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ON A FREE BOUNDARY PROBLEM ARISING IN CLIMATOLOGY

J. I. DÍAZ

Abstract

We present several results on the mathematical treatment of some diffusive energy balance models arising in Climatology. The model consists of a quasilinear parabolic equation with a discontinuous function of the unknown (the co-albedo) at the right hand side. The spatial domain is a two-dimensional compact connected Riemannian manifold representing the surface of the Earth. The free boundary corresponds to the level line $u = -10$ for which there is a discontinuity in the co-albedo function.

1 Introduction.

In this lecture we consider a diffusive energy balance model arising in Climatology given by a nonlinear parabolic problem formulated in the following terms

$$(P)\begin{cases} u_t - div(|\nabla u|^{p-2}\nabla u) \in QS(x)\beta(u) - \mathcal{G}(u) + f(x,t) & \text{in } (0,T)\times\mathcal{M}, \\ u(x,0) = u_0(x) & \text{in } \mathcal{M}, \end{cases}$$

where $\mathcal{M}$ is a C^∞ two-dimensional compact connected oriented Riemannian manifold without boundary and in consequence the differential operators must be understood in the usual sense associated to the Riemannian metric of $\mathcal{M}$. So, for instance, if $p = 2$ and $\mathcal{M}$ is the unit sphere the diffusion operator becomes the Laplace-Beltrami operator. We assume $T > 0$ arbitrarily fixed, $Q > 0$, $S \in L^\infty(\mathcal{M})$ and $p \geq 2$. The function $\mathcal{G}$ is increasing and β represents a bounded maximal monotone graph in $I\!R^2$ (of Heaviside

type). Through the paper we shall use the notation $div(|\nabla u|^{p-2}\nabla u) = \Delta_p u$. This kind of models were introduced, independently, in 1969 by M.I. Budyko [7] and W.D. Sellers [50]. The models have a diagnostic character and intended to understand the evolution of the global climate on a long time scale. Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters. This kind of models has been used in the study of the Milankovitch theory of the ice-ages (see, e.g. North, Mengel and Short [47]). The distribution of temperature $u(x,t)$ is expressed pointwise after a standard average process, where the spatial variable x is in the Earth's surface. The time scale is considered relatively long.

The model is obtained from the thermodynamics equation of the *atmosphere primitive equations* via averaging process (see, e. g. Lions, Temam and Wang [42] for a mathematical study of those equations, Kiehl [40] for the application of averaging processes and Remark 1 for some nonlocal variants of (P)). More simply, the model can be formulated by using the energy balance on the Earth's surface: internal energy flux variation $= R_a - R_e + D$, where R_a and R_e represent the absorbed solar and the emitted terrestrial energy flux, respectively and D is the horizontal heat diffusion.

The *absorbed energy* R_a depends on the planetary *coalbedo* β. The coalbedo function represents the fraction of the incoming radiation flux which is absorbed by the surface. In the energy balance climate models, a main change of the coalbedo occurs in a neighborhood of a critical temperature for which ice become white, usually taken as $u = -10^0C$. The different coalbedo is modelled as a discontinuous function of the temperature in the *Budyko model* and here it will be treated as a maximal monotone graph in $I\!R^2$, $\beta(u) = m$ if $u < -10$, $[m, M]$ if $u = -10$ and M if $u > -10$, where $m = \beta_i$ and $M = \beta_w$ represent the coalbedo in the ice-covered zone and the free-ice zone, respectively and $0 < \beta_i < \beta_w < 1$ (the value of these constants has been estimated by observation from satellites). In the *Sellers model*, β is assumed to be a more regular function. In both models, the absorbed energy is given by $R_a = QS(x)\beta(u)$ where $S(x)$ is the *insolation function* and Q is the so-called *solar constant*.

The Earth's surface and atmosphere, warmed by the Sun, reemit part of the absorbed solar flux as an infrared long-wave radiation. This energy R_e is represented, in the Budyko model, according to the Newton cooling law, that is, $R_e = Bu + C$. Here, B and C are positive parameters, which are obtained by observation, and can depend on the *greenhouse effect*. However, in the Sellers model, R_e is expressed according to the Stefan - Boltzman law $R_e = \sigma u^4$, where σ is called *emissivity constant* and now u is in Kelvin degrees.

The *heat diffusion* D is given by the divergence of the conduction heat flux F_c and the advection heat flux F_a. Fourier's law expresses $F_c = k_c\nabla u$ where k_c is the *conduction coefficient*. The advection heat flux is given by $F_a = \mathrm{v}\cdot\nabla u$ and it is known (see e.g. Ghil and Childress [24]) that, to the level of the planetary scale, it can be modeled in terms of $k_a \nabla u$ for a suitable diffusion coefficient k_a. So, $D = div(k\nabla u)$ with $k = k_c + k_a$. In the pioneering models, the diffusion coefficient k was considered as a positive constant. Nevertheless, in 1972, P.H. Stone [53] proposed a coefficient $k = |\nabla u|$, in order to consider negative feedback in the eddy fluxes. So, in that case the heat diffusion is represented by the quasilinear operator $D = div(|\nabla u|\nabla u)$. Our formulation (P) take into account such a case which corresponds to the special choice $p = 3$ (notice that the case $p = 2$ leads to the linear diffusion). These physical laws lead to problem (P) with $R_e(u) = \mathcal{G}(u) - f$. We mention that the two-dimensional model (P) can be reduced, under special conditions, to the one-dimensional problem

$$(P^1)\begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x \in QS(x)\beta(u) - R_e(u) & \text{in } (-1,1)\times(0,T),\\ u(x,0) = u_0(x) & \text{in } (-1,1),\end{cases}$$

with $\rho(x) = (1-x^2)^{\frac{p}{2}}$ where $x = \sin\theta$ and θ is the latitude. Notice that again there is no boundary condition since the meridional heat flux $(1-x^2)^{\frac{p}{2}}|u_x|^{p-2}u_x$ vanishes at the poles $x = \pm 1$.

In Section 2 we start by presenting some results on the existence and uniqueness of solutions. We also include in this section some comments on the *free boundaries* associate to the Budyko type model (the curves separating the regions $\{x : u(x,t) < -10\}$ and $\{x : u(x,t) > -10\}$). We end the section considering the stabilization of solutions as $t \to \infty$ to solutions of the associate stationary problem

$$(P_{Q,f}) \qquad -div(|\nabla u|^{p-2}\nabla u) + \mathcal{G}(u) \in QS(x)\beta(u) + f_\infty(x) \text{ on } \mathcal{M}.$$

Section 3 is devoted to the study of the number of stationary solutions according to the parameter Q, when β is not necessarily Lipschitz continuous and $p \geq 2$. We also study the bifurcation diagram of solutions of

$$(P_{Q,C}) \qquad -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \mathcal{G}(u) + C \in QS(x)\beta(u) \text{ on } \mathcal{M}.$$

2 The transient model

We introduce the following structure hypotheses: $p \geq 2$, $Q > 0$,

($H_{\mathcal{M}}$) $\mathcal{M}$ is a C^∞ two-dimensional compact connected oriented Riemannian manifold of $I\!R^3$ without boundary,

(H_β) β is a bounded maximal monotone graph in $I\!R^2$, i.e. $m \leq z \leq M$, $\forall z \in \beta(s)$.

($H_{\mathcal{G}}$) $\mathcal{G} : I\!R \to I\!R$ is a continuous strictly increasing function such that $\mathcal{G}(0) = 0$, and $|\mathcal{G}(\sigma)| \geq C|\sigma|^r$ for some $r \geq 1$,

(H_s) $S : \mathcal{M} \to I\!R$, $S \in L^\infty(\mathcal{M})$, $S_1 \geq S(x) \geq S_0 > 0$ a.e.$x \in \mathcal{M}$,

(H_f) $f \in L^\infty(\mathcal{M}\times(0,T))$, (resp. ($H_f^\infty$) $f \in L^\infty(\mathcal{M}\times(0,\infty))$),

(H_0) $u_0 \in L^\infty(\mathcal{M})$.

The possible discontinuity in the coalbedo function causes that (P) does not have classical solutions in general, even if the data u_0 and f are smooth. Therefore, we must introduce the notion of weak solution. The natural "energy space" associate to (P) is the one given by $V := \{u : \mathcal{M} \to I\!R,\ u \in L^2(\mathcal{M}).\ \nabla_{\mathcal{M}} u \in L^p(T\mathcal{M})\}$, which is a reflexive Banach space if $1 < p < \infty$. Here $T\mathcal{M}$ denotes the tangent bundle and, as mentioned before, any differential operator must be understood in terms of the *Riemannian metric* g given on $\mathcal{M}$ (see, e.g. Aubin [3] and Díaz and Tello [15]).

Definition 1 *We say that* $u : \mathcal{M} \to I\!R$ *is a* bounded weak solution of (P) *if i)* $u \in C([0,T]; L^2(\mathcal{M})) \cap L^p(0,T;V) \cap L^\infty(\mathcal{M}\times(0,T))$ *and ii) there exists* $z \in L^\infty(\mathcal{M}\times(0,T))$ *with* $z(x,t) \in \beta(u(x,t))$ *a.e.* $(x,t) \in \mathcal{M}\times(0,T)$ *such that*

$$\int_{\mathcal{M}} u(x,T)v(x,T)dA - \int_0^T < v_t(x,t), u(x,t) >_{V'\times V} dt +$$
$$+ \int_0^T\int_{\mathcal{M}} < |\nabla u|^{p-2}\nabla u,\ \nabla v > dAdt + \int_0^T\int_{\mathcal{M}} \mathcal{G}(u)vdAdt =$$
$$= \int_0^T\int_{\mathcal{M}} QS(x)z(x,t)vdAdt + \int_0^T\int_{\mathcal{M}} fvdAdt + \int_{\mathcal{M}} u_0(x)v(x,0)dA$$
$$\forall v \in L^p(0,T;V)\cap L^\infty(\mathcal{M}\times(0,T)) \text{ such that } v_t \in L^{p'}(0,T;V').$$

Theorem 1 *There exists at least a bounded weak solution of* (P). *Moreover, if* $T = +\infty$ *and* f *verifies* (H_f^∞), *the solution* u *of* (P) *can be extended to* $[0,\infty)\times\mathcal{M}$ *in such a way that* $u \in C([0,\infty), L^2(\mathcal{M})) \cap L^\infty(\mathcal{M}\times(0,\infty)) \cap L^p_{loc}(0,\infty;V)$. □

The above result can be proved in different ways. As in the case of the one-dimensional model (Díaz [9]) we can apply the techniques of Díaz and Vrabie [19] based on fixed point arguments which are useful for multivalued nonmonotone equations. Nevertheless, for different purposes it is useful to get existence results via regularization of the multivalued term $\beta(u)$. See, e.g., Xu [56] and Feireisl and Norbury [22] for some special formulations when $p = 2$. In our case it can be obtained as an easy adaptation of the results of Section 3. We also mention some results on the numerical approach due to Lin and North [41], Hetzer, Jarausch and Mackens [32], Bermejo [5] and Díaz, Bermejo and Tello [6].

The question of uniqueness has different answers for the different coalbedo functions under consideration depending on whether the coalbedo is supposed to be discontinuous or not. For the Sellers model (β locally Lipschitz), the uniqueness is obtained by standard methods (see e.g. Díaz [9]). Nevertheless, in the Budyko model (β multivalued), there are cases of nonuniqueness (in spite of the parabolic nature of (P)). The first nonuniqueness result in this context seems to be the one given in Díaz [9] where infinitely many solutions are found for the one-dimensional model (P^1) for any initial condition u_0 satisfying

$$\left.\begin{array}{l} u_0 \in C^\infty(I),\ u_0(x) = u_0(-x)\ \forall x \in [0,1], \\ u_0(0) = -10,\ u_0^{(k)}(0) = 0,\ k = 1,2 \\ u_0'(1) = 0,\ u_0'(x) < 0,\ x \in (0,1). \end{array}\right\} \quad (1)$$

Notice that these initial data u_0 are very "flat" at the level -10. A similar nonuniqueness result for the Budyko model with a suitable initial datum carries over to the two-dimensional model when $\mathcal{M} = S^2$. Each solution $u_1(x,t)$ of (P^1) generates a solution $u_2(x,y,t)$ of 2D model by rotation about the axis through the poles (notice that the initial datum $u_2(x,y,0)$ is independent of the longitude), i.e. $u_2(x,y,t) = u_1(\sin\theta, t)$ where $(x,y) \in S^2$ with latitude θ. It is not difficult to prove that u_2 is a solution of (P) for the initial datum $u_1(\text{sen}\theta, 0)$. Other nonuniqueness results can be found by using selfsimilar special solutions as in Gianni and Hulshof [26].

In order to obtain a criterion for the uniqueness of solutions for Budyko type models we introduce the notion of *nondegeneracy property* for functions defined on $\mathcal{M}$.

Definition 2 *Let $w \in L^\infty(\mathcal{M})$. We say that w satisfies the strong nondegeneracy property (resp. weak) if there exist $C > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$, $|\{x \in \mathcal{M} : |w(x) + 10| \le \epsilon\}| \le C\epsilon$ (resp. $|\{x \in \mathcal{M} : 0 < |w(x) + 10| \le \epsilon\}| \le C\epsilon$), where $|E|$ denotes the Lebesgue measure on the manifold $\mathcal{M}$ for all $E \subset \mathcal{M}$.*

Theorem 2 *i) Assume that there exists a solution u of (P) such that $u(.,t)$ verifies the strong nondegeneracy property for any $t \in [0,T]$. Then u is the unique bounded weak solution of (P). ii)There exists at most one solution of (P) verifying the weak nondegeneracy property.*

The proof is a modification of the results of Díaz [9], Díaz and Tello [15] where a slightly different of nondegeneracy property was used. We shall exploit the fact (adapted from Feireisl and Norbury [22]) that β generates a continuous operator from $L^\infty(\mathcal{M})$ to $L^q(\mathcal{M})$ $\forall q \in [1,\infty)$, although β is discontinuous. More precisely, we have (see Díaz [9], Díaz and Tello [15])

Lemma 1 *(i) Let $w, \hat{w} \in L^\infty(\mathcal{M})$ and assume that w satisfies the strong nondegeneracy property. Then for any $q \in [1,\infty)$ there exists $\tilde{C} > 0$ such that for any $z, \hat{z} \in L^\infty(\mathcal{M})$ with $z(x) \in \beta(w(x))$ and $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in \mathcal{M}$, we have that*

$$\| z - \hat{z} \|_{L^q(\mathcal{M})} \le (b_w - b_i)\min\{\tilde{C} \| w - \hat{w} \|^{1/q}_{L^\infty(\mathcal{M})}\ ,\ |\mathcal{M}|^{1/q}\}. \quad (2)$$

(ii) If $w, \hat{w} \in L^\infty(\mathcal{M})$ and satisfy the weak nondegeneracy property then

$$\int_{\mathcal{M}} (z(x) - \hat{z}(x))(w(x) - \hat{w}(x))dA \le (b_w - b_i)C \| w - \hat{w} \|^2_{L^\infty(\mathcal{M})}. \quad (3)$$

Idea of the proof of Theorem 2. Assume that there exist two bounded weak solutions u and $\hat{u}$ of (P), where u verifies the strong nondegeneracy property. Taking $(u - \hat{u})$ as the test function we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}} |u(t) - \hat{u}(t)|^2 dA + \int_{\mathcal{M}} (\mathcal{G}(u) - \mathcal{G}(\hat{u}))(u - \hat{u})dA +$$

$$\int_{\mathcal{M}} < |\nabla u(t)|^{p-2}\nabla u(t) - |\nabla \hat{u}(t)|^{p-2}\nabla \hat{u}(t),\ \nabla u(t) - \nabla \hat{u}(t) > dA =$$

$$= Q\int_{\mathcal{M}} S(x)(z(x,t) - \hat{z}(x,t))(u(x,t) - \hat{u}(x,t))dA. \quad (4)$$

for some $z \in \beta(u)$ and $\hat{z} \in \beta(\hat{u})$. By using the embedding $V \hookrightarrow L^\infty(\mathcal{M})$ if $p > 2$ and $V \subset L^\sigma(\mathcal{M})$ for all $\sigma \in [1,\infty)$ if $p = 2$ (recall that $\mathcal{M}$ is a two-dimensional compact Riemannian manifold: see, e.g. Aubin [3]) we arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \| u - \hat{u} \|^2_{L^2(\mathcal{M})} \le & \left(C_l Q \| S \|_{L^\infty(\mathcal{M})} - \frac{C_0 \| u - \hat{u} \|^{p-2}_{L^\infty(\mathcal{M})}}{C_{1,p,\infty}}\right) \| u - \hat{u} \|^2_{L^\infty(\mathcal{M})} + \\ & + \tilde{C}_0 \| u - \hat{u} \|^2_{L^2(\mathcal{M})}, \end{aligned} \quad (5)$$

in the case $p > 2$ and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \| u - \hat{u} \|^2_{L^2(\mathcal{M})} \le & \left(C_l Q \| S \|_{L^\infty(\mathcal{M})} - \frac{|\mathcal{M}|^{\frac{2}{\sigma}}}{C_{1,2,\sigma}}\right) \| u - \hat{u} \|^2_{L^\infty(\mathcal{M})} + \\ & + \| u - \hat{u} \|^2_{L^2(\mathcal{M})} + \frac{\epsilon}{C_{1,2,\sigma}}. \end{aligned} \quad (6)$$

for the case $p = 2$ where ϵ and $\sigma = \sigma(\epsilon)$. Now, we distinguish two cases:

CASE 1: if $C_l Q \| S \|_\infty - \dfrac{C_0 \| u - \hat{u} \|^{p-2}_{L^\infty(\mathcal{M})}}{C_{1,p,\infty}} \le 0$ and $p > 2$ the result holds by Gronwall's Lemma (the case $p = 2$ is similar).

CASE 2: if $C_l Q \| S \|_\infty - \frac{\|u-\hat{u}\|^{p-2}_{L^\infty(\mathcal{M})}}{C_{1,p,\infty}} > 0$, we consider a suitable rescaling $(\mathcal{M} \mapsto \mathcal{M}_\delta)$ given by the dilatation D of magnitude $\delta > 0$ on the manifold $(\mathcal{M}, \text{g})$, $D : \mathcal{M} \subset I\!R^3 \to I\!R^3$, $D(x) = \bar{x} = \delta x$. So problem (P) in the new coordinates becomes

$$(P_\delta)\left\{\begin{array}{ll} \bar{u}_t - \delta^p \text{div}_{\mathcal{M}_\delta}(|\nabla_{\mathcal{M}_\delta}\bar{u}|^{p-2}\nabla_{\mathcal{M}_\delta}\bar{u}) + \mathcal{G}(\bar{u}) \in QS\beta(\bar{u}) + f & \text{in } (0,T)\times\mathcal{M}_\delta \\ \bar{u}(0,\bar{x}) = u_0(\frac{\bar{x}}{\delta}) & \text{on } \mathcal{M}_\delta. \end{array}\right.$$

Clearly, if $\bar{u}$ is a solution of (P_δ) then $\bar{u}(\delta x, t)$ is a solution of (P). Moreover, the uniqueness of (P_δ) implies the uniqueness of (P), and conversely. A careful study of the dependence on δ of the involved constants (see Díaz and Tello [15]) allows to see that if we define the constant $K_{p,\delta}=C_{l,\delta}Q \parallel S_\delta \parallel_{L^\infty(\mathcal{M}_\delta)} - \frac{\delta^2|\mathcal{M}|^{\frac{2}{\sigma}}\|u_\delta-\hat{u}_\delta\|^{p-2}_{L^\infty(\mathcal{M}_\delta)}}{C_{1,2,\sigma,\delta}}$ we have that $\lim_{\delta\to 0} K_{p,\delta} = 0$. This fact allows us to reduce the proof to Case 1 and the proof of (i) follows. For the proof of (ii) we use the second part of Lemma 1 and so

$$\frac{1}{2}\frac{d}{dt} \parallel u-\hat{u} \parallel_2^2 \leq (C_d Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \frac{C_0 \parallel u-\hat{u} \parallel_\infty^{p-2}}{C_{1,p,q}}) \parallel u-\hat{u} \parallel_\infty^2 + \tilde{C}_0 \parallel u-\hat{u} \parallel_2^2$$

where C_d is the constant of the weak nondegeneracy property (Lemma 1). The uniqueness follows as in (i), by studying the sign of the constant $C_d Q \parallel S \parallel_{L^\infty(\mathcal{M})} - \frac{\parallel u-\hat{u} \parallel_\infty^{p-2}}{C_{1,p,q}}$ and by rescaling when it is negative.

Remark 1. It is possible to give several sufficient criteria for the nondegeneracy property. For instance, in the one-dimensional case, if $u_0 \in C^1((-1,1))$ is such that there exists $\epsilon_0 > 0$ satisfying that the set $\{x \in (-1,1) : |u_0(x)+10| \leq \epsilon_0\}$ has a finite number of connected components I_j with $j = 1,.., N$ and for any j there exists $x_j \in I_j$ such that $u_0(x_j) = -10$, and $|u_{0x}(x)| \geq \delta_0$ for some $\delta_0 > 0$ and any $x \in I_j$ close to x_j then there exists a solution $u(x,t)$ satisfying the strong nondegeneracy property on $(0,T^*)$ for some T^* (see Díaz and Tello [15]). Some results on solutions with $|\nabla u| \neq 0$ on the level where β becomes multivalued for a similar bidimensional problem are given in Gianni [25] (see also the recent results by Ham and Ko [28] for a related problem with $\beta \equiv 0$). Results on the continuous dependence with respect to the initial datum under nondegeneracy assumptions were obtained in Gianni [25] for the case in which $\mathcal{M}$ is an open regular set. Although the general case is technically more complex the same approach could be applied to this purpose.

The discontinuity of the albedo function assumed in the Budyko model (β multivalued) generates a natural *free boundary* or interface $\zeta(t)$ between the ice-covered area $(\{x \in \mathcal{M} : u(x,t) < -10\})$ and the ice-free area $(\{x \in \mathcal{M} : u(x,t) > -10\})$. The free boundary is then given as $\zeta(t) = \{x \in \mathcal{M} : u(x,t) = -10\}$. In Xu [56] the Budyko model for $p = 2$ is considered in the one-dimensional case. He shows that if the initial datum u_0 satisfies

$$\begin{aligned}&u_0(x) = u_0(-x),\ u_0 \in C^3([-1,1]),\ u_0'(x) < 0 \text{ for any } x \in (0,1)\\ &\text{and there exists } \zeta(0) \in (0,1) \text{ such that } (u_0(x)+10)(x-\zeta(0)) < 0\\ &\text{for any } x \in [0,\zeta(0)) \cup (\zeta(0),1],\end{aligned}$$

then there exists a bounded weak solution u of (P) for which the set $\zeta(t) = \{\zeta_+(t)\} \cup \{\zeta_-(t)\}$ with $x = \zeta_+(t)$ a smooth curve, $\zeta_-(t) = \zeta_+(t)$ and $\zeta_+(\cdot) \in C^\infty([0,T^*))$ where T^*

is characterized as the first time t for which $\zeta_+(t) = 1$. He also gives an expression for the derivative $\zeta_+'(t)$ (some related results for a model corresponding to $\rho(x) = 1$ can be found in Feireisl and Norbury [22], Gianni and Hulshof [26] and Stakgold [52]). We point out that the uniqueness result can be applied for such an initial datum. For the study of the free boundary in the bidimensional case Gianni [25].

The interpretation of the size of the separating zone $\zeta(t)$ for other models is in fact a controversial question. So, some satellite pictures (Image of the Weddell sea taken by the satellite Spot on December 10, 1987) show that the separating region between the ice-free and the ice-covered zones is not a simple line on the Earth but a narrow zone where ice and water are mixed. Mathematically it could correspond to say that the set $M(t) = \{x \in \mathcal{M}\colon u(x,t) = -10\}$ is a positively measured set. In the following we shall denote this set as the *mushy region* (since it plays the same role than in changing phase problems, see e.g. Díaz, Fasano and Meirmanov [12]).

Using the strong maximum principle it is possible to show that if $p = 2$ the interior set of the mushy region $M(t)$ is empty even if the interior of $M(0)$ is a nonempty open set (see Gianni and Hulshof [26]). As we shall see this is not the case when $p > 2$ (recall that $p = 3$ in Stone [53]). A necessary condition for the Budyko model (with $R_e = Bu + C$) for $\overset{\circ}{M}(t) \neq \emptyset$ is that

$$C - 10B \in [\beta_i QS(x), \beta_w QS(x)] \text{ for a.e. } x \in \mathcal{M}. \tag{7}$$

It is possible to show that if $p > 2$ this condition is also sufficient. Here we merely present a result for the one-dimensional case.

Theorem 3 *Let $p > 2$. Assume (7) and $u_0 \in L^\infty(I)$ such that there exist $x_0 \in I$ and $R_0 > 0$ satisfying $M(0) = \{x \in I : u_0(x) = -10\} \supset B(x_0, R_0)(= \{x \in I : |x - x_0| < R_0\})$. If u is the bounded weak solution of (P) satisfying the weak nondegeneracy property then there exists $T^* \in (0,T]$ and a nonincreasing function $R(t)$ with $R(0) = R_0$ such that $M(t) = \{x \in I : u(x,t) = -10\} \supset B(x_0, R(t))$ for any $t \in [0,T^*)$.*

Proof. We shall use an energy method as developed in Díaz and Veron [18]. Given u bounded weak solution of (P) we define $v = u + 10$. As in Lemma 3.1 of the above reference, by multiplying the equation by v we obtain that for a.e. $R \in (0, R_0)$ and $t \in (0,T)$ we have

$$\frac{1}{2}\int_{B(x_0,R)} |v(x,t)|^2 dx + \int_0^t \int_{B(x_0,R)} \rho(x)|v_x|^p dx d\tau + B\int_0^t \int_{B(x_0,R)} |v(x,\tau)|^2 dx d\tau \leq$$

$$\leq \int_0^t \int_{S(x_0,R)} \rho|v_x|^{p-2} v_x \cdot \vec{n} v ds d\tau + \int_0^t \int_{B(x_0,R)} \{QSz - C + 10B\} v dx d\tau = I_1 + I_2$$

where $\rho(x) = (1-x^2)^{\frac{p}{2}}$, $S(x_0, R) = \partial B(x_0, R) = \{x_0 - R\} \cup \{x_0 + R\}$ and $z(x,t) \in \beta(u(x,t))$ for a.e. $x \in B(x_0, R)$ and $t \in (0, T]$. We introduce the energy functions

$$
\begin{aligned}
E(R,t) &= \int_0^t \int_{B(x_0,R)} \rho(x) |v_x|^p \, dx d\tau \\
b(R,t) &= \sup_{0 \le \tau \le t} \operatorname{ess} \int_{B(x_0,R)} |v(x,\tau)|^2 dx.
\end{aligned}
$$

Using Holder's inequality and the interpolation-trace Lemma of Díaz-Veron [18] we get

$$
I_1 \le \left(\frac{\partial E}{\partial R}(R,t) \right)^{(p-1)/p} \left(\int_0^t \int_{S(x_0,R)} |v|^p dx d\tau \right)^{1/p} \le
$$

$$
\le C t^{(1-\theta)/p} \left(\frac{\partial E}{\partial R}(R,t) \right)^{(p-1)/p} \left(E(R,t)^{1/p} + R^{\delta} t^{1/p} b(R,t)^{1/2} \right)^{\theta} b(R,t)^{(1-\theta)/2},
$$

where $\theta = p/(3p-2)$ and $\delta = -(3p-2)/2p$. Using the assumption (7) we have that $\hat{z}(\cdot) = [(C - 10B)/QS(\cdot)] \in \beta(-10)$. Then applying Lemma 3 we get that

$$
I_2 \le (M - m) Q \parallel S \parallel_{L^\infty(I)} C' \int_0^t \parallel v(\tau) \parallel^p_{L^\infty(B(x_0,R))} d\tau.
$$

¿From Theorem 4 of Rakotoson and Simon [49] we have the estimate

$$
\parallel v \parallel_{L^\infty(J)} \le C_1 \parallel v_x \parallel_{L^p(J:\rho)} + C_2 \parallel v \parallel_{L^1(J:\rho)}, \quad \forall v \in V \tag{8}
$$

for some positive constants independent on the interval J. Then we obtain $I_2 \le (M - m) Q \parallel S \parallel_{L^\infty(I)} C'(C_1 E(R,t) + t C_3(R) b(R,t))$, where now

$$
C_3(R) = \frac{\left(\int_{B(x_0,R)} \rho(x)^2 dx \right)^{p-2}}{C_1^p \left(\int_{B(x_0,R)} \rho(x) dx \right)^p} \parallel u + 10 \parallel^p_{L^\infty((0,T):L^2(I))}.
$$

As in the proof of the uniqueness, we can assume C_1 small enough without loss of generality. Then, there exists $T^* \in (0, T]$ and $\lambda \in (0, 1]$ such that $\lambda(E(R,t) + b(R,t)) \le I_1$ which implies that $\lambda E^\mu \le t^{(1-\theta)/p} \frac{\partial E}{\partial R}$ for some $\mu \in (0,1)$ and for any $t \in [0, T^*)$ and the proof ends as in Díaz and Veron [18] (see also and Antonsev and Díaz [1]). ▪

Remark 2. The above proof can be adapted to the two-domensional problem by using local analysis. The existence of the mushy region (for any value of $p \in (1, \infty)$) can be proved for a different class of models by taking into account a discontinuous diffusivity (see Held, Linder and Suárez [30]). It would be interesting to find sufficient conditions implying the persistence of a mushy region for any time $t > 0$. The fact that a mushy region may exist for the stationary problem can be found from the results of Díaz [8] (see Theorem 1.14).

In order to analyze the stabilization of the solutions of (P), following Díaz, Hernández and Tello [13] we assume the additional condition

(H_∞) $f \in L^\infty((0,\infty) \times \mathcal{M})$ and there exists $f_\infty \in V'$ such that

$$
\int_{t-1}^{t+1} \| f(\tau, \cdot) - f_\infty(\cdot) \|_{V'} d\tau \to 0 \quad \text{as } t \to \infty.
$$

We start by recalling a global regularity of the solutions on $(0, \infty)$

Lemma 2 *Assume $u_0 \in V \cap L^\infty(\mathcal{M})$, $f \in L^\infty(\mathcal{M} \times (0,\infty)) \cap W^{1,1}_{loc}((0,\infty); L^1(\mathcal{M}))$ and $\int_t^{t+1} \| f_t(s, \cdot) \|_{L^1(\mathcal{M})} ds \le C_0$, $\forall t > 0$ where C_0 is a time independent constant. Then there exists a weak solution of (P) verifying*

$$
u \in L^\infty(0,\infty; V) \quad \text{and} \quad u_t \in L^2(0,\infty; L^2(\mathcal{M})). \tag{9}
$$

A key point in the proof is to check that $\varphi(t) = \frac{1}{p} \int_{\mathcal{M}} |\nabla u(x,t)|^p dA$ satisfies that $\varphi(t+1) \le C[\varphi(t) - \varphi(t+1)] + \theta(t)$ $\quad t > 0$, where C is a positive constant and $\theta(t) > 0$ when t is large enough with $\theta(t) = O(1)$ when $t \to \infty$. Then, thanks to a technical lemma due to Nakao [44] we conclude that $\varphi(t) = O(1)$ which is equivalent to $u \in L^\infty(0,\infty; V)$.

The following theorem proves the stabilization of the solutions u satisfying (9). As usual, given u bounded weak solution of (P), we define the ω-limit set of u by $\omega(u) = \{u_\infty \in V \cap L^\infty(\mathcal{M}) : \exists t_n \to +\infty \text{ such that } u(t_n, \cdot) \to u_\infty \text{ in } L^2(\mathcal{M})\}$.

Theorem 4 *Let $u_0 \in L^\infty(\mathcal{M}) \cap V$ and let u be any bounded weak solution satisfying (9). Then, i) $\omega(u) \ne \emptyset$ and if $u_\infty \in \omega(u)$ then $\exists t_n \to +\infty$ such that $u(\cdot, t_n + s) \to u_\infty$ in $L^2(-1,1; L^2(\mathcal{M}))$ and $u_\infty \in V$ is a weak solution of the stationary problem associate to f_∞ ; ii) in fact, if $u_\infty \in \omega(u)$ then $\exists \{\hat{t}_n\} \to +\infty$ such that $u(\cdot, \hat{t}_n) \to u_\infty$ strongly in V.*

Remark 3. If u_∞ is an isolated point of $\omega(u)$ it is easy to see that in fact the above convergences hold as $t \to \infty$ (and not merely for a sequence $t_n \to \infty$). The proof of this convergence is an open problem in the remaining cases. In fact, in some cases the set of stationary points is a continuum (see Remark 11) and the convergence when $t \to \infty$ is far from trivial (for some results in this direction see Feireisl and Simondon [23]).

Remark 4. A result on the convergence (in a suitable sense) of the free boundaries to the free boundary of the solution of the stationary problem is given in Gianni [25].

3 On the stationary problem

We consider the problem $(P_{Q,f})$ obtained in the last subsection. Following Díaz, Hernández and Tello [13] we made in this section the additional assumptions

($H_{\mathcal{G}}^{*}$) $\mathcal{G}$ satisfies ($H_{\mathcal{G}}$) and $\lim_{|s|\to\infty}|\mathcal{G}(s)| = +\infty$.

(H_{f_∞}) $f_\infty \in L^\infty(\mathcal{M})$ and there exist $C_f > 0$ such that $-\|f_\infty\|_{L^\infty(\mathcal{M})} \leq f(x) \leq -C_f$ a.e. $x \in \mathcal{M}$

(H_β^*) there exists two real numbers $0 < m < M$ and $\epsilon > 0$ such that $\beta(r) = \{m\}$ for any $r \in (-\infty, -10-\epsilon)$ and $\beta(r) = \{M\}$ for any $r \in (-10+\epsilon, +\infty)$.

(H_{C_f}) $\mathcal{G}(-10-\epsilon) + C_f > 0$ and $\dfrac{\mathcal{G}(-10+\epsilon) + \|f_\infty\|_{L^\infty(\mathcal{M})}}{\mathcal{G}(-10-\epsilon)+C_f} \leq \dfrac{S_0 M}{S_1 m}$.

A function $u \in V \cap L^\infty(\mathcal{M})$ is called a *bounded weak solution* of $(P_{Q,f})$ if there exists $z \in L^\infty(\mathcal{M})$, $z(x) \in \beta(u(x))$ a.e. $x \in \mathcal{M}$ such that for any $v \in V$

$$\int_{\mathcal{M}} (|\nabla u|^{p-2}\nabla u)\cdot\nabla v dA + \int_{\mathcal{M}} \mathcal{G}(u) v dA = \int_{\mathcal{M}} QS(x) z v dA + \int_{\mathcal{M}} f_\infty v dA.$$

We start with a multiplicity result given in Díaz, Hernández and Tello [13]

Theorem 5 *Let u_m, u_M be the (unique) solutions of the problems*

$$(P_m) \quad -\Delta_p u + \mathcal{G}(u) = QS(x)m + f_\infty(x) \quad on\ \mathcal{M},$$

$$(P_M) \quad -\Delta_p u + \mathcal{G}(u) = QS(x)M + f_\infty(x) \quad on\ \mathcal{M},$$

respectively. Then: i) for any $Q > 0$ there is a minimal solution $\underline{u}$ (resp. a maximal solution $\overline{u}$) of problem $(P_{Q,f})$. Moreover any other solution u must satisfy

$$u_m \leq \underline{u} \leq u \leq \overline{u} \leq u_M \tag{10}$$

$$\mathcal{G}^{-1}(QS_0 m - \|f_\infty\|_{L^\infty(\mathcal{M})}) \leq u_m \leq \mathcal{G}^{-1}(QS_1 m - C_f), \tag{11}$$

$$\mathcal{G}^{-1}(QS_0 M - \|f_\infty\|_{L^\infty(\mathcal{M})}) \leq u_M \leq \mathcal{G}^{-1}(QS_1 M - C_f). \tag{12}$$

ii) for any Q there is, at least, a solution u of $(P_{Q,f})$ which is a global minimum of the functional

$$J(w) = \frac{1}{p}\int_{\mathcal{M}} |\nabla w|^p dA + \int_{\mathcal{M}} G(w) dA - \int_{\mathcal{M}} f_\infty w dA - \int_{\mathcal{M}} QS(x) j(w) dA,$$

on the set $K = \{w \in V,\ G(w) \in L^1(\mathcal{M})\}$, where $\beta = \partial j$.
Moreover, if (H_{C_f}) holds, then: iii) if $0 < Q < Q_1$, then $(P_{Q,f})$ has a unique solution $u = u_m$, $u < -10$, u is the minimum of J on K, and

$$\mathcal{G}^{-1}(-\|f_\infty\|_{L^\infty(\mathcal{M})}) \leq \lim_{Q\searrow 0}\ inf\ \|u\|_{L^\infty(\mathcal{M})} \leq \lim_{Q\searrow 0}\ sup\ \|u\|_{L^\infty(\mathcal{M})} u \leq \mathcal{G}^{-1}(-C_f),$$

iv) if $Q_2 < Q < Q_3$, then $(P_{Q,f})$ has at least three solutions, u_i, $i = 1,2,3$ with $u_1 = u_M$, $u_1 > -10$, $u_2 = u_m$, $u_2 < -10$ and $u_1 \geq u_3 \geq u_2$ on $\mathcal{M}$. Moreover u_1 and u_2 are local minima of J on $K \cap L^\infty(\mathcal{M})$ and, if $p > 2$, on K, and v) if $Q_4 < Q$, then $(P_{Q,f})$ has a unique solution $u = u_M$, $u > -10$, u is the minimum of J on K and $\|u\|_{L^\infty(\mathcal{M})} \to +\infty$ when $Q \to +\infty$, where

$$Q_1 = \frac{\mathcal{G}(-10-\epsilon)+C_f}{S_1 M} \qquad Q_2 = \frac{\mathcal{G}(-10+\epsilon) + \|f_\infty\|_{L^\infty(\mathcal{M})}}{S_0 M} \tag{13}$$

$$Q_3 = \frac{\mathcal{G}(-10-\epsilon)+C_f}{S_1 m} \qquad Q_4 = \frac{\mathcal{G}(-10+\epsilon) + \|f_\infty\|_{L^\infty(\mathcal{M})}}{S_0 m}. \tag{14}$$

Corollary 1 *Let $R_e(u) = Bu + C$ with β given as presented at Introduction, $-10B + C > 0$ and $\frac{S_1}{S_0} \leq \frac{M}{m}$. Then we have i) if $0 < Q < \frac{-10B+C}{S_1 M}$, then $(P_{Q,f})$ has a unique solution, ii) if $\frac{-10B+C}{S_0 M} < Q < \frac{-10B+C}{S_1 m}$ then $(P_{Q,f})$ has at least three solutions, iii) if $\frac{-10B+C}{S_0 m} < Q$, then $(P_{Q,f})$ has a unique solution.*

Following Arcoya, Díaz and Tello [2] we shall describe more precisely the bifurcation diagram and in particular, we shall prove that the principal branch (emanating from $(0, \mathcal{G}^{-1}(-C)) \in \mathbb{R}^+ \times L^\infty(\mathcal{M})$) is S-shaped, i.e. it contains at least one turning point to the left and another one to the right. By a turning point to the left (respectively, to the right), we understand a point (Q^*, u^*) in the principal branch such that in a neighborhood in $\mathbb{R}^+ \times L^\infty(\mathcal{M})$ of it the principal branch is contained in $\{(Q,u) \in \mathbb{R}^+ \times L^\infty(\mathcal{M}) \,/\, Q \leq Q^*\}$ (respectively, $\{(Q,u) \in \mathbb{R}^+ \times L^\infty(\mathcal{M}) \,/\, Q \geq Q^*\}$). A previous result is due to Hetzer [33] for the special case of $p = 2$ and β a C^1 function. He proves that the principal branch of the bifurcation diagram has an even number (including zero) of turning points. The following result already improves this information showing that indeed this number of turning points is greater than or equal to two.

We make the additional assumption

(H_C) $\mathcal{G}(-10-\epsilon) + C > 0$ and $\dfrac{\mathcal{G}(-10+\epsilon)+C}{\mathcal{G}(-10-\epsilon)+C} \leq \dfrac{S_2 M}{S_1 m}$.

We start by considering the problem with β a Lipschitz function (as in the Sellers model).

Theorem 6 *Let β be a Lipschitz continuous function verifying (H_β^*). Then Σ contains an unbounded connected component which is S-shaped containing $(0, \mathcal{G}^{-1}(-C))$ with at least one turning point to the right contained in the region $(Q_1, Q_2) \times L^\infty(\mathcal{M})$ and another one to the left in $(Q_3, Q_4) \times L^\infty(\mathcal{M})$.* □

The proof has several steeps: we first prove that Σ has an unbounded component containing the point $(0, G^{-1}(-C))$ by using a resul due to Rabinowitz [48]. The we study the bifurcation diagram for the auxiliary zero - dimensional models

$$\begin{array}{lrcll} (P_1) & \mathcal{G}(u)+C & = & QS_1\beta(u) & u \in I\!R, \\ (P_2) & \mathcal{G}(u)+C & = & QS_2\beta(u) & u \in I\!R. \end{array}$$

Finally, by a comparison argument we get that the branch is necessarily S-shaped. ∎

Our next result avoids the Lipschitz assumption made in Theorem 6.

Theorem 7 *Let β a general maximal monotone graph satisfying (H_β^*) and assume (H_C). Then Σ has an unbounded S-shaped component containing $(0, \mathcal{G}^{-1}(-C))$ with at least one turning point to the right contained in the region $(Q_1, Q_2) \times L^\infty(\mathcal{M})$ and another one to the left in $(Q_3, Q_4) \times L^\infty(\mathcal{M})$, respectively.* □

To prove Theorem 7 we approximate problem $(P_{Q,C})$ when β is not Lipschitz continuous and show the *convergence* of the principal branches C_n of these approximating problems to a S-shaped unbounded connected set C of solutions of $(P_{Q,C})$ by applying a topological lemma due to Whyburn [55]

Remark 5. By using a shooting method it is possible to show (see Díaz and Tello [16] that there exist infinitely many equilibrium solutions for some values of Q when we study the one-dimensional problem

$$(P_{1,Q,C}) \begin{cases} -(|u'|^{p-2}u')' + Bu + C \in Q\beta(u) & x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases}$$

If $Q_1 < Q < Q_2$ then $(P_{1,Q,C})$ has infinitely many solutions. Moreover, there exists $K_0 \in I\!N$ such that for every $K \in I\!N$, $K \geq K_0 \in I\!N$ there exists at least a solution which crosses the level $u_K = -10$, exactly K times.

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J.I. Díaz
Departamento de Matemática Aplicada
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain

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Preface

This volume contains the proceedings of the international conference on *Free Boundary Problems: Theory and Applications* (FBP'99), held in Chiba, Japan, at the Keyaki-Kaikan of Chiba University, during November 7-13, 1999. The conference was sponsored by

- Chiba University;
- the Commemorative Association for the Japan World Exposition (1970);
- Gakkōtosho CO. LTD.;
- Chiba Convention Bureau.

The series of FBP conferences was held in Montecatini (Italy, 1981), Maubuisson (France, 1984), Irsee (Germany, 1987), Montreal (Canada, 1990), Toredo (Spain, 1993), Zakopane (Poland, 1995), Crete (Greece, 1997) and recently in Chiba (Japan, 1999). The basic idea for the organization of the FBP'99 conference was inherited from the past ones, and one of the main ideas was to bring together many scientists in various fields of sciences in order to discuss diverse problems listed below from the interdisciplinary point of view.

The conference consisted of the invited lectures, focus sessions, short communications and poster presentations which were given by more than one hundred scientists. The main topics were

(1) *Material Sciences*	(2) *Interface Dynamics*
(3) *Free Boundary (Stefan) Problems*	(4) *Phase Transitions*
(5) *Fluid Mechanics*	(6) *Reaction – Diffusion Systems*
(7) *Porous Media Equations*	(8) *Finance Problems*
(9) *Environment Problems*	(10) *Optimization and Control*
(11) *Numerical Methods*	(12) *Algorithms in Computer Science*

Especially, in this conference, the mathematical approach to environmental problems was taken up as one of the main topics and many realistic modelings for them were proposed and excitedly discussed .

N. Kenmochi
Chiba University
Japan

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