

# On the Haïm Brezis Pioneering Contributions on the Location of Free Boundaries

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## 1. Introduction

Starting in the seventies, and simultaneously to his beautiful results on the existence and regularity of solutions of many nonlinear PDEs, Haïm Brezis produced a series of papers in which, in a pioneering way, he rigorously found new qualitative phenomena as, for instance, the compactness of the support of the solution of suitable problems posed on unbounded domains and, more generally, on the location of this type of free boundaries (sometimes unexpected from the original formulation).

In this paper, we shall recall some of his results indicating their great impact in the literature which remains being relevant and useful thirty years later.

Our presentation starts by making mention to his results on the support of the solution of Variational Inequalities, specially on some ones arising in Fluid Mechanics (Section 2). Some of his results on the support of the solution of semi-linear equations are collected in Section 3. Finally, in Section 4, we shall recall his works connecting compact support properties and the abstract theory of monotone operators.

As Haïm Brezis commented at the official dinner of the Gaeta meeting, this set of results looks like a set of geological, or archeological, layers (almost the first ones among the generated by him) in his very vast production. Nevertheless, as in Geology, the time and the life use to fracture such set of initially well-ordered layers producing unexpected changes and mixtures. Something similar is produced also in Mathematics and so, for instance, the study of some special obstacle problem became of great interest to understand some limit behavior in the Ginzburg-Landau model in superconductivity (see Sandir and Serfaty [62]).

## 2. The support of the solution of a variational inequality in fluid mechanics

Starting in 1973, Haïm Brezis and Guido Stampacchia studied in a series of papers (see [31], [32], [20], [68] and the presentation made in [56]) a very classical problem of the Fluid Mechanics introducing a new approach. They considered the problem of a flow past a given profile with prescribed velocity at the infinity.<sup>1</sup>

At the beginning of the seventies, the literature on the problem was very vast, with important contributions by many authors: P. Molenbroeck (1890), S.A. Chaplyng (1902), J. Leray (1935), H. Bateman (1938), T. von Karman (1941), R. Courant and K.O. Friedrich (1948), L. Crocco (1951), L. Bers (1954), P. Germain (1954), M.J. Lighthill (1955), R. Finn and D. Gilbarg (1957), R. Finn and J. Serrin (1958) (see a larger and detailed list of references in the book by L. Bers [16]). From the mathematical point of view, the study of the incompressible case was essentially complete after the works by R. Finn. The situation was entirely different for the study of the compressible fluids.

Besides of studying the compressible case, another goal of the works by Brezis and Stampacchia was to get some sharp estimates on the maximum velocity by means of some method leading to some easy application of numerical algorithms (Stampacchia mentioned in [68] the suggestion received from the Istituto per le Applicazioni del Calcolo dall'Istituto di Meccanica Razionale del Politecnico di Torino). In fact, with their works, they initiated the development of the study of solutions with compact support on unbounded domains which would be extended later to a general class of semilinear and quasilinear partial differential equations.

The new approach by Brezis and Stampacchia was to show, rigorously, how the study of the associate hodograph plane (in the study of steady subsonic flow for a non viscous fluid, past a given symmetric convex profile in the plane) leads to a suitable obstacle problem on an unbounded domain.

They considered a closed convex profile  $\mathcal{P}$  in  $\mathbb{R}^2$ , symmetric with respect to the  $x$ -axis. They assumed the fluid to be irrotational and so, the velocity  $\mathbf{q} = (u, v)$  verifies the equations

$$\operatorname{div}(\rho\mathbf{q}) = 0, \operatorname{rot}(\mathbf{q}) = \mathbf{0}$$

where  $\rho$  denotes the density of the fluid (a constant in the incompressible case). It is also assumed that  $\mathbf{q} \rightarrow \mathbf{q}_\infty = (q_\infty, 0)$  as  $|(x, y)| \rightarrow +\infty$  and  $\mathbf{q} \cdot \mathbf{n} = 0$  on  $\partial\mathcal{P}$ . Then, it is possible to define the stream function  $\psi$  given by

$$\psi_x = -\rho v \quad \psi_y = \rho u.$$

Using Bernoulli's equation, there exists a decreasing function  $\rho = h(q)$  relating  $\rho$  with  $q = |\mathbf{q}|$  which depends on the physical properties of the fluid (for instance,  $h(q) = (1 - Cq^2)^{1/(\gamma-1)}$  for barotropic gases). Then  $q$  can be considered as a function

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<sup>1</sup>This subject already attracted the attention of scientists and artists (as for instance, Leonardo da Vinci (1452–1519)) since the beginnings of our culture.

of  $\psi_x$  and  $\psi_y$  and we get the equation

$$\left(1 - \frac{u^2}{a^2(q)}\right)\psi_{xx} + \left(1 - \frac{v^2}{a^2(q)}\right)\psi_{yy} - \frac{2uv}{a^2(q)}\psi_{xy} = 0, \tag{1}$$

where

$$a^2(q) = -q \frac{h(q)}{h'(q)},$$

$a(q)$  is the local speed of sound. In particular, (1) reduces to  $\Delta\psi = 0$  when the fluid is incompressible. The boundary condition along  $\partial\mathcal{P}$  is  $\psi = 0$ . Equation (1) is a mixed type quasilinear equation which is elliptic in the subsonic range ( $q < q_c$ ) and hyperbolic in the supersonic range ( $q > q_c$ ). Here  $q_c$  is the speed of sound, solution of  $a(q_c) = q_c$ .

It is well known that if we consider  $\psi$  as a function of  $\mathbf{q}$  instead of  $(x, y)$  (*the hodograph plane*) then equation (1) becomes linear in the new variables. More precisely, the *hodograph transform*, in polar coordinates,  $\mathcal{T} : (x, y) \rightarrow (u, v) \rightarrow (\theta, q)$

$$tg \theta = \frac{v}{u},$$

leads (1) to the Chaplying equation, which, by introducing

$$\sigma = \int_q^{q_c} \frac{h(\tau)}{\tau} d\tau$$

and

$$k(q) = \frac{1}{h^2(q)} \left(1 - \frac{q^2}{a^2(q)}\right) = k(\sigma)$$

can be written as

$$\psi_{\sigma\sigma} + k\psi_{\theta\theta} = 0. \tag{2}$$

This becomes the Tricomi equation when  $k(q)$  is replaced by a linear function near  $\sigma = 0$ . Notice that  $k(\sigma) > 0$  in the subsonic range ( $\sigma > 0$ ) and  $k(\sigma) < 0$  in the supersonic one ( $\sigma < 0$ ).

Although the main interest of the hodograph transform lies in the fact that we deal with a linear equation, this equation has to be solved on a domain which is a priori unknown (the image of the profile  $\mathcal{P}$  under  $\mathcal{T}$  is not known since we do not know the distribution of velocities along  $\mathcal{P}$ ). Because of the symmetry, we have  $\psi = 0$  along the  $x$ -axes and it is sufficient to study the problem in the upper half plane where  $\psi > 0$ . Assuming that the flow is *totally subsonic*, the hodograph transform leads the profile  $\mathcal{P}$  into a curve  $\Gamma$  (*a free boundary*) contained in the region  $[\sigma > 0]$ . If we denote by  $\sigma = l(\theta)$  to this free boundary, it was shown in Ferrari and Tricomi [51] that the boundary conditions satisfied by  $\psi$  along  $\Gamma$  are the following

$$\frac{\partial\psi}{\partial\sigma} = -\frac{R(\theta)q(\sigma)}{1 + k(\sigma)\left(\frac{dl}{d\theta}\right)^2} \text{ and } \frac{\partial\psi}{\partial\theta} = -\frac{R(\theta)q(\sigma)\frac{dl}{d\theta}}{1 + k(\sigma)\left(\frac{dl}{d\theta}\right)^2},$$

with  $R(\theta)$  the radius of curvature of  $\mathcal{P}$  at the point  $P \in \mathcal{P}$  where the tangent makes an angle  $\theta$  with the  $x$ -axis (we take  $R(\theta) < 0$  since  $\mathcal{P}$  is convex).

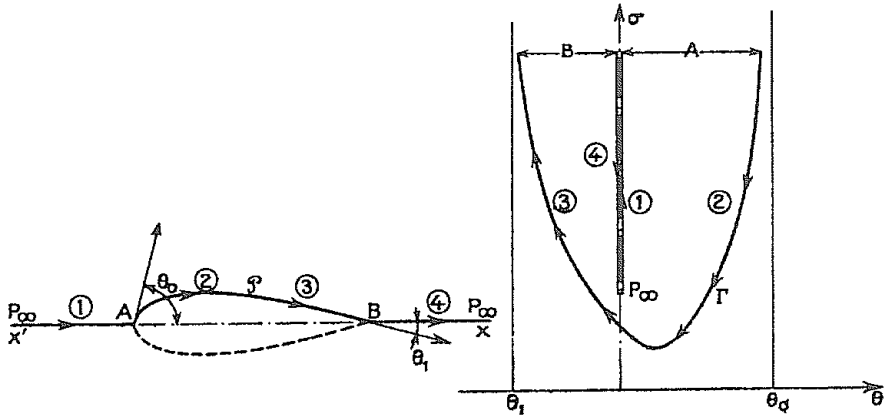


FIGURE 1

Inspired by the work of C. Baiocchi [6] on a different hydrodynamics problem, Brezis and Stampacchia introduced the change of unknown

$$u(\theta, \sigma) = \int_{l(\theta)}^{\sigma} \frac{k(\tau)}{q(\tau)} \psi(\theta, \tau) d\tau, \tag{3}$$

for  $\sigma > l(\theta)$  and  $\theta_1 < \theta < \theta_0$ . In order to identify the properties satisfied by  $u(\theta, \sigma)$  it is useful to introduce the set

$$\mathcal{D} = \{(\theta, \sigma) : \theta_1 < \theta < \theta_0, \sigma > l(\theta)\} \setminus \{(0, \sigma) : \sigma \geq \sigma_{\infty}\}$$

where

$$\sigma_{\infty} = \int_{q_{\infty}}^{q_c} \frac{h(\tau)}{\tau} d\tau$$

( $q_{\infty}$  being the x-component of the prescribed velocity at the infinity). Then, they show (see the exposition made in [20]) that  $u$  verifies  $u > 0$  in  $\mathcal{D}$  and

$$\begin{cases} -\frac{1}{q^2} \left( \frac{q^2}{k} u_{\sigma} \right)_{\sigma} - u_{\theta\theta} - u = R & \text{in } \mathcal{D}, \\ u = 0 & \text{on } \Gamma, \\ \nabla u = 0 & \text{on } \Gamma, \\ u(0, \sigma) = \text{Constant} = H_{\mathcal{P}} & \sigma \geq \sigma_{\infty}, \end{cases}$$

where  $2H_{\mathcal{P}}$  coincides with the height of the profile. To get a *complementary formulation* (i.e., without any explicit mention to the free boundary  $\Gamma$ ) they introduce the set  $\Omega = \{(\theta, \sigma) : \theta_1 < \theta < \theta_0, \sigma > 0\}$  and extend  $u$  to  $\Omega$  by choosing  $u(0, \sigma) = 0$  for  $0 < \sigma \leq l(\theta)$ . Then, they show that  $u$  satisfies an obstacle problem by introducing the functional space

$$V = \{v : qv \in L^2(\Omega), qv_{\theta} \in L^2(\Omega), \frac{q}{\sqrt{k}}v_{\sigma} \in L^2(\Omega), v = 0 \text{ on } \partial\Omega\}$$

with the canonical norm and the closed convex subset

$$K_H = \{v \in V : v \geq 0 \text{ on } \Omega \text{ and } u(0, \sigma) = H_{\mathcal{P}} \text{ for } \sigma \geq \sigma_\infty\}.$$

Then, they define the bilinear form

$$a(u, v) = \int_{\Omega} \left( \frac{1}{k} u_\sigma v_\sigma + u_\theta v_\theta - uv \right) q^2(\sigma) d\theta d\sigma.$$

After proving that  $a(u, v)$  is coercive on  $K_H$ , i.e.,

$$\lim_{\substack{\|u\| \rightarrow \infty \\ u \in K_H}} \frac{a(u, u)}{\|u\|_V} = \infty,$$

they conclude that *function  $u$  defined by (3) is the unique solution of the variational inequality*

$$\begin{cases} u \in K_H \\ a(u, v - u) \geq \int_{\Omega} R(\theta) v q^2(\sigma) d\theta d\sigma \quad \text{for all } v \in K_H. \end{cases} \tag{4}$$

Having solved (4), if we denote by

$$D^+ = \{(\theta, \sigma) \in \Omega : u(\theta, \sigma) > 0\},$$

when  $\overline{D^+}$  does not intersect the axis  $\{\sigma = 0\}$ , the curve  $\Gamma$ , boundary of  $D^+$ , represents the distribution of velocities along  $\mathcal{P}$ . If  $\overline{D^+}$  intersects the axis  $\{\sigma = 0\}$  we conclude that  $q_\infty$  is too large and there exists no totally subsonic flow past  $\mathcal{P}$ .

In this way, their treatment<sup>2</sup> allows to apply, in an automatic way, well-known algorithms for the numerical approximation of  $u$  (see, for instance [53]).

But this nice results were not entirely complete since in order to estimate the maximum of the speed  $q_{\max} := \max q$  it was needed to get some lower estimate on the location of the free boundary  $\Gamma$ . They proved that if  $q_A \geq q_\infty$  is the solution of the equation

$$\frac{H}{R_m} - 1 = \frac{q_A}{q_\infty} \left[ -1 + \frac{1}{h(q_\infty)} \int_{q_\infty}^{q_A} \frac{h(\tau)}{\tau} d\tau \right] \tag{5}$$

with  $R_m := \min_\theta |R(\theta)| > 0$  and if  $q_A \leq q_c$  then, the maximum velocity satisfies that  $q_{\max} \leq q_A$ . To do that, they construct the auxiliary function

$$\phi(\sigma) = \begin{cases} R_m q_A \int_A^\sigma \frac{k(\tau)}{q(\tau)} (\tau - A) d\tau & \text{if } A \leq \sigma \leq \sigma_\infty, \\ 0 & \text{if } 0 \leq \sigma \leq A, \end{cases}$$

where

$$A = \int_{q_A}^{q_c} \frac{h(\tau)}{\tau} d\tau,$$

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<sup>2</sup>In my modest opinion, this new approach to such a classical problem has many common intellectual points with some other cultural creations of the value as, for instance, the *Rhapsody on a Theme of Paganini*, Op. 43 by Sergei Vasilyevich Rachmaninov or *Les Femmes d'Alger* (1907) by Pablo Picasso (oeuvre in which many people find some motivations on *The Visitation* (1610–14) by Domenicos Theotocopoulos “El Greco”).

and they prove that it is a supersolution of problem (4). They also proved that the comparison principle holds for this problem and so the inequality  $u \leq \phi$  leads to a lower estimate of the free boundary,  $D^+ \subset [\sigma > A]$ , and, finally, to the conclusion  $q_{\max} \leq q_A$ .

In the incompressible case, equation (5) reduces to

$$\frac{H}{R_m} - 1 = \frac{q_A}{q_\infty} \left[-1 + \log \frac{q_A}{q_\infty}\right],$$

and, in the particular case of an sphere ( $H = R_m$  and  $\log \frac{q_A}{q_\infty} = 1$ ) it is obtained that  $q_{\max} \leq eq_\infty$  (some explicit computation shows that  $q_{\max} = 2q_\infty$ ).

Before passing to recall other results by Brezis on the location of free boundaries, we must mention some other papers on the study of subsonic flows inspired by the articles by Brezis and Stampacchia. The previous study was extended to the case in which the flow presents a free boundary  $S$  (the *sillage*, boundary of a wake) where  $q = q_S$  in Brezis and Duvaut [26]). They proved that if  $q_S < q_\infty$  then the wake disappears at a finite distance of the profile but that when  $q_S = q_\infty$  the free boundary converges to  $(0, +\infty)$  as  $|(x, y)| \rightarrow +\infty$ . The problem was later developed, from the numerical point of view, in Bourgat and Duvaut [17]. Some sharper estimates on the location of the free boundary in the hodograph plane were obtained in [37] and [41]. The problem concerning an obstacle in a channel was considered in Tomarelli [69] (see also Bruch and Dormiani [33]). The case of non-symmetric convex profiles was studied in Hummel [55] and later extended by Shimborsky [66] to plane channels, Venturi tubes and flow around a Joukowski airfoil. A careful study of the convergence of solutions and free boundaries was given in Santos [63], [64] (see also the presentation made in Rodrigues [61]). Many references on the collision of two jets of compressible fluids can be found in the books Friedman [52] and Antontsev, Díaz and Shmarev [4].

### 3. The support of the solution of semilinear (multivalued or sublinear) second order equations

Simultaneously to his works with Stampacchia on the above fluid mechanics problem (the paper [32] was received on June 28, 1975), Brezis found that the support of the solution of other variational inequalities (of obstacle type) for a general second order elliptic operator verifies also similar compactness properties. So, in Brezis [18] (see also [19]) he studied the compactness of the support of the solution of the multivalued semilinear equation

$$\begin{cases} Lu + \beta(u) \ni f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where  $\Omega$  is a smooth *unbounded* domain of  $\mathbb{R}^N$ ,  $L$  is a second-order elliptic operator

$$L = - \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i a_i \frac{\partial}{\partial x_i} + a$$

and  $\beta$  is a maximal monotone graph in  $\mathbb{R}^2$  such that  $0 \in \beta(0)$ . He assumed that

$$\begin{aligned}
 & a_{ij} \in C^1(\overline{\Omega}) \cap L^\infty(\Omega); \quad a_i, a \in L^\infty(\Omega), \\
 & \left\{ \begin{array}{l} \text{for every } r > 0 \text{ there is } \alpha(r) > 0 \text{ such that} \\ \sum_{i,j} a_{ij} \xi_i \xi_j \geq \alpha(r) |\xi|^2 \text{ for } x \in \Omega, \quad |x| \leq r, \quad \xi \in \mathbb{R}^N, \\ a(x) \geq \delta > 0 \text{ for } x \in \Omega. \end{array} \right.
 \end{aligned}$$

It is clear that if problem (6) has a solution with compact support then, necessarily  $\varphi$  has also compact support and

$$\beta^-(0) \leq f(x) \leq \beta^+(0) \text{ for } |x| \text{ large,}$$

where  $[\beta^-(0), \beta^+(0)]$  denotes the interval  $\beta(0)$ . These conditions are not sufficient but he proved in [18] that they “almost” sufficient. More precisely, he proved that if

$$\varphi \in C^2(\partial\Omega), \quad \varphi \text{ has compact support and } \beta^0(\varphi) \in L^\infty(\partial\Omega),$$

$$f \in L^\infty_{loc}(\overline{\Omega}) \text{ and } \beta^-(0) < \liminf_{|x| \rightarrow \infty} \text{ess} f(x) \leq \limsup_{|x| \rightarrow \infty} \text{ess} f(x) < \beta^+(0), \quad (7)$$

then (6) has a unique solution with compact support,  $u \in W^{2,p}(\Omega)$  for all  $p < \infty$ . The proof was based in the explicit construction of suitable radially symmetric super and subsolutions defined in the whole space  $\mathbb{R}^N$ . Besides to study the optimality of assumption (7), by particularizing  $\beta$  as different multivalued maximal monotone graphs in  $\mathbb{R}^2$ , Brezis stated, as corollaries, the existence and uniqueness of a solution with compact support to some minimization problems of the type

$$\text{Min}_{\substack{u \in H^1_0(\Omega), \quad u \geq 0 \\ u = \varphi \text{ on } \partial\Omega, \quad \text{supp } u \text{ compact}}} \int \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 - fu \right) dx,$$

and

$$\text{Min}_{\substack{u \in H^1(\Omega) \cap L^1(\Omega) \\ u = \varphi \text{ on } \partial\Omega}} \int \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 + |u| \right) dx.$$

In that paper, he wrote the following remark:

It has been shown by several authors that some nonlinear variational problems have a solution with compact support (see [5], [15], [59]). It would be of interest to unify these various results.

He added a footnote to this remark:

A new result in that direction has been obtained very recently by M. Crandall.

At this time he also knew the results on the support of the solutions of the porous media equation by Oleinik, Kalahnikov and Yui Lin, Barenblatt, Aronson, Peletier and many others<sup>3</sup>.

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<sup>3</sup>As a matter of fact, the study of this subject was one of the several points suggested by Haïm Brezis to this author as thesis subjects, during their first meeting, on April 1974. Roughly speaking I could summarize a large part of my own scientific production as an attempt of to give an answer to the above mentioned remark by Brezis. To be more specific, the reader is sent to the monographs Díaz [38] and Antontsev, Díaz and Shmarev [4].

The interest of Brezis on the support of solutions of variational inequalities was extended to the parabolic case in his paper with A. Friedman [27]. They study the obstacle problem

$$\begin{cases} u_t - \Delta u + \beta(u) \ni 0 & \text{in } (0, \infty) \times \mathbb{R}^N, \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N. \end{cases} \quad (8)$$

with  $\beta$  the maximal monotone graph in  $\mathbb{R}^2$  given by

$$\beta(r) = \begin{cases} \phi & \text{if } r < 0, \\ (-\infty, 0] & \text{if } r = 0, \\ 0 & \text{if } r > 0. \end{cases}$$

Besides proving the compactness of the support of the solution  $u(t, \cdot)$ , for any fixed  $t > 0$  (once  $u_0$  has a compact support), they proved, by first time in the literature, the property of *support shrinking* (concerning positive initial data  $u_0$  such that  $u_0(x) \rightarrow 0$  when  $|x| \rightarrow \infty$ ). They also give fine estimates on the support of  $u(t, \cdot)$  and prove the *extinction in finite time* (i.e., the existence of  $t^* < \infty$  such that  $u(x, t) \equiv 0$ , on  $\mathbb{R}^N$ , for any  $t \geq t^*$ ). This paper was the inspiration of many subsequent researches by different authors (Tartar, Evans, Knerr, Veron, J.I. Díaz, Herrero, Vázquez, G. Díaz, Gilding, Kersner and many others: see, e.g., references in the monographs [38] and [4]). We must mention also the study of first order hyperbolic Variational Inequalities made in Bensoussan and J.L. Lions [11] for linear operators and Díaz and Veron [50] for nonlinear balance laws.

In a paper with A. Bensoussan and A. Friedman [12], Brezis reconsidered the question of the location of the free boundary for variational and quasi variational inequalities but now by means of the construction of local supersolutions which, in particular, allows to get estimates on the support of the solution also for bounded domains. This technique was extended to a very general class of nonlinear equations in [39] and [38].

We cannot end this part of the section dealing with multivalued equations without making mention to the results on qualitative properties of solutions (independently of his deep results on the regularity of the solution) obtained by Haïm Brezis on other different (but typical) free boundary problems. This was the case of the *dam problem* (considered firstly under general geometry conditions in Brezis, Kinderlehrer, Stampacchia [28] and later improved by Brezis' students J. Carrillo and M. Chipot [35]). Brezis returned on this problem in [24].

The interest of Brezis on mathematical problems suggested by the Environment was recently illustrated with the organization (jointly to this author) of the meeting between the Académie des Sciences and the Real Academia de Ciencias on Mathematics and Environment held at Paris, 23–24 May, 2002 ([25]). The meeting was additionally an occasion to render homage to the memory of Jacques-Louis Lions.

Another different problem studied by him was the magnetic confinement of a plasma in a Tokamaks. In collaboration with H. Berestycki [13], he introduced some variations to a previous formulation by Mercier and Temam giving



many qualitative properties for the solutions. The problem was latter considered by many authors: Ambrosetti, Mancini, Damlamian, Caffarelli, Friedman, Kinderlehrer, Nirenberg, Stakgold, Bandle, Marcus, Sermonge, Mossino, Rakotoson, Blum, Gallouet and Simon, among them. Let us mention that the modelling of other types of magnetic confinement plasma fusion machines, the so called Stellarators (as, for instance, the TJ-II of the CIEMAT, Madrid) presents important differences with respect to the usual model for Tokamaks (see Díaz and Rakotoson [49]).

As a natural continuation of the Brezis result on the multivalued semilinear problem (6) and in connection with the above mentioned footnote of his paper, he studied with Ph. Benilan and M.G. Crandall, the support of the solution of the equation

$$-\Delta u + \beta(u) \ni f \quad \text{in } \mathbb{R}^N,$$

when  $f \in L^1(\mathbb{R}^N)$  improving his results of [18] and considering also the case in which  $f$  has a compact support. They proved that the necessary and sufficient condition on  $\beta$  in order to get a solution with compact support is that

$$\int_0^\infty \frac{ds}{\sqrt{j(s)}} < +\infty \tag{9}$$

where  $j$  is the convex primitive of  $\beta$  (i.e., such that  $\partial j = \beta$ ). This criterion was extended to the case of quasilinear problems of the type

$$-\Delta_p u + \beta(u) \ni f \quad \text{in } \mathbb{R}^N, \tag{10}$$

in Díaz and Herrero ([43], [44]) where  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $p > 1$ , to the criterion

$$\int_0^\infty \frac{ds}{\sqrt[p]{j(s)}} < +\infty \tag{11}$$

which, for instance, now applies to Lipschitz functions  $\beta(u)$  if  $p > 2$ . The above results were extended in many directions in the literature. For instance, the study of the semilinear elliptic equation (6), but now on a bounded domain  $\Omega$  and with  $f = 0$  on  $\Omega$  and  $\varphi = 1$  on the boundary, was studied by Bandle, Sperb and Stakgold [8] (see also [42]) showing that condition (9) is, again, the necessary and sufficient condition on  $\beta$  for the formation of a internal free boundary (the boundary of the *dead core*). The most general result in connection with the necessity of condition (11) was due to Vázquez who extended, in [70], the *Höpf strong maximum principle*. Many other contributions on this subject were produced by many authors (Veron, Serrin, Lanconelli, Díaz, Saa, Thiel, Kamin, Pucci, Zou, . . . : we send the reader to the monographs [38] and [4], and the recent survey Pucci and Serrin [57] for detailed references).

It seems interesting to point out that in Brezis and Nirenberg [30] the authors use the transformation  $u = e^{-v}$  to study the singularity of  $v$ , solution of  $-\Delta v + |\nabla v|^2 = h^2(v)$  for a suitable function  $h^2(v)$ , by analyzing the vanishing at a single point of  $u$ , solution of a semilinear equation of the type (6).

In collaboration with E. Lieb [29], Brezis also studied the support of a (vector) solution  $\mathbf{u}$  of some nonlinear elliptic systems arising in the study of the Minimum Action to some Vector Field Equations. They proved that, under suitable conditions,  $|\mathbf{u}|$  is a nonnegative subsolution of a semilinear equations similar to (6). The study of the support of solutions of nonlinear systems and higher order equations was carried out by many authors: (Bidaut-Veron, Bernis, Antontsev, Bertsch, Dal Passo, Shishkhov, Andreucci, Tedev, Cirimi, . . . : see [38] and [4] for detailed references).

We briefly mention here that besides the use of the super and subsolutions method we also know other useful tools to this purpose such as appropriate energy methods [4], the application of rearrangement techniques leading to measure estimates on the dead core and coincidence sets ([38], [48], [9]), etc.

#### 4. Compact support properties and the abstract theory of monotone operators

The fundamental contributions of Haïm Brezis to the abstract theory of maximal monotone operators on Hilbert spaces (and accretive operators in Banach spaces) are well known (see, for instance [22]). Even in that period of full dedication to that line of research he also was interested in many different applications to non linear partial differential equations (see, for instance, his lecture at the Vancouver *International Congress of Mathematicians* [23]). This abstract theory allows to get, also, general results for the numerical analysis of difficult problems generating a free boundary (see, for instance [14]) and can be applied to show the connections on the behavior of the free boundaries associated to some parabolic problems and the ones associated to the family of elliptic problems generated by time-implicit discretization [1].

But which I would like to illustrate here is the way in which such special problem, as the flow past a given profile mentioned in Section 2, seems to have been the starting point of an abstract result in the framework of the maximal operators in Hilbert spaces.

Although it was not explicitly said anywhere, it seems to me that his results on the support of the solution of second order elliptic variational inequalities could be the motivation for the study of the abstract Cauchy problem

$$\begin{cases} \frac{du}{dt}(t) + Au(t) \ni f(t) & \text{in } X, \\ u(0) = u_0, \end{cases}$$

in the case in which  $X = H$  is a Hilbert space and  $A : D(A) \rightarrow \mathcal{P}(H)$  a maximal monotone operator multivalued at 0 (with  $0 \in \text{int}D(A)$ ). So, in a pioneering way, he obtain in [23] the first abstract result on the finite extinction time property. He proved that if we assume  $f(t)$  such that

$$B(f(t), \epsilon) \subset A0, \text{ for a.e. } t \geq t_f, \text{ for some } \epsilon > 0 \text{ and } t_f \geq 0, \quad (12)$$

then the property of *finite extinction time* holds (there exists  $t^* \in [t_f, +\infty)$  such that  $u(t) \equiv 0$ , in  $H$ , for any  $t \geq t^*$ ) in a similar way to his results with A. Friedman on the semilinear equation (8). In contrast to the use of the comparison principle made in his previous results for elliptic and parabolic partial differential equations, now he merely used the fact that  $A$  is a maximal monotone operator and assumption (12).

Brezis considered in [23] a classical pursuit problem (already proposed by Leibnitz but modelled, now, in terms of a multivalued system associated to some suitable ordinary differential equations, i.e., with  $H = \mathbb{R}^N$ ) as a simple application of the above abstract result. It turns out that assumption (12) is difficult to be checked in order to get some possible applications to partial differential equations (where, for instance  $H = L^2(\Omega)$ ). This was the motivation of the work [39] in which the property of finite extinction time was proved for Banach spaces  $X$  and  $A : D(A) \rightarrow \mathcal{P}(X)$  a multivalued  $m$ -accretive operator. Several applications for the special case of  $X = L^\infty(\Omega)$ , to some parabolic problems of the type (8) with  $\beta$  a multivalued maximal monotone graph of  $\mathbb{R}^2$  (including second-order parabolic obstacle problems) were given in that paper. By working, again, on the space  $X = L^\infty(\Omega)$  and using a certain duality with some fully nonlinear parabolic equation, the above abstract result yields to the extinction in a finite time of solutions to multivalued nonlinear diffusion equations of the form

$$u_t - \Delta\beta(u) \ni f,$$

arising in several contexts ([36]).

The finite extinction property can be proved also (via this abstract result) for other nonlinear multivalued parabolic problems of the type

$$\begin{cases} u_t - \nu\Delta u - g\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f(t, x) & \text{in } Q_\infty, \\ u = 0 & \text{on } \Sigma_\infty, \\ u(0, x) = u_0(x) & \text{on } \Omega, \end{cases}$$

for  $\nu \geq 0$  and  $g > 0$  and  $f(t, x) \neq 0$ . Such formulation arises in very different applied problems (non-Newtonian fluids of Bingham type, image processing, microgranular structures: see references, for instance, in [3]). Moreover, coming back to the similarity with the unexpected mixtures of geological layers mentioned at the Introduction, it seems interesting to point out that the above multivalued operator is also related to some very old works in Differential Geometry ([60]).

A different problem which looks quite similar to the previous ones (since it deals with a multivalued operator) but for which the above abstract results does not apply directly is the multivalued hyperbolic dry friction type problem as, for instance,

$$\begin{cases} u_{tt} - u_{xx} + \beta(u_t) \ni 0 & \text{in } (0, 1) \times (0, +\infty), \\ u(t, 0) = u(t, 1) = 0 & t \geq 0, \\ u(0, \cdot) = u_0(\cdot) & t > 0, \\ u_t(0, \cdot) = v_0(\cdot) & \text{in } (0, 1), \end{cases}$$

where now  $\beta$  denotes the maximal monotone graph of  $\mathbb{R}^2$  given by

$$\beta(u) = \{1\} \text{ if } u > 0, \beta(0) = [-1, 1] \text{ and } \beta(u) = \{-1\} \text{ if } u < 0. \tag{13}$$

This problem was already considered by Haïm Brezis in his paper [21]. Later, he proposed to his student A. Haraux (as one of the main thesis goals) the study of the dynamics of solutions of this problem. Haraux [54] proved that  $u(t, x) \rightarrow \zeta(x)$  in  $H_0^1(0, 1)$  as  $t \rightarrow +\infty$ , with  $\zeta$  verifying  $-1 \leq \zeta_{xx} \leq 1$  and then (at the beginnings of the seventies) Brezis proposed the conjecture that the equilibrium position  $\zeta$  is reached after a finite time (*stabilization in finite time*). Although some partial results in this direction were obtained by H. Cabannes [34] (for some special initial data  $u_0$  and  $v_0$ ) the case of arbitrary initial data seems to be still an open problem.

Motivated by this, and also suggested by the numerical approach of solutions, some easier formulations were considered in the literature, as, for instance, the spatially discretized vibrating string via a finite differences. The resulting system also arises in the study of the vibration of  $N$ -particles of equal mass  $m$ . In fact, it was by passing to the limit in the number of particles (in absence of any friction) how the wave equation was obtained in 1746 by Jean Le Rond D'Alembert.

If we denote the located positions, along the interval  $(0, 1)$  of the  $x$  axis, by  $x_i(t)$  and we assume that each particle is connected to its neighbors by two harmonic springs of strength  $k$ , then the equations of motion can be written as the vectorial problem

$$(\mathbf{P}_N) \begin{cases} m\ddot{\mathbf{x}}(t) + k\mathbf{A}\mathbf{x}(t) + \mu_\beta\mathbf{B}(\dot{\mathbf{x}}(t)) + \mu_\beta\mathbf{G}(\dot{\mathbf{x}}(t)) \ni \mathbf{0}, \\ \mathbf{x}(0) = \mathbf{x}_0, \dot{\mathbf{x}}(0) = \mathbf{v}_0, \end{cases}$$

where  $\mathbf{x}(t) := (x_1(t), x_2(t), \dots, x_N(t))^T$  (here  $\mathbf{h}^T$  means the transposed vector of  $\mathbf{h}$ ),  $\mathbf{A}$  is the symmetric positive definite matrix of  $\mathbb{R}^{N \times N}$  given by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & -1 & \dots \\ \dots & -1 & 2 & -1 \\ 0 & \dots & -1 & 2 \end{pmatrix},$$

and  $\mathbf{B} : \mathbb{R}^N \rightarrow \mathcal{P}(\mathbb{R}^N)$  (respectively  $\mathbf{G} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ ) denotes the (multivalued) maximal monotone operator (respectively the Lipschitz continuous function) given by  $\mathbf{B}(y_1, \dots, y_N) = (\beta(y_1), \dots, \beta(y_N))^T$  (resp.  $\mathbf{G}(y_1, \dots, y_N) = (g(y_1), \dots, g(y_N))^T$ ). The term  $\mu_\beta\beta(\dot{x}_i(t))$  represents the Coulomb friction and  $\mu_\beta\mathbf{G}$  represents other type of frictions such as, for instance, the one due to the viscosity of an surrounding fluid. We point out that this type of friction arises very often in the applications and that its consideration was already proposed by Lord Rayleigh (see, e.g., [58]).

The study of the special case of a single oscillator,  $N = 1$ , without viscous friction

$$m\ddot{x} + 2kx + \mu_\beta\beta(\dot{x}) \ni 0 \tag{14}$$

can be found in many textbooks. The motion stops definitively after a finite time  $T_e < +\infty$  ( $x(t) \equiv x_\infty$  for any  $t \geq T_e$  for some  $x_\infty \in [-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]$ ). As in the case of the damped wave equation, it is not difficult to prove ([47]) that for any

$(\mathbf{x}_0, \mathbf{v}_0) \in \mathbb{R}^{2N}$ , problem  $(P_N)$  admits a unique weak solution  $x \in C^1([0, +\infty) : \mathbb{R}^N)$  and that there exists a unique equilibrium state  $x_\infty \in \mathbb{R}^N$  (i.e., satisfying that  $Ax_\infty \in ([-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]^N)^T$ ) such that  $\|\dot{x}(t)\| + \|x(t) - x_\infty\| \rightarrow 0$  as  $t \rightarrow +\infty$ . The stabilization in a finite time, in absence of viscous friction ( $\mu_g = 0$ ) was proved in Bamberger and Cabannes [7]. It was proved in [47] that the presence of a viscous friction (with a suitable behavior of  $g$  near 0) may originate a qualitative distinction among the orbits in the sense that the state of the system may reach an equilibrium state in a finite time or merely in an asymptotic way (as  $t \rightarrow +\infty$ ), according the initial data  $\mathbf{x}(0) = \mathbf{x}_0$  and  $\dot{\mathbf{x}}(0) = \mathbf{v}_0$ . This dichotomy seems to be new in the literature and contrasts with the phenomena of *finite extinction time* for first order ODEs and parabolic PDEs. More precisely, the following was proved in [47]: i) if  $g(r)r \leq 0$  in some neighborhood of 0 then all solutions of  $(P_N)$  stabilize in a finite time, ii) if  $g(r) = \lambda r$  with  $\lambda \geq 2\sqrt{\lambda_1 mk}/(\mu_\beta \mu_g)$ , where  $\lambda_1$  denotes the first eigenvalue of  $A$  then there exist solutions of  $(P_N)$  which do not stabilize in any finite time, and iii) if  $N = 1, A = 1 \in R$  and  $g'(0) < 2\sqrt{mk}/(\mu_\beta \mu_g)$  any solution stabilize in finite time but if  $g'(0) \geq 2\sqrt{mk}/(\mu_\beta \mu_g)$  there exist solutions which do not stabilize in any finite time.

Another dynamical question raised by Haïm Brezis concerns the study of the damped oscillator

$$m\ddot{x} + \mu |\dot{x}|^{\alpha-1} \dot{x} + kx = 0, \tag{15}$$

when now  $\alpha \in (0, 1)$ . Here  $\mu$  and  $k > 0$  are fixed parameters. In fact we can simplify the above formulation to

$$\ddot{x} + |\dot{x}|^{\alpha-1} \dot{x} + x = 0, \tag{16}$$

by dividing by  $k$  and by introducing the rescaling  $\tilde{x}(\tilde{t}) = \beta^{1/(\alpha-1)}x(\lambda\tilde{t})$  where  $\lambda = \sqrt{m}/\sqrt{k}$  and  $\beta = \mu/(k^{(2-\alpha)/2}m^{\alpha/2})$ . Notice that the  $x$ -rescaling fails for the linear case  $\alpha = 1$  since there is not any defined scale for  $x$  and the equation is merely reduced to  $\ddot{x} + \beta\dot{x} + x = 0$  with  $\beta = \mu/(\sqrt{km})$ , a parameter which characterizes the dynamics. Notice also that the limit case  $\alpha \rightarrow 0$  corresponds to the Coulomb friction equation (14).

We recall that, even if the nonlinear term  $|\dot{x}|^{\alpha-1} \dot{x}$  is not a Lipschitz continuous function of  $\dot{x}$ , the existence and uniqueness of solutions of the associate Cauchy problem

$$(P_\alpha) \begin{cases} \ddot{x} + |\dot{x}|^{\alpha-1} \dot{x} + x = 0 & t > 0, \\ x(0) = x_0, \dot{x}(0) = v_0 \end{cases}$$

is well known in the literature: see, e.g., Brezis [21]. The asymptotic behavior, for  $t \rightarrow \infty$ , of solutions of the Coulomb and linear problems  $(P_0)$  and  $(P_1)$  (limit cases when  $\alpha \rightarrow 0$  or  $\alpha \rightarrow 1$ ) was well known. In the second case the decay is exponential. In the first one, as already mentioned, given  $x_0$  and  $v_0$  there exist a finite time  $T = T(x_0, v_0)$  and  $\zeta \in [-1, 1]$  such that  $x(t) \equiv \zeta$  for any  $t \geq T(x_0, v_0)$ . When  $\alpha \in (0, 1)$  it was also well known that the solutions of  $(P_\alpha)$  verify  $(x(t), \dot{x}(t)) \rightarrow (0, 0)$  as  $t \rightarrow \infty$  (see, e.g., Haraux [54]). The question of knowing if this convergence is in fact an identity after a finite time was proposed by Brezis.

This time the answer to his question (almost thirty years later) was not as the one expected by him. In a series of papers ([45], [46] and [2]) it was shown that the generic asymptotic behavior above described for the limit case ( $P_0$ ) is only exceptional for the sublinear case  $\alpha \in (0, 1)$  since the generic orbits  $(x(t), \dot{x}(t))$  decay to  $(0, 0)$  in a infinite time and only two one-parameter families of them decay to  $(0, 0)$  in a finite time: in other words, when  $\alpha \rightarrow 0$  the exceptional behavior becomes generic. For a different approach see [71].

We end by remarking that in some other nonlinear partial differential systems it arises a feature very different from the case of scalar dissipative equations: the vector solution has some components which stabilize in finite time, and others for which this phenomenon does not occur. This property occurs, for instance, for the linear heat equation with a multivalued nonlinear dynamical boundary condition (for more details and other examples see [40]).

## 5. Special acknowledgements

If most of the papers ends with some acknowledgements, this presentation could not finish without expressing here, in this special occasion, the deep recognition and gratitude of many Spaniards mathematicians towards Haïm Brezis by the support and encouragements received from him since 1974. It was thanks to his generous help as the panorama of the mathematics in Spain, specially in the field of the nonlinear analysis, started to enjoy an activity and recognition nonexistent before. Fortunately, this was later extended to many other fields of the mathematics.

This singular contribution was officially recognized to him, in April 2000, when he received with two days of difference the nomination as foreign member of the Real Academia de Ciencias de España and the distinction as Doctor Honoris Causa by the Universidad Autónoma de Madrid.

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