Global bifurcation and continua of nonnegative solutions for some nonlinear elliptic eigenvalue type problems

Jesús Ildefonso DÍAZ* and Jesús HERNÁNDEZ*

Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040–Madrid, Spain
ji_diaz@mat.ucm.es

Departamento de Matemáticas
Universidad Autónoma de Madrid
28049–Madrid, Spain
jesus.hernandez@uam.es

A Enrique Outerelo, de dos alumnos agradecidos:
por haber puesto un poco de orden en sus mentes matemáticas,
que era altamente conveniente.

ABSTRACT

In this note, we study the existence and multiplicity of solutions, strictly positive or nonnegative having a *dead core* (where the solution vanishes) of several nonlinear problems of eigenvalue type.

2000 Mathematics Subject Classification: 35J60, 35B52.
Key words: Semilinear and quasilinear elliptic equations, bifurcation diagrams, index theory, free boundary problems, super and subsolutions.

*Research partially supported by project REN2003-0223-C03 of the DGISGPI (Spain) and RTN HPRN-CT-2002-00274 of the EC.

1. Introduction.

Semilinear elliptic equations of reaction-diffusion type have been used widely in the last thirty years as models for a variety of problems arising in applications (population dynamics, combustion, chemical reactions, etc.). They give rise to many interesting mathematical problems. in particular for what concerns existence and (maybe) multiplicity of positive (or nonnegative) solutions, which are often the only meaningful in the physical situation. A simple example is given by

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, Δ is the usual Laplacian modeling (linear) diffusion, λ is a real parameter and f is a nonlinear reaction term satisfying usually $f(0) \geq 0$. Of course, f can also depend on x (and on ∇u), other boundary conditions, even nonlinear, can be imposed, etc. (see the books [10] and [22] and its references).

If the nonlinearity f is C^1, or at least locally Lipschitz, a simple argument involving the Maximum Principle shows that *nonnegative* solutions (i.e. $u \geq 0$ in Ω) are actually *positive* ($u > 0$ in Ω). But if f is not Lipschitz at zero (with $f(0) = 0$) then it may happen that nonnegative solutions have a "dead core", i.e., regions of positive measure in Ω where the solution vanishes. The same situation may arise in models with nonlinear diffusion which give rise, after a change of unknown, to problems with non-Lipschitz nonlinearities. These free boundary problems were extensively studied in the early eighties (see [10]).

Here we deal with a class of problems which, maybe, posses both positive and "dead core" solutions. More precisely, in Section 2 we consider the quasilinear elliptic one-dimensional problem

$$\mathcal{P}_{\lambda,\alpha} \begin{cases} -(|u'|^{p-2} u')' + \alpha u^m = \lambda u^q & \text{in } (-1,1), \\ u(\pm 1) = 0 \end{cases} \tag{1.1}$$

where $p > 1$ and α, λ are positive numbers and $0 < m < q < p-1$, which corresponds to the case of strong absorption with respect to the diffusion (see [10]). Hence this includes semilinear equations with non-Lipschitz nonlinearities, as $-\Delta u + u^\alpha = \lambda^\beta u$, $0 < \alpha < \beta < 1$, and quasilinear degenerate equations as as $-(|u'|^2 u')' + \alpha u = \lambda u^2$. Some comments on previous work are given below.

In Section 2 we sketch the results we have obtained and partially written in [11]. Here we use phase plane arguments for the associated ODE and get a complete picture of the solution set. We show how these methods also work in our case and get results close to [4], [23], [22]. There is, however, a remarkable difference, since the "lower branch" of positive solutions "stops" at some critical value λ_1 giving rise to "dead core solutions". A more detailed version of these results will be given in [12].

In Section 3 we sketch how similar ideas can be developed when Ω is a smooth bounded domain in $\mathbb{R}^N$. We consider the problem

$$\begin{cases} -\Delta u = \lambda(u^2 - u^3) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see [20], [7]) and sketch a proof of the existence of at least two positive solutions for λ large which is still valid when $-\Delta$ is replaced by a second order linear operator not necessarily in divergence form. The essential tool here is a result by Dancer [9] (see also [18]) saying that in the interval between ordered sub and supersolutions $[u_0, u^0]$ there is always a solution $u_0 \le u \le u^0$ with index 1, $i(u) = 1$. This extends a result by Brezis and Nirenberg [6] for operators in divergence form. Finally, in a series of Remarks we give indications about how to modify the arguments (and results) in order to deal with non-Lipschitz (and singular) semilinear elliptic problems and degenerate operators involving the p-Laplacian.

2. The one-dimensional problem

To state our main result for problem (1.1) it is useful to introduce the notation $f(w) = w^q - w^m$ and $F(r) := \int_0^r f(s)ds$. We also introduce $r_F = (q/m)^{1/(q-m)}$ (the unique zero of $F(r)$). Let $\lambda_1 > 0$ be given by

$$\lambda_1 := \alpha^{(p-1-q)/(p-1-m)} \left(\frac{(p-1)^{1/p}}{p^{1/p}} \int_0^{r_F} \frac{dr}{(-F(r))^{1/p}} \right)^{(p-1)(q-m)/(p-1-m)} \tag{2.1}$$

Notice that $\lambda_1 < \infty$ thanks to the assumption $m < q$. We have

Theorem 2.1 *There exists a $\lambda_0 \in (0, \lambda_1)$ such that: a) if $\lambda \in (0, \lambda_0)$ there is no positive solution, b) if $\lambda = \lambda_0$, there is a unique positive solution $u(\cdot, \mu_+(\lambda_0))$ ($\mu_+(\lambda_0) := \|u\|_\infty$), c) if $\lambda \in (\lambda_0, \lambda_1]$, there are two positive solutions $p(\cdot,\lambda) = u(\cdot, \mu_-(\lambda))$ and $q(\cdot,\lambda) = u(\cdot, \mu_+(\lambda))$ ($\mu_+(\lambda) := \|p\|_\infty$, $\mu_-(\lambda) := \|q\|_\infty$). Moreover, $p(\cdot,\lambda) < q(\cdot,\lambda)$ on $(-1,+1)$ and $u'(\pm 1, \mu_-(\lambda_1)) = 0$. d) if $\lambda \ge \lambda_1$ there is one positive solution $q(\cdot,\lambda) = u(\cdot, \mu_+(\lambda))$. e)Finally, if $\lambda > \lambda_1$ there is a family of nonnegative solutions which are generated by $u(\cdot, \mu_-(\lambda_1))$ and for $\lambda > \lambda_1$ we have $\mu_-(\lambda) = (\frac{\alpha}{\lambda})^{1/(q-m)}C$ with $C = \|u(\cdot, \mu_-(\lambda_1))\|_\infty$. More precisely, let v_1 be the function defined on $|z| \le L(\alpha, \lambda_1)$, $L(\alpha,\lambda_1) := \alpha^{-(p-1-q)/(q-m)}\lambda_1^{(p-1-m)/(q-m)}$ by the identity*

$$v_1(z) = (\frac{\lambda_1}{\alpha})^{1/(q-m)} u(zL(\alpha,\lambda_1)^{-1}; \mu_-(\lambda_1)).$$

Then, for any y, $|y| \le 1 - l(\lambda)$, $l(\lambda) := (\lambda_1/\lambda)^{(p-1-m)/(q-m)(p-1)}$, the function

$$r(x;y) = \begin{cases} (\frac{\alpha}{\lambda})^{1/(q-m)} v_1((x-y)L(\alpha,\lambda_1), \textit{ for } |x-y| \le l(\lambda) \\ 0, \textit{ for } |x-y| > l(\lambda) \end{cases}$$

is a solution of $\mathcal{P}_{\lambda,\alpha}$. *In fact, if* N *is a positive integer and* $\lambda \geq \lambda_1 N^{(q-m)/(p-1-m)}$, *given a vector* $\mathbf{y} = (y_1, y_2, .., y_N)$ *with*

$$-1 \leq y_i - l(\lambda),\ y_i + l(\lambda) \leq y_{i+1} - l(\lambda),\ i = 1, .., N-1, y_N + l(\lambda) \leq 1$$

and if we define the set of solutions of $\mathcal{S}_N(\lambda)$ *as the one given by functions of the form*

$$r(x, \mathbf{y}) = \begin{cases} (\frac{\alpha}{\lambda})^{1/(q-m)} v_1((x-y_i)L(\alpha,\lambda_1), \text{ for } |x-y_i| \leq l(\lambda), \\ 0, \text{ for } |x-y_i| > l(\lambda), \end{cases}$$

then the set of nontrivial and nonnegative solutions of $\mathcal{P}(\lambda)$ *is formed by* $\mathcal{S}(\lambda)$ *jointly with* $q(\cdot, \lambda)$ *where* $\mathcal{S}(\lambda)$ *is the set defined by* $\mathcal{S}(\lambda) = \cup_{j=1}^N \mathcal{S}_j(\lambda)$.

The proof will be obtained as a consequence of the study of the bifurcation diagram for the auxiliary problem

$$\mathcal{P}(L) \begin{cases} -(|v'|^{p-2} v')' = v^q - v^m & \text{in } (-L, L), \\ v(\pm L) = 0 \end{cases}$$

in which we consider the equation on a general interval $(-L, L)$ and take L as variable parameter. We shall prove that

Theorem 2.2 *We define*

$$\gamma(\mu) := \frac{1}{[p/(p-1)]^{1/p}} \int_0^\mu \frac{dr}{(F(\mu) - F(r))^{1/p}}. \tag{2.2}$$

Then $\gamma'(\mu) = 0$ *has a unique root* $\mu_0 \in (r_F, \infty)$. *We introduce the numbers* $L_0 = \gamma(\mu_0)$ *and* $L_1 = \gamma(r_F)$. *For* $L \geq L_0$ *we denote by* $\mu_+(L)$ *to the largest solution of the nonlinear equation* $L = \gamma(\mu)$, *and for* $L_1 \geq L \geq L_0$ *let* $\mu_-(L)$ *be the smallest solution. Then we have the following cases: i) if* $L \in (0, L_0)$ *there is no positive solution, ii) if* $L = L_0$, *there is a unique positive solution* $v(\cdot, \mu_+(L_0))$, *iii) if* $L \in (L_0, L_1]$, *there are two positive solutions* $P(\cdot, L) = v(\cdot, \mu_-(L))$ *and* $Q(\cdot, L) = v(\cdot, \mu_+(L))$, *Moreover,* $P(\cdot, L) < Q(\cdot, L)$ *on* $(-L, L)$ *and* $v'(\pm 1, \mu_-(L_1)) = 0$, *iv) if* $L \geq L_1$ *there is one positive solution* $Q(\cdot, L) = v(\cdot, \mu_+(L))$, *v) for any* $L > L_1$ *there is a family of nonnegative solutions which is generated by* $v(\cdot, \mu_-(L_1))$. *In fact, for any* h, $|h| \leq L - L_1$, *the function*

$$s(x, h) = \begin{cases} v(x-h, \mu_-(L_1)) & \text{for } |x-h| \leq L_1, \\ 0 & \text{for } |x-h| > L_1, \end{cases}$$

is also a nonnegative solution. If N *is a positive integer and* $L \geq NL_1$, *given a vector* $\mathbf{y} = (y_1, y_2, .., y_N)$ *with* $-L \leq y_i - L_1$, $y_i + L_1 \leq y_{i+1} - L_1$, $i = 1.., N-1$, $y_N + L_1 \leq L$ *the function*

$$s(x, \mathbf{y}) = \begin{cases} v(x-y_i, \mu_-(L_1)) & \text{for } |x-y_i| \leq L_1, \\ 0 & \text{for } |x-y_i| > L_1, \text{ for } i = 1, .., N \end{cases}$$

is a nonnegative solution. We call $\mathcal{S}_N(L)$ *the set of such solutions* $r(x, \mathbf{y})$. *Finally, for* $L > L_1$ *let* N *be the integral part of* L/L_1 *and let* $\mathcal{S}(L) = \cup_{j=1}^N \mathcal{S}_j(L)$. *Then the set of nontrivial solutions of* $\mathcal{P}(L)$ *is formed by* $\mathcal{S}(L)$ *jointly with* $Q(\cdot, L)$.

2.1. Proof of Theorem 1 from Theorem 2.

Let $u_{\lambda,\alpha}$ be a solution of $\mathcal{P}_{\lambda,\alpha}$ Then the change of variables

$$u_{\lambda,\alpha}(x) = (\frac{\alpha}{\lambda})^{1/(q-m)} v(x\alpha^{-(p-1-q)/(q-m)}\lambda^{(p-1-m)/(q-m)}) \tag{2.3}$$

transforms $u_{\lambda,\alpha}$ into a solution v of the problem $\mathcal{P}(L)$ with $L := \alpha^{-\frac{p-1-q}{q-m}}\lambda^{\frac{p-1-m}{q-m}}$ and, conversely, any solution v of problem $\mathcal{P}(L)$ into a solution of $\mathcal{P}_{\lambda,\alpha}$. We define $\lambda_0 := [\gamma(\mu_0)\ \alpha^{(p-1-q)/(q-m)}]^{(q-m)/(p-1-m)}$. From Theorem 2 we get that the bifurcation equation $L = \gamma(\mu)$ for the solutions of $\mathcal{P}(L)$ leads to the equivalent bifurcation equation

$$\alpha^{-(p-1-q)/(q-m)}\lambda^{(p-1-m)/(q-m)} = \gamma(\|u_{\lambda,\alpha}\|_\infty (\frac{\lambda}{\alpha})^{1/(q-m)}) \tag{2.4}$$

for the solutions of $\mathcal{P}_{\lambda,\alpha}$. Since $\gamma(\mu_0)$ is the minimum value of γ we deduce that if $\lambda < \lambda_0$ equation (2.4) has no solution and for $\lambda = \lambda_0$ there is only one solution. This proves a) and b). Since the range of the branch γ_+ is $[\gamma(\mu_0), +\infty)$ we deduce that for any $\lambda > \lambda_0$ the equation (2.4) has, at least, a solution which implies the existence of a solution of $\mathcal{P}_{\lambda,\alpha}$, $q(\cdot,\lambda)$. If $\lambda \in (\ \lambda_0, \lambda_1]$, from the continuity of the branch γ_-, we deduce the existence of a second solution of the equation (2.4) which implies the existence of the solution $p(\cdot,\lambda)$. Both roots correspond to the two solutions $\mu_- < \mu_+$ of the equation $L = \gamma(\mu)$ and then

$$\mu_- = \|P(\cdot,\lambda)\|_\infty (\frac{\lambda}{\alpha})^{1/(q-m)}, \mu_+ = \|Q(\cdot,\lambda)\|_\infty (\frac{\lambda}{\alpha})^{1/(q-m)}$$

for a suitable λ which proves that $\|p(\cdot,\lambda)\|_\infty < \|q(\cdot,\lambda)\|_\infty$. Moreover, since $p(\cdot,\lambda) = (\frac{\alpha}{\lambda})^{1/(q-m)}P(x\alpha^{-(p-1-q)/(q-m)}\lambda^{(p-1-m)/(q-m)} : L)$ and
$q(\cdot,\lambda) = (\frac{\alpha}{\lambda})^{1/(q-m)}Q(x\alpha^{-(p-1-q)/(q-m)}\lambda^{(p-1-m)/(q-m)} : L)$, using iii) of Theorem 2, we get c). Part d) is proved in a similar way.

Remark 1 Some related results in the literature are the following: The case of $m > 1$ was studied in [15] in the larger class of possible changing sign solutions. Their results are of a completely different nature to our Theorem 1. A closer result can be found in Section 2 of [5] where the authors consider the same equation for $p = q = 2$ but looking for 2π-periodic solutions on R. We point out that several points of the description made in Theorem 1 remain true for the same type of problems in higher dimensions (see [12]). In this last direction it is interesting to mention the paper [8] where the authors consider the case $p = 2, 0 < m < 1 < q < (N+2)/(N-2)$ in R^N, $N \geq 3$. Nevertheless, no multiplicity study is made there. □

2.2. Proof of Theorem 2

Lemma 1. *A function v is a positive solution of problem $\mathcal{P}(L)$ if and only if*

$$\frac{1}{[p/(p-1)]^{1/p}}\int_{v(x)}^{\mu}\frac{dr}{(F(\mu)-F(r))^{1/p}}=|x|\,,\ \text{for}\ |x|\le L,$$

where $\mu\in[r_F,\infty)$ and $L>0$ are related by the equation $\gamma(\mu)=L$.
PROOF. If a positive solution exists then necessarily it will have a maximum $\mu>0$ in some point $\zeta\in(-L,L)$. So, let us consider

$$\mathcal{CP}\left\{\begin{array}{l}-(|v'|^{p-2}v')'=f(v)\\ v(\zeta)=\mu,\ v'(\zeta)=0.\end{array}\right.$$

If $\mu<r_f$ (the zero of f) no solution of $\mathcal{CP}$ may satisfy $\mathcal{P}(L)$. Multiplying by v', integrating by parts and using the initial conditions and that $v'\le 0$ near $x=\zeta$ we find

$$-v'(x)=A^{-1}(F(\mu)-F(v(x)) \tag{2.5}$$

where $A(r):=[(p-1)/p]r^p$. It is easy to see that if r_F is the (unique) positive number such that $F(r_F)=0$ then if $\mu\in(r_f,r_F)$ no solution of $\mathcal{CP}$ may satisfy $\mathcal{P}$. So, let $\mu\in[r_F,\infty)$.When $\mu=r_F$ the integral of the function γ may have a second singularity at $r=0$ which is integrable. For a *positive solution* v of problem $\mathcal{CP}$, $v=0$ only at $r=\pm L$. Therefore $\zeta=0$ and the proof holds. □

The next result shows some general qualitative behavior of the graph of $\gamma(\mu)$.

Proposition 2.3 *We have (i) $\gamma\in C[r_F,\infty)\cap C^1(r_F,\infty)$; (ii) $\gamma(\mu)\to+\infty$ and $\gamma'(\mu)\to+\infty$ as $\mu\to+\infty$. (iii) $\gamma'(\mu)\to-\infty$ as $\mu\downarrow r_F$.*

PROOF. It is useful to introduce the function

$$\Lambda(\mu)=\frac{p^{1/p}}{[p-1]^{1/p}}\gamma(\mu)\ =\mu\int_0^1\frac{d\tau}{(F(\mu)-F(\tau\mu))^{1/p}}.$$

Then

$$\Lambda'(\mu)=\frac{\Lambda(\mu)}{\mu}-\frac{\mu}{p}\int_0^1\frac{F'(\mu)-\tau F'(\tau\mu)d\tau}{(F(\mu)-F(\tau\mu))^{(p+1)/p}}. \tag{2.6}$$

For $\mu\in(r_F,\infty)$ we have that $F'(\mu)\neq 0$ and it is not difficult to verify that the integral in (2.6) is convergent and that $\Lambda'(\mu)\in C(r_F,\infty)$, $\Lambda\in C([r_F,\infty)\times[0,\infty))$. For the rest of the proof it is useful to introduce the auxiliary function $\theta(t):=pF(t)-tf(t)$. Then we get

$$\Lambda'(\mu)=\frac{1}{\mu p}\int_0^{\mu}\frac{(\theta(\mu)-\theta(r))dr}{(F(\mu)-F(r))^{(p+1)/p}} \tag{2.7}$$

The proof that $\Lambda'(\mu) \to -\infty$ as $\mu \downarrow r_F$ uses Fatou's lemma and that $\int_0^\delta \frac{dr}{(F(\mu)-F(r))^{(p+1)/p}}$ is not integrable. Property (iii) is proved in a similar way. □

In order to get a more precise information on the number of zeros of function $\gamma'(\mu)$ we need some different arguments.

Proposition 2.4 *$\gamma'(\mu) = 0$ has a unique root $\mu_0 \in (r_F, \infty)$.*

PROOF. We follow closely the proof given in the nondegenerate case [23]. The following properties hold: A) There is a $\mu_1 \in (r_F, \infty)$ such that $\theta(r) < 0$ on $(0, \mu_1)$ and $\theta(r) > 0$ on (μ_1, ∞). B) There is a $\mu_2 \in (0, \mu_1)$ such that $\theta'(r) < 0$ on $(0, \mu_2)$ and $\theta'(r) > 0$ on (μ_2, ∞), and C) There exists a $\mu_3 \in (0, \mu_2)$ such that $(r\theta(r))' < 0$ on $(0, \mu_3)$ and $(r\theta(r))' > 0$ on (μ_3, ∞). It follows from properties A and B that $\Lambda'(\mu) > 0$ on (μ_1, ∞) and, if $r_F < \mu_2, \Lambda'(\mu) < 0$ on $(r_F,\ \mu_2)$. It is clear that necesarily $\Lambda'(\mu)$ has at least one zero in the interval $J := [\max(r\ _F, \mu_2), \mu_1]$. In fact, there can be at most one by proving that

$$\Lambda''(\mu) + C\Lambda'(\mu) > 0 \tag{2.8}$$

on this interval J, for some $C > 0$ (notice that then $\Lambda''(\mu) > 0$ on any of such zero). The proof uses the formula

$$\Lambda''(\mu) = \frac{1}{\mu^2 p} \int_0^\mu \frac{\{(\delta_2\theta')(\delta_1 F) - (p+1/p)(\delta_1\theta)(\delta_2 f)\}}{(\delta_1 F)^{(2p+1)/p}} dr$$

where $(\delta_1 h)(r) = h(\mu) - h(r)$ and $(\delta_2 h)(r) = \mu h(\mu) - rh(r)$, $0 \leq r < \mu$. □

The crucial point in the rest of the proof of Theorem 2 is that $v'(\pm 1, \mu_-(L_1)) = 0$. This follows from (2.5). Similar ideas can be found in Proposition 3 of [4]. □

Remark 2 Theorem 2 holds for a larger class of functions f (see [12]). A very interesting situation occurs when $p > 2$ and, for instance of $f(v) = v(1-v)(v-a)$, for some $a < 1$. In that case it is possible to show the existence of nontrivial solutions taking its maximum on a positive measured subset (see Díaz and Kichenassamy [13]).

3. The case of a bounded N-dimensional domain

In this paragraph we deal with the general case of any smooth bounded domain in $\mathbb{R}^N$. Except in the case of a ball, the ODE methods of Section 2 do not work any more and should be replaced by variational or topological arguments.

In our exemple below we obtain immediately a supersolution, which provides a very simple a priori estimate as well, and then, by a rather intrincate argument we get a subsolution for λ large. Hence the existence of a maximal positive solution will follow from the usual argument ([1]) telling that between ordered sub and supersolutions of

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with f smooth enough, there is always at least a solution u, $u_0 \leq u \leq u^0$. Brezis and Nirenberg [6] have proved that there is always a solution of the associated functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(x,u) dx,$$

with $F(x,u) = \int_0^u f(x,s) ds$. A very nice application of this result was used in [3], together with the Montain Pass Lemma of Ambrosetti-Rabinowitz, in order to exhibit a second positive solution for the corresponding problem. Since it is known that local minima have index 1 (a result due to Rabinowitz [21] for C^2 functionals and to Amann [2] in the case C^1), it seems reasonable to guess that the corresponding result should be true for general second order operators with smooth coefficients. Such result was proved by Dancer, it is actually a corollary of a general abstract result in Banach spaces (namely Theorem 2 or Corollary 1 in [9]) whose assumptions seem not easy to check in concrete examples. A different proof for the case of equation (3.1), which is still valid for singular problems (see [16], [17] and its references) can be found in [18].

We consider now the problem

$$\begin{cases} -\Delta u = \lambda(u^2 - u^3) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We have the following

Theorem 3.1 *There exists a $\overline{\lambda} > 0$ such that for any $\lambda > \overline{\lambda}$ problem (3.2) has at least two positive solutions.*

Proof (Sketch). It is obvious that $u^0 \equiv 1$ is a supersolution to (3.2) and,moreover, an easy argument using the Maximum Principle shows that it provides an a priori estimate for positive solutions as well. Then a rather intrincated argument provides for $\lambda > 0$ large enough a positive subsolution $u_0(\lambda) < 1$. Then, by the above mentioned result there is a solution $u(\lambda)$ such that $u_0(\lambda) < u(\lambda) < 1$, with $i(u(\lambda)) = 1$ if $\lambda > \overline{\lambda}$, for some $\overline{\lambda} > 0$.
On the other side $u \equiv 0$ is a local minimum for (3.2), and, by a simple linearization computation around 0, $i(0) = 1$. But it is known ([1]) that $i([0, 1]) = 1$ and then the properties of the index ([1]) and a simple counting argument show that there exists a solution $0 < v(\lambda) < u(\lambda)$ such that $i(v(\lambda)) \neq 0$. □

Remark 3 This result was proved by Rabinowitz [20] by using a combination of variational methods and degree theory. See also the works of Clément and Sweers [7], Gardner and Peletier [14] and its references. Problem (3.2) is not actually a particular instance of the genral framework in the Introduction, but we prefer to illustrate *the method* in this case. □

Remark 4 The above proof still work if we replace $u^2 - u^3$ by $u^r - u^s$ with $1 < r < s$, and even allow $0 < r < s < 1$, even if now the linearization around zero cannot be

performed. On the other side, $-\Delta$ can be replaced by a second order uniformly elliptic linear differential operator in general form with smooth coefficients.
The first part of the proof could still work in the singular case (i.e., $-1 < r < s < 0$) but zero is not a solution and the second part is meaningless, at least in this version. The suitable tools are given in [16] and then applied in [17]. □

Remark 5 The method of sub and supersolutions also works for quasilinear problems with the p-Laplacian, but it remains the difficulty of getting a suitable subsolution (for an index result see Kichenassamy [19]). □

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., **18** (1976), 620–709.

[2] H. Amann, A note on degree theory for gradient mappings, Proc. Amer. Math. Soc., **85** (1982), 591–595.

[3] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal., **122** (1994), 519–543.

[4] D.G. Aronson, M.G. Crandall and L.A. Peletier, Stabilization of solutions of a degenerate nonlinear diffusion problem, Nonlinear Analysis, **6** (1982), 1001–1022.

[5] M.-F. Bidaut-Veron, V. Galaktionov, P. Grillot and L. Veron, Singularities for a 2-dimensional semilinear elliptic equation with a non Lipschitz nonlinearity, J. Differential Equations, **154**, 1999, no. 2, (1997), 318–338.

[6] H. Brezis and L. Nirenberg, H^1 versus C^1 minimizers, C. R. Acad. Sci. Paris, **317**, Série I, (1993), 467–472.

[7] P. Clément and G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, Ann. Sc. Norm. Sup. Pisa, **13**, (1987), 97–121.

[8] C. Cortázar, M. Elgueta and P. Felmer, On a semilinear elliptic problem in R^N with a non-Lipschitz nonlinearity, Advances in Diff. Eqns., **1**, (1996), 199–218.

[9] E.N. Dancer, Multiple fixed points of positive mapping, J. Reine Ang. Math., **171**, (1986), 46–66.

[10] J.I. Díaz, Nonlinear Partial Differential Equations and Free Boundaries. Pitman, London. 1985.

[11] J.I. Díaz and J. Hernández, Global bifurcation and continua of nonnegative solutions for a quasilinear elliptic problem, C. R. Acad. Sci. Paris, **329**, Série I, (1999), 587–592.

[12] J.I. Diaz and J. Hernández. In preparation.

[13] J.I. Diaz and S. Kichenassamy, Unpublished manuscript (1990).

[14] R. Gardner and L.A. Peletier, The set of positive solutions in large balls, Proc. Roy. Soc. Edinburgh, **104 A**, (1999), 587–592.

[15] M. Guedda and L. Veron, Bifurcation phenomena associated to the p-Laplace operator, Trans. Amer. Math. Soc., **310**, (1988), 419–431.

[16] J. Hernández, F. Mancebo and J.M. Vega de Prada, On the linearization of some singular nonlinear elliptic problems and applications, Ann. I.H.Poincaré. Anal.. non Lin., **19**, (2002), 777–813.

[17] J. Hernández, F. Mancebo and J.M. Vega de Prada, Positive solutions for singular nonlinear elliptic equations, Preprint (2003).

[18] J. Hernández, F. Mancebo F. and J.M. Vega de Prada, In Preparation.

[19] S. Kichenassamy, Local index of potential operators of monotone type, Houston J. Math., **16** (1990) no. 1, 139–149.

[20] P.H. Rabinowitz, Pairs of positive solutions of nonlinear elliptic partial differential equations, Indiana Univ. Math. J., **23** (1973), 172–185.

[21] P.H. Rabinowitz, A note on topological degree for potential operators, . Math. Anal. Appl., **51** (1975) 483–492.

[22] J. Smoller J. Shock-waves and Reaction-Diffusion Equations, Springer, Berlin, 1983.

[23] J. Smoller and J. Wasserman J., 1981, Global bifurcation of steady-state solutions, J. Diff. Eqns., **39**, (1981), 269–290.