dfrac{\partial P}{\partial x}) = 0, & \text{in } I \times (0, T), \\[2mm] P(x, 0) = P_0(x), & \text{in } I, \\[2mm] P(0, t) = P(L, t) = P_a, & \\[2mm] F(t) = \int_\Omega (P(x, y, t) - P_a) dx dy, & \text{for } t \in (0, T), \end{cases}$$

(32)

where P_a, α, β, ϵ and U are known positive constants and T is small enough. Some results for this problem were given in ([39]).

5. ASYMPTOTICS IN COULOMB FRICTION TYPE PROBLEMS

In a series of joint works ([32], [33], [34]) we study the asymptotic behavior of solutions of the damped oscillator

$$m x_{tt} + \mu |x_t|^{\alpha - 1} x_t + kx = 0,$$

(33)

where $\alpha \in (0, 1)$ and $\mu, k > 0$. In fact our work was related to the formulation

$$x_{tt} + |x_t|^{\alpha - 1} x_t + x = 0$$

(34)

which is attained by dividing by k and by introducing the rescaling $\tilde{x}(\tilde{t}) = \beta^{1/(\alpha-1)} x(\lambda \tilde{t})$ where $\lambda = \frac{\sqrt{m}}{\sqrt{k}}$ and $\beta = \frac{\mu}{k^{(2-\alpha)/2} m^{\alpha/2}}$. Notice that the x-rescaling fails for $\alpha = 1$. In that case there is no well defined scale for x and the equation is reduced to $x_{tt} + \beta x_t + x = 0$ with $\beta = \frac{\mu}{\sqrt{km}}$ remaining as a parameter to characterize the dynamics. The limit case $\alpha \to 0$ corresponds to the Coulomb friction equation

$$x_{tt} + sign(x_t) + x \ni 0$$

(35)

where $sign$ is the maximal monotone graph of $\mathbb{R}^2$ given by $sign(r) = -1$, if $r < 0$, $[-1, 1]$ if $r = 0$, and 1 if $r > 0$. The limit equation when $\alpha \to 1$ corresponds with the linear damping equation

$$x_{tt} + x_t + x = 0.$$

(36)

We recall that, even if the nonlinear term $|x_t|^{\alpha-1} x_t$ is not a Lipschitz continuous function of x_t, the existence and uniqueness of solutions of the associate Cauchy problem

$$P_\alpha \begin{cases} x_{tt} + |x_t|^{\alpha-1} x_t + x = 0 & t > 0, \\ x(0) = x_0, \; x_t(0) = v_0 \end{cases}$$

(and of the limit problems P_0 and P_1 corresponding to the equations (35) and (36) respectively) is well known in the literature: see, e.g. Brezis [17]. An easy application of the results of the above reference yields to a rigorous proof of the convergence of solutions when $\alpha \to 0$ and $\alpha \to 1$.

The asymptotic behavior, for $t \to \infty$, of solutions of the limit problems P_0 and P_1 is well known (see, for instance, Jordan and Smith [50]). In the first case the decay is exponential. In the second one it is easy to see that "given x_0 and v_0 there exist a finite time $T = T(x_0, v_0)$ and a number $\zeta \in [-1, 1]$ such that $x(t) \equiv \zeta$ for any $t \geq T(x_0, v_0)$". For problem P_α it is well-known that $(x(t), x_t(t)) \to (0, 0)$ as $t \to \infty$ (see, e.g. Haraux [41]).

The main result of papers [32] and [33] was to show that the generic asymptotic behavior above described for the limit case P_0 is only exceptional for the sublinear case $\alpha \in (0, 1)$ since the generic orbits $(x(t), x_t(t))$ decay to $(0, 0)$ in a infinite time and only two one-parameter families of them decay to $(0, 0)$ in a finite time: in other words, when $\alpha \to 0$ the exceptional behavior becomes generic.

We started with some formal results via asymptotic arguments. We can rewrite the equation (34) in as the planar system

$$\begin{cases} x_t = y \\ y_t = -x - |y|^{\alpha-1} y \end{cases}$$

(37)

which, by eliminating the time variable, for $y \neq 0$, leads to the differential equation of the orbits in the phase plane

$$y_x = \frac{-x - |y|^{\alpha-1} y}{y}$$

(38)

and that allows us to carry out a phase plane description of the dynamics.

We remark that the plane phase is antisymmetric since if $y = \varphi(x)$ is a solution of (38) then the function $y = -\varphi(-x)$ is also solution. So, it is enough to describe a semiplane (for instance $x \geq 0$). By multiplying by x and y, respectively, we get that $(x^2 + y^2)_t = 2|y|^{\alpha+1}$. On the other hand, it is easy to see that $(1/x, 1/y)$ satisfy a system which has the point $(0, 0)$ as a spiral unstable critical point. For values of $x^2 + y^2 >> 1$ the orbits of the system are given, in first approximation, by $x^2 + y^2 = C$ because $|y|^{\alpha-1} y$ is small compared with x. The effect of this term is to decrease slowly C with time giving the trajectory a spiral character. For $\alpha = 1$ the character of the trajectories close to the origin depends on the parameter β. For $\beta > \beta_c := 2$ the origin is a stable mode and for $\beta < \beta_c$ is a stable spiral corresponding to underdamped oscillations. It should be noticed that for $\alpha > 1$ the origin becomes a stable spiral point. The limit case $\alpha \to +\infty$ can be described analytically with two-time scale methods (see [34]).

We proved that there are two modes of approach to the origin and so that the origin $(0, 0)$ is a node for the system (37). The lines of zero slope are given by

$$-x = |y|^{\alpha-1} y.$$

(39)

So the convergence to $(0, 0)$ is only possible through the regions $\{(x, y) : x > 0, y < -x^{1/\alpha}\} \cup \{(x, y) : x < 0, y > (-x)^{1/\alpha}\}$. Let us see that the "ordinary" mode corresponds to orbits that are very close to the ones corresponding to small effects of the inertia. Due to the symmetry it is enough to describe this behavior for the orbits approaching the origin with values of $x > 0$ and $y < 0$. Let $-y = \bar{y} > 0$. Equation (38) takes the form

$$\bar{y} \bar{y}_x = -x + \bar{y}^\alpha.$$

(40)

The line of zero slope is $\bar{y} = x^{1/\alpha}$ and we search for orbits obeying, for $0 < x << 1$, to the expression $\bar{y} = x^{1/\alpha} + z(x)$ for some function $z(x)$. If we anticipate the

condition $0 < z(x) << x^{1/\alpha}$, equation (38) takes the "linearized form" $\frac{1}{\alpha}x^{(\frac{1}{\alpha}-1)}z + x^{\frac{1}{\alpha}}z_x - \alpha x^{(1-\frac{1}{\alpha})}z = 0$. Thus the first term can be neglected, compared with the last one, and then the solution can be written as $z(x) \sim C \exp\{-[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\}$ with C an arbitrary constant (which explain the name of "ordinary" orbits). This type of orbits are given, close to the origin, by the approximate equation (39), which for the orbits that reach the origin from below implies that $\bar{y} \sim x^{1/\alpha} \sim -\frac{dx}{dt}$ and so, integrating the simplified equation

$$\frac{dx}{dt} = -x^{1/\alpha} \qquad (41)$$

we get that

$$x(t) \sim [\frac{\alpha}{(1-\alpha)(t+t_1)}]^{\alpha/(1-\alpha)} \qquad (42)$$

and so that it takes an infinite time to reach the origin.

Some different orbits approaching the origin can be found by searching among solutions with large values of $|y|$ compared with $|x|^{1/\alpha}$. Thus, close to the origin, the orbits with negative y are "very close" to the solutions of the equation found by replacing (40) by the simplified the equation

$$\bar{y}\bar{y}_x = \bar{y}^\alpha \qquad (43)$$

corresponding to a balance of inertia and damping. The solution ending at the origin ($\bar{y}(0) = 0$) is given by

$$\bar{y}(x) = -\{(2-\alpha)x\}^{1/(2-\alpha)}. \qquad (44)$$

Notice that it involves no arbitrary constant. So this curve is unique (a symmetric curve arises for $y > 0$ and $x < 0$) which justifies the term of "extraordinary" orbit. The time evolution of this orbit is given, for $x << 1$, by integrating the equation

$$-\frac{dx}{dt} = [(2-\alpha)x]^{1/(2-\alpha)} \qquad (45)$$

and so

$$x(t) = \frac{1}{(2-\alpha)}[\frac{(2-\alpha)(1-\alpha)}{2\alpha}(t_0-t)_+]^{(2-\alpha)/(1-\alpha)},$$

where in general $h(t)_+ = \max\{0, h(t)\}$. This indicate that the motion (of this approximated solution) ends at a finite time, t_0, determined by the initial conditions which, by (45) must satisfy that $v_0 \sim \pm[(2-\alpha)|x_0|]^{1/(2-\alpha)}$ We point out that the two exceptional orbits emanating from the origin spiral around the origin when $x^2 + y^2$ grows toward infinity and so each of them is a separatrix curve in the phase plane. Notice that due to the autonomous nature of the equation, if $x(t)$ is the solution of the Cauchy problem (P_α) of initial data (x_0, v_0) then for any parameter $\tau \geq 0$ the function $\bar{x}(t) := x(t+\tau)$ coincides with the solution of (P_α) of initial data $(x(\tau), x_t(\tau))$. In this way, the above extraordinary orbits give rise to two curves of initial data for which the corresponding solutions of (P_α) vanish after a finite time.

We end this section by pointing out that the solution of problem (P_α) for $0 < \alpha << 1$ takes an asymptotic form which can be easily described. The differential equations of the orbits "simplify" if $y \neq 0$ is finite and $\alpha \to 0$ to $yy_x = -x-1$ for $y > 0$ and $\bar{y}\bar{y}_x = -x+1$ for $\bar{y} = -y > 0$. The solutions are circles with center at $x = -1$ if $y > 0$ and center $x = 1$ if $y < 0$ joined. An orbit formed with half

circles with centers at $x = -1$ or $x = 1$ when it hits the interval $(0,1)$ from below it is transformed into an orbit that reaches the origin following very closely that segment, governed by the equation (41) of solution (47). In the limit $\alpha \to 0$ we found that any point $\zeta \in [-1,1]$ is an asymptotically stable stationary state of (P_0).

In a second part of the papers we proved some rigorous estimates on the decay. In ([32]) we used a fixed point argument to show that there exists two curves Γ_+ and Γ_- of initial data (x_0, v_0) for which the solutions $x(t)$ of the corresponding Cauchy problem (P_α) vanish after a finite time. Moreover, we give some additional results on these two curves:

(i) Near the origin the curves Γ_+ and Γ_- can be represented by two functions, $y = \varphi_+(x)$ and $y = \varphi_-(x)$, solutions of the equation (38), where $\varphi_+ : [0, \varepsilon] \to (-\infty, 0]$ and $\varphi_- : [-\varepsilon, 0] \to [0, +\infty)$, for some $\varepsilon > 0$.

(ii) Functions φ_+ and φ_- satisfy that $\varphi_\pm(0) = 0$

$$-\infty < \int_0^\varepsilon \frac{ds}{\varphi_+(s)} \text{ and } \int_{-\varepsilon}^0 \frac{ds}{\varphi_-(s)} < +\infty. \qquad (46)$$

In particular, $\varphi'_+(x) \downarrow -\infty$ when $x \downarrow 0$ and $\varphi'_-(x) \uparrow +\infty$ when $x \uparrow 0$.

(iii) We have

$$-Cx^{\frac{1}{2-\alpha}} \leq \varphi_+(x) \leq -x^{\frac{1}{\alpha}} \text{ for } x \in [0, \varepsilon] \text{ and}$$
$$(-x)^{\frac{1}{\alpha}} \leq \varphi_-(x) \leq C(-x)^{\frac{1}{2-\alpha}} \text{ for } x \in [-\varepsilon, 0],$$

for some $C > 0$.

(iv) There exists a $x_s \in (0, \varepsilon]$ such that $\varphi_+(x_s) = -(x_s)^{\frac{1}{\alpha}}$ and $(-x_s)^{\frac{1}{\alpha}} = \varphi_-(x_s)$. Moreover the regions $D_+ := \{(x,y): x \in [0, x_s] \text{ and } \varphi_+(x) \leq y \leq -x^{\frac{1}{\alpha}}\}$, $D_- := \{(x,y): x \in [-x_s, 0] \text{ and } (-x)^{\frac{1}{\alpha}} \leq y \leq \varphi_-(x)\}$ are time invariants for equation (38).

In order to prove that the decay to zero in an infinite time is more generic than the decay to zero in a finite time we obtain sharper invariants regions

(i) There exists a $\delta \in (0, x_s)$ small enough such that the regions $D_+^\delta := \{(x,y) \in D_+ : x \in [0, x_s-\delta] \text{ and } -x^{\frac{1}{\alpha}} - \delta \exp\{-[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\} < y \leq -x^{\frac{1}{\alpha}}\}$, $D_-^\delta := \{(x,y) \in D_- : x \in [-x_s+\delta, 0] \text{ and } (-x)^{\frac{1}{\alpha}} \leq y \leq (-x)^{\frac{1}{\alpha}} + \delta \exp\{[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\}\}$ are time invariants for equation (38).

(ii) If $(x_0, v_0) \in D_+^\delta$ (respectively D_-^δ) then the solution $x(t)$ of (P_α) satisfies that $x(t) \geq C\ t^{-\alpha/(1-\alpha)}$ (respectively $x(t) \leq -C\ t^{-\alpha/(1-\alpha)}$) for some $C > 0$ and any $t > 0$.

$$x(t) \sim [\frac{\alpha}{(1-\alpha)(t+t_1)}]^{\alpha/(1-\alpha)} \qquad (47)$$

and so that it takes an infinite time to reach the origin.

The above results were improved in Amann and Díaz [3] and, specially, in Vázquez [61] which contain a completely rigorous proof of the asymptotic part. For some results on a related system see Diaz and Millot [35].

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