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Special Finite Time Extinction in Nonlinear Evolution Systems: Dynamic Boundary Conditions and Coulomb Friction Type Problems

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Dedicated to Herbert Amann on the occasion of his 65th birthday

1. Introduction

The finite extinction time phenomenon (the solution reaches an equilibrium after a finite time) is peculiar to certain nonlinear problems which solutions exhibit an asymptotic behavior entirely different from the typical behavior of solutions of linear problems.

After recalling, in Section 2, some general results and methods on this property, we shall consider several examples of nonlinear systems where finite extinction time is not an universal property of all the solutions of the problem: a feature very different from the case of scalar dissipative equations.

For instance, sometimes the vector solution has some components which stabilize in finite time, and others for which this phenomenon does not occur. This is the case for the linear heat equation with a suitable nonlinear dynamical boundary condition. In Section 3 we present an unpublished result by H. Amann and the author (Madrid, October 1988), the vector solution has two components, one is the solution of the pde in the interior, the other one its trace on the boundary which vanishes after a finite time.

In other nonlinear systems, finite extinction time is peculiar to a finite set of orbits. Such behavior arises, for instance, in the case of oscillation under strong friction which is close to Coulomb dry friction for small values of the velocity. This will be presented in Section 4, where, in particular, we shall recall the results by H. Amann and the author on this type of problems.

2. A survey on finite extinction time properties

In order to fix ideas, let $\Omega \subset \mathbf{R}^N$, $N \geq 1$, be a general open set, let $Q_\infty = \Omega \times (0, +\infty)$, $\Sigma_\infty = \partial\Omega \times (0, +\infty)$, and consider an evolution boundary value problem formulated as

$$\left\{ \begin{array}{lcll} u_t + Au & = & f(x,t) & \text{in } Q_\infty, \\ Bu & = & g(x,t) & \text{on } \Sigma_\infty, \\ u(x,0) & = & u_0(x) & \text{on } \Omega. \end{array} \right. \tag{1}$$

Here, Au denotes a nonlinear operator and Bu denotes a boundary operator (we assume, for simplicity, that A and B are *autonomous operators*).

In the study of the stabilization of solutions, as $t \to +\infty$, it is usually assumed that $f(x,t) \to f_\infty(x)$ and $g(x,t) \to g_\infty(x)$ as $t \to +\infty$, in some functional spaces and the main task is to prove that $u(x,t) \to u_\infty(x)$, as $t \to +\infty$, in some topology of a suitable functional space, with $u_\infty(x)$ solution of

$$\left\{ \begin{array}{lcll} Au_\infty & = & f_\infty(x) & \text{in } \Omega, \\ Bu_\infty & = & g_\infty(x) & \text{on } \partial\Omega. \end{array} \right. \tag{2}$$

Here we are interested in a stronger property. We start by assuming that

$$\begin{array}{ll} f(x,t) = f_\infty(x) & \text{for } t \geq T_f, \\ g(x,t) = g_\infty(x) & \text{for } t \geq T_g. \end{array} \tag{3}$$

We arrive to the following natural concept

Definition 2.1. *Let u be a solution of problem* (1)*. We will say that $u(x,t)$ possesses the property of "stabilization in a finite time" to the stationary solution $u_\infty(x)$ if there exists $t^* < \infty$ such that $u(x,t) \equiv u_\infty(x)$, on Ω, for any $t \geq t^*$.*

In many cases $f_\infty(x) \equiv 0$, $g_\infty(x) \equiv 0$, and then $u_\infty(x) \equiv 0$. Then, the above property is known as the *finite time extinction property.*

Most of the material collected in this Section is devoted to the case of the extinction property. For a different survey on this property and other applications, in particular to some problems in Fluid Mechanics, see Chapters 2 and 4 of the monograph [5].

2.1. Some abstract results for multivalued operators: finite extinction time and a dichotomy for hyperbolic equations

Perhaps, the pioneer abstract result on finite extinction time of solutions is due to H. Brezis [41]. He proved that if $X = H$ is a Hilbert space, and $A : D(A) \to \mathcal{P}(H)$ is a maximal monotone operator multivalued at 0 (with $0 \in \operatorname{int} D(A)$) then the solution of the abstract Cauchy problem

$$\left\{ \begin{array}{l} \frac{du}{dt}(t) + Au(t) \ni f(t) \quad \text{in } X, \\ u(0) = u_0, \end{array} \right.$$

possesses the property of finite extinction in finite time once we assume $f(t)$ such that

$$B(f(t), \epsilon) \subset A0, \text{ for a.e. } t \geq t_f, \text{ for some } \epsilon > 0 \text{ and } t_f \geq 0. \tag{4}$$

It turns out that if, for instance, $H = L^2(\Omega)$ this property is difficult to be checked for most multivalued operators. This was the motivation of the work [50] in which the property of finite extinction time was proved for Banach spaces X and $A : D(A) \to \mathcal{P}(X)$ a multivalued m-accretive operator. Several applications for the special case of $X = L^\infty(\Omega)$, to some parabolic problems of the type

$$u_t - \Delta u + \beta(u) \ni f,$$

with β a maximal monotone graph of $\mathbb{R}^2$ (including second-order parabolic obstacle problems) were given in that paper. The application to multivalued nonlinear diffusion equations (of the porous media type)

$$u_t - \Delta\beta(u) \ni f,$$

was carried out in [54].

The finite extinction property can be proved also (via this abstract result) for other nonlinear multivalued parabolic problems of the type

$$\begin{cases} u_t - \nu\Delta u - g\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f(t,x) & \text{in } Q_\infty, \\ u = 0 & \text{on } \Sigma_\infty, \\ u(0,x) = u_0(x) & \text{on } \Omega, \end{cases}$$

for $\nu \geq 0$ and $g > 0$ and $f(t,x) \neq 0$. Such problems arise in several applied contexts (non-Newtonian fluids of Bingham type, image processing, microgranular structures: see references in [2]). The above multivalued operator is also related to some old works in Differential Geometry ([81]).

A problem which looks quite similar to the previous ones but for which the above abstract results does not apply is the multivalued (second-order in time) hyperbolic dry friction type problem as, for instance,

$$\text{(DSP)} \quad \begin{cases} u_{tt} - u_{xx} + \beta(u_t) \ni 0 & \text{in } (0,1) \times (0,+\infty), \\ u(t,0) = u(t,1) = 0 & t \geq 0, \\ u(0,.) = u_0(.) & \text{in } (0,1), \\ u_t(0,.) = v_0(.) & \text{in } (0,1), \end{cases}$$

where now β denotes the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta(u) = \{1\} \text{ if } u > 0,\ \beta(0) = [-1,1] \text{ and } \beta(u) = \{-1\} \text{ if } u < 0. \tag{5}$$

According well-known results (see, e.g., Haraux [70] and also [39]) $u(t,x) \to \zeta(x)$ in $H_0^1(0,1)$ as $t \to +\infty$, with ζ verifying $-1 \leq \zeta_{xx} \leq 1$. Since the beginnings of the seventies, H. Brezis proposed a conjecture: the equilibrium position is reached after a finite time (*stabilization in finite time*). Some partial results in this direction were obtained by H. Cabannes [42], [43] for some special initial data u_0 and v_0. The case of arbitrary initial data, u_0 and v_0, seems to be still an open problem.

Due to the difficulty of the above problem (and also suggested by the numerical approach of solutions) some easier formulations were considered in the literature, as, for instance, the spatially discretized vibrating string via a finite differences:

$$(\mathrm{DDSP})\quad \begin{cases} \dfrac{d^2u_i}{dt^2} - \dfrac{u_{i+1}-2u_i+u_{i-1}}{h^2} + \beta\left(\dfrac{du_i}{dt}\right) \ni 0 \quad i=1,\dots,n, \\ u_0 = u_{n+1} = 0, \\ u_i(0) = a_i \quad \text{and} \quad \dfrac{du_i}{dt}(0) = b_i. \end{cases}$$

This system also arises in the study of the dynamics vibration of N-particles of equal mass m. Indeed, we denote the located positions, along the interval $(0,1)$ of the x axis, by $x_i(t)$ and we assume that each particle is connected to its neighbors by two harmonic springs of strength k. We also assume that the particles are subject to a resultant friction force which is the composition of a Coulomb (or solid) friction and other type of frictions such as, for instance, the one due to the viscosity of an surrounding fluid. Then, the equations of motion for this mechanical system are

$$(P_N)\begin{cases} m\ddot{x}_i(t) + k(-x_{i-1}(t) + 2x_i(t) - x_{i+1}(t)) + \mu_\beta\beta(\dot{x}_i(t)) + \mu_g g(\dot{x}_i(t)) \ni 0 \\ x_i(0) = u_{0,i},\ \dot{x}_i(0) = v_{0,i} \end{cases}$$

$i = 1,\dots,N$, where we are assuming that

$$x_0(t) = 0,\ x_{N+1}(t) = 1 \quad \text{for any } t \in (0,+\infty),$$

μ_β, μ_g are positive constants, the term $\mu_\beta\beta(\dot{x}_i(t))$ represents the Coulomb friction, g is a Lipschitz continuous function such that $g(0) = 0$ and the initial data $(u_{0,i})$, $(v_{0,i})$ are given in $\mathbb{R}^N$.

Notice that if we write, for simplicity, $k = \frac{1}{h^2}$ (with $h = 1/(N+1)$) and $m = 1$, then problem (P_N) coincides with the spatial discretization, by finite differences, of the damped string equation

$$(P_\infty)\begin{cases} u_{tt} - u_{xx} + \mu_\beta\beta(u_t) + \mu_g g(u_t) \ni 0 & \text{in } (0,1)\times(0,+\infty), \\ u(0,t) = u(1,t) = 0, & t \in (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = v_0(x) & x \in (0,1). \end{cases} \tag{6}$$

In fact, it was by passing to the limit, $N \to \infty$ in (P_N) (in absence of any friction) how the wave equation was obtained in 1746 by Jean Le Rond D'Alembert.

In order to give several criteria to have the stabilization in a finite time for (P_N) it is useful to start with the study of the special case of a single oscillator, $N = 1$, without viscous friction

$$m\ddot{x} + 2kx + \mu_\beta\beta(\dot{x}) \ni 0. \tag{7}$$

The study of this elementary equation can be found in many textbooks (see, for instance, [73]). It is easy to see then that the motion stops definitively after a finite time: i.e., there exists $T_e < +\infty$ and $x_\infty \in [-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]$ such that $x(t) \equiv x_\infty$ for any $t \geq T_e$.

Concerning the case of N-particles we can mention the work by Bamberger and Cabannes [29] in which they prove the stabilization in a finite time in absence of viscous friction ($\mu_g = 0$). We point out that this type of friction arises very often in the applications and that its consideration was already proposed by Lord Rayleigh (see, e.g., [80]). Concrete expressions for g can be found also in [73].

System (P_N) can be written, in short, as a vectorial problem

$$(\mathbf{P}_N) \begin{cases} m\ddot{\mathbf{x}}(t) + k\mathbf{A}\mathbf{x}(t) + \mu_\beta \mathbf{B}(\dot{\mathbf{x}}(t)) + \mu_\beta \mathbf{G}(\dot{\mathbf{x}}(t)) \ni \mathbf{0}, \\ \mathbf{x}(0) = \mathbf{x}_0, \ \dot{\mathbf{x}}(0) = \mathbf{v}_0 \end{cases}$$

where $\mathbf{x}(t) := (x_1(t), x_2(t), \ldots, x_N(t))^T$, $\mathbf{A}$ is the symmetric positive definite matrix of $\mathbb{R}^{N\times N}$ given by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots \\ 0 & -1 & 2 & -1 & 0 \\ \ldots & 0 & -1 & 2 & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{pmatrix},$$

$\mathbf{B} : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ and $\mathbf{G} : \mathbb{R}^N \to \mathbb{R}^N$ denote the (multivalued) maximal monotone operator and the Lipschitz continuous function given by

$$\mathbf{B}(y_1, \ldots, y_N) = (\beta(y_1), \ldots, \beta(y_N))^T \text{ and } \mathbf{G}(y_1, \ldots, y_N) = (g(y_1), \ldots, g(y_N))^T$$

(here $\mathbf{h}^T$ means the transposed vector of $\mathbf{h}$). As in the case of the damped wave equation, it is not difficult to prove ([63]) that for any $(x_0, v_0) \in R^{2N}$, problem (P_N) admits a unique weak solution $x \in C^1([0, +\infty) : R^N)$ and that there exists a unique equilibrium state $x_\infty \in R^N$ (i.e., satisfying that $Ax_\infty \in ([-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]^N)^T$) such that $\| \dot{\mathbf{x}}(t) \| + \| x(t) - x_\infty \| \to 0$ *as* $t \to +\infty$.

Sharper results on the asymptotic behavior were obtained in ([63]). It was proved there that the presence of a viscous friction (with a suitable behavior of g near 0) may originate a qualitative distinction among the orbits in the sense that the state of the system may reach an equilibrium state in a finite time or merely in an asymptotic way (as $t \to +\infty$), according the initial data $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$. This dichotomy seems to be new in the literature and contrasts with the phenomena of *finite extinction time* for first-order (in time) ordinary and parabolic nonlinear equations. More precisely, the following was proved in ([63]): i) if $g(r)r \leq 0$ in some neighborhood of 0. then all solutions of (P_N) stabilize in a finite time, ii) if $g(r) = \lambda r$ with $\lambda \geq 2\sqrt{\lambda_1 mk}/(\mu_\beta \mu_g)$, where λ_1 denotes the first eigenvalue of A then there exist solutions of (P_N) which do not stabilize in any finite time, and iii) if $N = 1$, $A = 1 \in R$ and $g'(0) < 2\sqrt{mk}/(\mu_\beta \mu_g)$ any solution stabilize in finite time but if $g'(0) \geq 2\sqrt{mk}/(\mu_\beta \mu_g)$ there exist solutions which do not stabilize in any finite time.

We point out that the positive results on stabilization in a finite time remain true under the presence of some *impulsive forces* $\mathbf{f}(t)$ leading to the system

$$m\ddot{\mathbf{x}}(t) + k\mathbf{A}\mathbf{x}(t) + \mu_\beta \mathbf{B}(\dot{\mathbf{x}}(t)) + \mu_\beta \mathbf{G}(\dot{\mathbf{x}}(t)) \ni \mathbf{f}(t)$$

assuming that their amplitude is small enough: more precisely

$$\mathbf{f}(t)^T \in [-\frac{\mu_\beta}{2k}+\epsilon, \frac{\mu_\beta}{2k}-\epsilon]^N \text{for a.e.} t \geq T_f, \text{ for some } T_f, \epsilon > 0.$$

Some sharper results are the main goal of some work in progress with G. Hetzer [59]. This behavior contrasts with the case in which the amplitude of $\mathbf{f}(t)$ becomes large and $g'(v) < 0$ for any $v \neq 0$. It was proved in [45] that, then, the dynamics generates a wide range of events leading to the chaos.

As mentioned before, the simultaneous possibility of the occurrence of stabilization in a finite or infinite time does not hold for solutions of scalar first-order in time equations of the form

$$u_t - d\Delta u + \beta(u) \ni 0 \tag{8}$$

for $\beta(u)$ multivalued at $u = 0$ and $d \geq 0$ (see, for instance, [50] and its references). We assume given some Dirichlet and initial conditions. Moreover, if we add an extra term of the form $g(u)$, such that, $g(u)u \geq 0$ for any $u \in \mathbb{R}$, then the solutions of

$$U_t - d\Delta U + \beta(U) + g(U) \ni 0 \tag{9}$$

satisfy that $\|u(t,.)\|_{L^p(\Omega)} \geq \|U(t,.)\|_{L^p(\Omega)}$ and, so, the extinction in a finite time of $u(t,.)$ implies the same property for $U(t,.)$. The opposed comparison holds when $g(u)u \leq 0$. This explain the important different behaviors among the solutions of problems of first- and second-order in time. Notice that if we assume $k = 0$ in $(\mathbf{P}_1)$ then we get that $U(t) = \dot{x}(t)$ satisfies an equation similar to (9) with $d = 0$. Notice, also, that if m is very small then problem $(\mathbf{P}_1)$ becomes a *quasi-static problem* (in the terminology of [69]) and then the solutions are closed to the solutions of the first-order in time problem

$$(QSP_1) \begin{cases} 2kx + \mu_\beta \beta(\dot{x}) + \mu_g g(\dot{x}(t)) \ni 0, \\ x(0) = x_0. \end{cases}$$

In that case, $g(u)u \geq 0$ implies a comparison opposite to the above mentioned one with respect the solutions with $g = 0$. Nevertheless, now the multivalued character of β at $u = 0$ does not imply the stabilization in a finite time for the solutions of (QSP_1).

The mentioned dichotomy may arise for other dynamical systems, as, for instance, the damped wave equation

$$u_{tt} - u_{xx} + \beta(u_t) + \lambda u_t \ni 0 \quad \text{in } (0,1) \times (0,+\infty), \tag{10}$$

with Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ for $t \in (0,+\infty)$. If we assume $\lambda \geq 2\pi$ then we can find some solutions of (10) which does not stabilize in any finite time although there are solutions which stabilize in a finite time. Indeed: we can construct solutions of the first type in the form $u(x,t) = a(t)sin(\pi x) + \frac{1}{2}x(x-1)$, with $a(t)$ such that

$$\ddot{a} + \pi^2 a + \lambda \dot{a} = 0, \tag{11}$$

and $\dot{a}(t) > 0$ for all time (which is possible since $\lambda \geq 2\pi$). By the contrary, if we choose $b(t)$ solution of (11) such that $\dot{b}(t) > 0$ for all $t \in [0,1)$, $\dot{b}(1) = 0$,

$b(1) = 1/\pi^2$ and take $a(t) = b(t)$ if $t \leq 1$ and $a(t) = 1/\pi^2$ for $t \geq 1$ then we get a solution which attains the stationary state $u_\infty \equiv 1$ after $t = 1$.

It is possible to obtain some abstract results for the localization in a finite time of solutions of some second-order in time Cauchy problem of the type

$$\begin{cases} \frac{d^2u}{dt^2}(t) + \partial\Psi(u(t)) + \partial\Phi(\frac{du}{dt}(t)) \ni 0 & \text{in } H, \\ u(0) = u_0, \ \frac{du}{dt}(0) = v_0, \end{cases}$$

in the framework of the subdifferential operators of convex functions Ψ and Φ on an Hilbert space H when we assume

$$\Phi(u) \geq \alpha \|u\| \text{ for some } \alpha > 0,$$

$$0 \in \operatorname{int} \partial\Phi(0) \text{ and } \partial\Psi \text{ is single-valued.}$$

As before, we assume proved, as a previous step, that there exists $\mathbf{u}_\infty \in H$ such that

$$\| \frac{du}{dt}(t) \| + \| \mathbf{u}(t) - \mathbf{u}_\infty \| \to 0 \quad \textit{as } t \to +\infty. \tag{12}$$

Then, if

$$-\partial\Psi(\mathbf{u}_\infty) \in \operatorname{int} \partial\Phi(0) \tag{13}$$

there exists $t^* \geq 0$ such that $u(t) = u_\infty$ for any $t \geq t^*$.

This abstract theorem ([64]) can be applied, once we assume (13), not only to the finite-dimensional system of the vibration of N damped particles (already given in [1] and [44]) but also to the damped string equation (6) and to some viscoelastic Bingham materials leading to the multivalued equation

$$u_{tt} - \Delta u - \Delta\beta(u_t) \ni 0$$

(this time with the choice $H = H^{-1}(\Omega)$ as in [54]).

The last remark of this section concerns the case of periodic solutions of scalar second-order in time equations as (7). Several results on solutions presenting a *dead core* can be found in [47], [37] and the monograph [75] where many other references can be found. On the other hand, when the friction is replaced by $\beta(\dot{x} - \widehat{v})$ containing a given excitation $\widehat{v}$ it is possible to prove the existence of a periodic solution (a limit cycle) which is *attaint in a finite* time by other orbits. This behavior can be also present for damped wave equations of the form (10) [64].

2.2. Finite extinction time for single-valued operators via semidiscretization

One of the possible motivations of a different set of results on the extinction in finite time property for single-valued operators comes from the study of one of the archetype of quasilinear partial differential operators: the p-Laplacian

$$= \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty.$$

During some time, this operator was considered as an academic illustration of nonlinear diffusion operators but without any special relevant role in applied frameworks. Perhaps this was the reason (and because the often use of it in the J.L. Lions' school literature: see, e.g., [78]) why the operator was sometimes called as the "French nonlinear Laplacian". As we shall mention later, the situation changed soon and many important applications were modeled in terms of such quasilinear operator.

If, to fix ideas, we consider the problem

$$(P^p)\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_p u & \text{in } Q_\infty, \\ u(t,x) = 0 & \text{on } \Sigma_\infty, \\ u(0,x) = u_0(x) & \text{on } \Omega \end{cases}$$

very different qualitative properties were shown for its solutions according the degenerate or singular nature of the operator. So, for instance, if $p > 2$ there is *finite speed of propagation* (i.e., if $supp(u_0) \subset B(0,r) \subset\subset \Omega$, then the solution of problem (P^p) satisfies that $supp(u(t))$ is a compact set for any $t > 0$ ([57], [58]). In contrast to that, if $1 < p \leq 2$ and $u_0 \geq 0$, $u_0 \neq 0$, then $u(t) > 0$ or $u(t) = 0$ in Ω for all $t > 0$ ([58]). The finite time extinction of the solutions of (P^p) when $\frac{2N}{N+2} \leq p < 2, \quad N \geq 2$ was proved in [28], and, for $1 < p < \frac{2N}{N+1}$, in [72] (see also [5]).

Problems of this type are connected to some problems in fluid mechanics as, for instance, the discharge of a turbulent and perfect gas in a pipeline. Indeed, if we assume the pipeline occupying the interval $(0, L)$ and with a section of diameter D very small in comparison with L, the *hydraulic approximation* leads to a system of equations for the density ρ, velocity u, pressure p and temperature T

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0, \\ \rho\dfrac{\partial u}{\partial t} + \rho u \dfrac{\partial u}{\partial x} = -\dfrac{\partial p}{\partial x} - \dfrac{1}{2}\rho\,|u|\,u, \\ (\dfrac{\partial u}{\partial t} + u\dfrac{\partial}{\partial x})(\dfrac{\gamma}{\gamma-1}T + \dfrac{1}{2}u^2) - \dfrac{\partial p}{\partial t} = -\dfrac{1}{2}\,|u|\,(\dfrac{\gamma}{\gamma-1}T - \dfrac{1}{2}u^2), \\ \dfrac{p}{\rho} = T. \end{cases} \tag{14}$$

Replacing the third equation (the enthalpy equation) by

$$(\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x})Ln(p/\rho^\gamma) = -\frac{|u|}{2}\frac{\gamma}{T}(T - 1 - \frac{\gamma-1}{2\gamma}u^2) \tag{15}$$

and assuming the initial and boundary conditions corresponding to an static initially full pipeline with one closed boundary point and other in which the discharge make take place at the pressure p_0 ($p_0 > p_a$) and temperature T_0 (for any time) it was shown in ([60]) that when $t >> 1/f$ the second and fourth equation can be simplified, by neglecting lower-order terms and using some suitable variable scales), to

$$0 = -\frac{\partial p}{\partial x} - \frac{1}{2}\rho\,|u|\,u \text{ and } \frac{p}{\rho} = T = 1.$$

Then, from the first equation we deduce that p satisfies that

$$\begin{cases} \dfrac{\partial p}{\partial t} - \dfrac{\partial}{\partial x}(\left|\dfrac{\partial p^2}{\partial x}\right|^{1/2} \operatorname{sign}\left(\dfrac{\partial p^2}{\partial x}\right)) = 0 & t > 0,\ x \in (0,1), \\ \dfrac{\partial p}{\partial x}(0,t) = 0,\ p(1,t) = p_a & t > 0, \\ p(x,0) = 1 & x \in (0,1). \end{cases}$$

Notice that since $u \geq 0$, making $p^2 - p_a^2 := w$ we arrive to the doubly nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial \psi(w)}{\partial t} - \Delta_q w = 0 & t > 0,\ x \in (0,1), \\ \dfrac{\partial w}{\partial x}(0,t) = 0,\ w(1,t) = 0 & t > 0, \\ w(x,0) = w_0 & x \in (0,1), \end{cases} \tag{16}$$

with $\psi(w) = (w + p_a^m)^{1/m}$ which is a nondecreasing function of w. The correct exponents here are $m = 2$ and $q = 3/2$ nevertheless other interesting cases are $m = 7/4$ and $q = 11/7$ (case of very polished pipes) and $m = 1$ and $q = 3/2$ (laminar regime). The existence and uniqueness of solutions of a larger class of problems of this type was the main motivation of the paper [68]. The *finite extinction property* (here $w(x,t) \equiv p_a$ for any $t \geq t_0$ and $x \in (0,1)$) was proved [60].

A family of more general problems, including as special case problem (16), corresponds to the following formulation, also arising in the study of the nonlinear heat equation with absorption

$$(\mathcal{P}) \begin{cases} \frac{\partial}{\partial t}\left(u\,|u|^{\gamma-1}\right) - \Delta_p u + |u|^{\sigma-1}\,u = f + \operatorname{div}\mathbf{g} & \text{in } Q_\infty, \\ u = 0 & \text{on } \Sigma_\infty, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\gamma > 0$, $\sigma > 0$, $1 \leq p < \infty$, $\lambda \geq 0$ (if $p = 1$, ∇u represents the *total variation*). In order to study the extinction in finite time of solutions we assume that there exists $T_0 \geq 0$ such that $f(t,.) = 0$, $\mathbf{g}(t,.) = \mathbf{0}$ in Ω, if $t > T_0$.

Under suitable conditions, the "solution" (notion to be made precise in each case) satisfy an integral energy inequality leading to the extinction in a finite time of the function

Theorem 1. *Let $u \in L^1_{\mathrm{loc}}(T_0,+\infty : W_0^{1,p}(\Omega))$ for some $p > 1$ (or $u \in L^1_{\mathrm{loc}}(T_0,+\infty : BV_0(\Omega))$, if $p = 1$) such that $\exists\ \gamma, k, c > 0, \lambda \geq 0, \sigma > k-1$ for which*

$$|u|^{\gamma+k}, |u|^{\sigma+k}, |\nabla u|^p\,|u|^{k-1} \in L^1_{\mathrm{loc}}(T_0,+\infty : L^1(\Omega))$$

and

$$y(t) + c\int_s^t \int_\Omega |\nabla u|^p\,|u|^{k-1} + \lambda \int_{t_f}^t \int_\Omega |u|^{\sigma+k} \tag{17}$$

$$\leq y(s) \ \textit{a.e. } s,t \in (T_0,+\infty), \tag{18}$$

where $y(t) = \int_\Omega |u|^{\gamma+k}\,dx$. *Assume that*

$$1 \le p < \gamma + 1 \quad \text{and} \quad \lambda = 0 \tag{19}$$

or

$$1 \le p, \quad \sigma < \gamma \quad \text{and} \quad 0 < \lambda, \tag{20}$$

and let

$$k = \begin{cases} 1 & \text{if } N \le p \text{ or } (\gamma+1) \le \dfrac{Np}{N-p}, \\ \dfrac{N-p}{p}\left(1+\gamma-\dfrac{p(N-1)}{N-p}\right) > 1 & \text{if } 1 < p < N \text{ and } \gamma+1 > \dfrac{Np}{(N-p)}. \end{cases}$$

Then $u \in C^{0,\alpha}_{\text{loc}}(T_0, +\infty : L^{\gamma+k}(\Omega))$ *for some* $\alpha \in (0,1)$ *and there exists a* $T_e \in (T_0, +\infty)$ *such that* $u(t,.) \equiv 0$ *in* Ω $\forall t \ge T_e$.

The proof uses an integral version of the Torricelli-Bernoulli energy inequality found in [5] (Proposition 1.1. or Theorem 2.1, Chapter 2) for $p > 1$ and [2] for $p = 1$. More precisely $y(t) + C\int_s^t y(\tau)^\mu dt \le y(s)$, for a.e. $s, t \in (T_0, +\infty)$ for some $\mu \in (0,1)$. Some relevant choices of the parameters γ, p, σ which provide the fulfillment of the above conditions are: $p = 2$, $\gamma = 1$ and $\sigma < 1$; $\sigma = 1$, $p = 2$ and $\gamma > 1$; $\sigma = 1$, $\gamma = 1$ and $p < 2$.

Several notions of solutions are possible (for simplicity, we assume now $p > 1$). The "variational theory" search for solutions in the "energy space" $u \in L^{p'}(0,T; W_0^{1,p}(\Omega))$, and use that (if $p \ge \frac{2N}{N+2}$) $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$. At least for $k = 1, u \in L^p(0,T; W_0^{1,p}(\Omega)), \forall T > 0$ implies that $|\nabla u|^p |u|^{k-1} \in L^1_{\text{loc}}(0, +\infty : L^1(\Omega))$. A first problem arises with the zero-order term $|u|^{\sigma-1} u$ since $u \in L^{p'}(0,T; W_0^{1,p}(\Omega)) \Rightarrow |u|^{\sigma+k} \in L^1_{\text{loc}}(0,+\infty : L^1(\Omega))$. Then, if the equation takes place in $D'(\Omega)$ the natural regularity for u_t is $|u_t|^{\gamma-1} u_t \in L^{p'}_{\text{loc}}(0,+\infty : W^{-1,p'}(\Omega)) + L^1_{\text{loc}}(0,+\infty : L^1(\Omega))$. In that case the test functions must be taken in $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T : L^\infty(\Omega))$. The existence of solutions in the above framework is due to many authors (see references in [5]) assumed $|u_0|^{(\gamma-1)}u_0 \in L^2(\Omega)$ and $f, \mathbf{g} \in L^{p'}_{\text{loc}}(0,+\infty : L^{p'}(\Omega))$ and the, so-called, *weak solution* satisfies that $u \in C([0,+\infty) : L^2(\Omega))$. The regularity $|u|^{\gamma+k}, |u|^{\sigma+k}, |\nabla u|^p |u|^{k-1} \in L^1_{\text{loc}}(T_0,+\infty : L^1(\Omega))$ can be obtained by asking some extra regularity to the data (see, e.g., [36]). A nontrivial fact is the justification of the time integration by parts formula

$$\left\langle (|u|^{\gamma-1} u)_t, |u|^{k-1} u \right\rangle = \frac{\gamma}{\gamma+k}\int_0^T \left[\frac{d}{dt}\int_\Omega |u(t,.)|^{\gamma+k}\,dx\right] dt.$$

It could be easily justified for the case of *strong solutions* (i.e., $\frac{\partial}{\partial t}\left(u|u|^{\gamma-1}\right) \in L^1(Q)$) but it is know that this class of solutions are quite exceptional. More in general (but for $\gamma = k = 1, \lambda = 0$) this was proved in a pioneering paper by J.L. Lions [77]. See other references, in particular for $\gamma \ne 1$ and $k \ne 1$, in [46].

In some cases the *extinction energy* $y(t) = \int_\Omega |u|^{\gamma+k}\,dx$ may be not well defined for solutions $u(t) \in W_0^{1,p}(\Omega) \subset L^2(\Omega)$. For instance, this is the case if

$\gamma = 1, \lambda = 0$ and $1 \leq p < 2N/(N+2)$. Due to this difficulty, following to [34], it is useful to justify the energy inequality (for $k = 1$) by working in the space $W = W_0^{1,p}(\Omega) \cap L^{\gamma+1}(\Omega)$ if $\gamma < p-1$ or $1 < \sigma \leq p$ or Ω bounded, otherwise W is defined as the closure of $C_0^\infty(\Omega)$ in the Banach space $\{u \in L^{\gamma+1}(\Omega), \nabla u \in L^p(\Omega)\}, \|u\| = \|u\|_{\gamma+1} + \|\nabla u\|_p$. The existence of *an energy solution* (i.e., $u \in C([0,+\infty) : L^{\gamma+1}(\Omega)) \cap L^p(0,T:W) \cap L^{\sigma+1}((0,T)\times\Omega))$ for any finite T, satisfying the equation in D' and with $u(0,.) = u_0(.))$ was proved by assuming that $u_0 \in L^{\gamma+1}(\Omega)$, and $f + \operatorname{div} \mathbf{g} \in L^{p'}(0,T:W') + L^{(\sigma+1)'}((0,T)\times\Omega)$.

But the notion of solution can be found out of the energy space W. Among the several types of solutions in this framework we could mention, specially, the so called *mild solutions* motivated by the numerical analysis and the *abstract semigroup theory*: given $\epsilon > 0$ and a time discretization $t_0 = 0 < t_1 < \cdots < t_n \leq T, t_i - t_{i-1} < \epsilon, T - t_n < \epsilon$, and given $f_i \in L^\infty(\Omega)$, $w_0 \in L^\infty(\Omega)$ we consider the implicit time-discretization,

$$(DP) \begin{cases} \dfrac{b(w_i) - b(w_{i-1})}{t_i - t_{i-1}} - \Delta_p w_i + \lambda |w_i|^{\sigma-1} w_i = f_i \text{ in } \Omega, \end{cases}$$

where $b(u) = |u|^{\gamma-1} u$. Notice that $w_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Now, let $u_0 \in L^\gamma(\Omega)$, $f \in L^1(0,T:L^1(\Omega)), \mathbf{g} = \mathbf{0}$.

Definition. A *mild solution of (P)* is a function u such that $b(u) \in C([0,+\infty) : L^1(\Omega))$, $u(0,.) = u_0(.)$, and, for any $\epsilon > 0$ there exists $(t_0, t_1, \ldots, t_n, f_0, f_1, \ldots, f_n, w_0, w_1, \ldots, w_n)$ satisfying (DP) with

$$\|b(u_0) - b(w_0)\|_1 \leq \epsilon,\ \textstyle\sum_i \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_1 \, dt \leq \epsilon \text{ and } \|b(u(t)) - b(w_0)\|_1 \leq \epsilon$$

for any $t \in (t_{i-1}, t_i]$, $i = 1, \ldots, n$.

The existence of a mild solution was due to [33]. Moreover, it was proved there that if, in addition, $u_0 \in L^{\gamma+1}(\Omega), f \in L^{p'}(0,T:W') + L^{(\sigma+1)'}((0,T)\times\Omega)$ then the mild solution is also an energy solution.

Now we can study the finite extinction time for the step function $w_\epsilon(t) := w_i$ if $t \in (t_{i-1}, t_i]$, $i = 1, \ldots, n$.

Definition. *We say that $w_\epsilon(t)$ extinct in a finite time if there exists $T_{\epsilon,e} = t_j$, for some $j \leq n$ such that $\|w_\epsilon(t)\|_\infty > 0$ for $t \in [0, T_{\epsilon,e})$ and $\|w_\epsilon(t)\|_\infty = 0$ for $t \in [T_{\epsilon,e}, T]$.*

Since $w_\epsilon(t)$ satisfies the integral energy inequality (17) we get to the following result (due to [32] for $p = 2, \lambda = 0$ and [72] for $p > 1$ and $\lambda = 0$)

Corollary 1. *Assume that there exists $T_0 = t_m$, $m \leq n$ such that $f_\epsilon(t,.) = 0$, in Ω, if $t > T_0$ ($f_\epsilon(t,.)$ defined in a similar way to $w_\epsilon(t)$). Then, under the assumption of Theorem 1 on γ, p, k, and σ, function $w_\epsilon(t)$ extincts in a finite time $T_{\epsilon,e}$. Moreover, if u is a mild solution and assume that there exists $T_0 \geq 0$ such that $f(t,.) = 0, \mathbf{g}(t,.) = \mathbf{0}$ in Ω, if $t > T_0$, and that $u(T_0) \in L^{\gamma+k}(\Omega)$) then $u(t)$ extincts in a finite time T_e(only dependent on $\|u(T_0)\|_{\gamma+k}$).*

Notice that due to the regularizing effects (see [33]), it is possible to have a finite energy at time T_0 ($u(T_0) \in L^{\gamma+k}(\Omega)$)) even if $u_0 \in L^{\gamma}(\Omega)$.

An unpleasant fact of mild solutions is the lake of an easy characterization in terms of test functions and the lake of information on their spatial regularity. A different notion of solutions corresponds to the so called *renormalized solutions* (introduced, for second-order equations, in [36]). Since the general integral energy holds, the finite extinction time phenomenon can be obtained also for such solutions assumed, again, $u(T_0) \in L^{\gamma+k}(\Omega)$).

The assumption $u(T_0) \in L^{\gamma+k}(\Omega)$ is, in some sense, necessary. A counterexample can be done in other case: take $\gamma = 1, \lambda = 0$, $p = 1$, assume that $0 \in \Omega$ and $u(0,.) = \delta_0$ (the Dirac delta at the origin). Then, it is possible to show ([3]) that there is not any regularizing effect and $u(t,.) = C(t)\delta_0$ with $C(t) > 0$.

The extinction time also exists for other time-discretizations (now of semi-implicit type). We write (assuming now $w \geq 0$)

$$(w^{\gamma})_t = \frac{\gamma}{\gamma+1-p}(w^{p-1})(w^{\gamma+1-p})_t \approx \frac{\gamma}{\gamma+1-p}(w_i^{p-1})\frac{(w_i^{\gamma+1-p} - w_{i-1}^{\gamma+1-p})}{t_i - t_{i-1}}.$$

Given $\epsilon > 0$, a time discretization $t_0 = 0 < t_1 < \cdots < t_n \leq T, t_i - t_{i-1} < \epsilon, T - t_n < \epsilon$, and given $f_i \in L^{\infty}(\Omega)$, $w_0 \in L^{\infty}(\Omega)$ we consider the semi-implicit time-discretization,

$$\frac{\gamma}{\gamma+1-p}(w_i^{p-1})\frac{(w_i^{\gamma+1-p} - w_{i-1}^{\gamma+1-p})}{t_i - t_{i-1}} - \Delta_p w_i + \lambda |w_i|^{\sigma-1} w_i = f_i \text{ in } \Omega.$$

When f_i and w_0 are nonnegative, the existence and uniqueness of a nonnegative w_i is consequence of the results of [67]. The convergence of the scheme was given in [76] for $p = 2, \lambda = 0$ and in [79] for $p \neq 2$ and $\lambda = 0$. We have

Corollary 2. *Assume that there exists $T_0 = t_m$, $m \leq n$ such that $f_\epsilon(t,.) = 0$, in Ω, if $t > T_0$ ($f_\epsilon(t,.)$ defined as done for $w_\epsilon(t)$). Then, under the assumption of the Theorem 1 on γ, p, k, and σ function $w_\epsilon(t)$ extincts in a finite time $T_{\epsilon,e}$.*

The existence of a finite extinction time can be also proved for another type of semi-implicit time-discretization (see [35])

$$\frac{b(w_i) - b(w_{i-1})}{t_i - t_{i-1}} - \operatorname{div}(|\nabla w_{i-1}|^{p-2}\nabla w_i) + \lambda |w_i|^{\sigma-1} w_i = f_i in \;\; \Omega.$$

A collection of results on finite extinction time for other type of schemes in which the discretization is not in time but in space can be found in [53].

We end this section by pointing out that, some times, the finite extinction time property can be proved by many other methods that the mentioned above. For instance, it is quite usual to get it via comparison principle (see references, e.g., in [5] and the contribution by Ph. Souplet to this volume). The comparison of symmetrical rearrangements allows, sometimes, to improve pointwise criteria (see, e.g., [65]). Spectral arguments can be also used to this end ([31]). Finally, we mention that the finite extinction time property is clearly connected with the question of the controllability to zero property in control theory (see, for instance, [66]).

3. Extinction by components

In this section we present an unpublished result by H. Amann and the author [26], lost for sometime among the files of both authors but found by one of them recently (joking, almost as the case of the 1519 Leonardo's manuscript about the anatomy lost since 1630 discovered, by chance, on 1966 in the archives of the Spanish National Library at Madrid). The main goal of this study was to exhibit some nonlinear system for which only one of the two components satisfies the finite extinction time property.

One of such systems can be formulated as follows

$$\begin{cases} u_t - \Delta u = f(t,x) & \text{in } Q_\infty, \\ u_t + u_\nu + \beta(u) \ni g(t,x) & \text{on } \Sigma_\infty, \\ u(0) = u_0^\Omega & \text{in } \Omega, \\ u(0) = u_0^\Gamma & \text{in } \Gamma. \end{cases}$$

Here Ω denotes a *convex* bounded domain in $\mathbb{R}^N$ and $\beta : D(\beta) \to \mathcal{P}(\mathbb{R})$ is the Signorini maximal monotone graph

$$\beta(r) = \begin{cases} \phi & \text{if } r < 0, \\ (-\infty, 0] & \text{if } r = 0, \\ 0 & \text{if } r > 0, \end{cases}$$

$u_0^\Omega \in L^\infty(\Omega)$, $u_0^\Gamma \in L^\infty(\Gamma)$ and u_ν is the conormal derivative of u at points of Γ.

The existence of solutions is well known in the literature, specially. after the deep contributions by H. Amann and collaborators (a large list of references can be found, for instance, in [30]). The dynamics of the problem becomes quite peculiar when β is not Lipschitz continuous near the origin.

Theorem 2. *Assume that* $f(t,x) \leq -\varepsilon^2$ *near* $\partial\Omega$ *for* $t \geq t_f, g(t,x) \leq -\varepsilon^2$ on $\partial\Omega \times (t_g, +\infty)$. *Then the trace* $u(t,\cdot)$ *on* Γ *vanishes after a finite time* t_0.

The proof uses the explicit construction of a local barrier function of the form $\overline{u}\,(t,x) = \Phi(t-t_0) + C\,|x-x_0|^2$ for any $x_0 \in \partial\Omega$ and for some suitable Φ, $t_0$and C (see some related coincidence results in [55] and [56]).

Notice that we can reformulate the problem in terms of a vectorial abstract problem: take the Hilbert spaces V and H, with $V = \{(u,v) \in H^1(\Omega) \times H^{1/2}(\Gamma);\ u_{|\Gamma} = v\}$ which is a real separable Hilbert space isomorphic to $H^1(\Omega)$ where the latter is endowed with the equivalent norm

$$\|u\|_{H^1(\Omega)} = \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u_{|\Gamma}\|^2_{L^2(\Gamma)} \right)^{1/2},$$

$H = L^2(\Omega) \times L^2(\Gamma)$ which endowed with the usual inner product

$$\langle (u,v), (\tilde{u}, \tilde{v}) \rangle = \langle u, \tilde{u} \rangle_{L^2(\Omega)} + \langle v, \tilde{v} \rangle_{L^2(\Gamma)},$$

is a real Hilbert space. Let us define $A : V \to V^*$ by

$$(A(u,v), (\varphi, \psi)) = \int_\Omega \nabla u \cdot \nabla \varphi \, dx,$$

where $(\cdot,\cdot)$ is the usual pairing between V and V^*. We define the restriction $A_H : D(A_H) \subset H \to H$ of A to H by $D(A_H) = \{(u,v) \in V;\ A(u,v) \in H\}$ and $A_H(u,v) = A(u,v)$, for each $(u,v) \in D(A_H)$. It is easy to see that

$$D(A_H) = \{(u,v) \in L^2(\Omega) \times L^2(\Gamma);\ \Delta u \in L^2(\Omega),\ u_\nu \in L^2(\Gamma),\ u_{|\Gamma} = v\}$$

and $A_H(u,v) = (-\Delta u, u_\nu)$. Then the solution is associated to the vector $U(t,.) = (u(t,\cdot), u(t,\cdot)_{|\Gamma}) \in L^2(\Omega) \times L^2(\Gamma)$ can be formulated as

$$\begin{array}{c} \frac{d}{dt}U(t,.) + AU(t,.) + BU(t,.) \ni F(t,.) \\ U(0,.) = U_0 \end{array}$$

with $BU(.) = (0, \beta(u))$ and the above theorem shows that the component $u(t,\cdot)_{|\Gamma}$ vanishes in a finite time although, by the strong maximum principle for the linear heat equation, $u(t,\cdot)$ cannot vanish in any subset of positive measure of Ω for any time $t > 0$.

Another nonlinear system for which only one of the two components satisfies the finite extinction time property is the hybrid problem

$$\left\{ \begin{array}{ll} v_{tt}(t,x) - v_{xx}(t,x) = 0, & \text{in } Q_\infty, \\ v(t,0) = 0,\ v(t,1) = z(t), & t > 0, \\ v(0,\cdot) = 0, v_t(0,\cdot) = 0, & \text{in } \Omega, \\ \ddot{z}(t) + \alpha v_x(t,1) + z(t) + \beta(\dot{z}) \ni 0, & \\ z(0) = 0,\ \dot{z}(0) = 1, & \end{array} \right. \tag{21}$$

where $\Omega = (0,1)$ and β is given by (5). This system corresponds, for instance, to the modelling of a point block which lies on a rough surface and is connected to a fixed support by a spring and one end of a flexible string is attached to the block, the other end is fixed above the surface so that the string is tense. For small horizontal displacements $v(t,x)$ the dynamics of the string are governed by the wave equation and $z(t)$ denotes the displacement of the block which is assume to be in contact, without any lubrication with a surface, so dry friction occurs (see [38]).

4. Finite extinction time for a finite set of orbits

A very special dynamics arise in the case of the damped oscillator

$$m\ddot{x} + \mu\,|\dot{x}|^{\alpha-1}\,\dot{x} + kx = 0, \tag{22}$$

when $\alpha \in (0,1)$. Here μ and $k > 0$ are fixed parameters. In fact we can simplify the above formulation to

$$\ddot{x} + |\dot{x}|^{\alpha-1}\,\dot{x} + x = 0, \tag{23}$$

by dividing by k and by introducing the rescaling $\widetilde{x}(\widetilde{t}) = \beta^{1/(\alpha-1)}x(\lambda\widetilde{t})$ where $\lambda = \sqrt{m}/\sqrt{k}$ and $\beta = \mu/(k^{(2-\alpha)/2}m^{\alpha/2})$. Notice that the x-rescaling fails for the linear case $\alpha = 1$ since there is not any defined scale for x and the equation is merely reduced to $\ddot{x} + \beta\dot{x} + x = 0$ with $\beta = \mu/(\sqrt{km})$, a parameter which characterizes the dynamics. Notice also that the limit case $\alpha \to 0$ corresponds to the Coulomb friction equation (7).

We recall that, even if the nonlinear term $|\dot{x}|^{\alpha-1}\dot{x}$ is not a Lipschitz continuous function of $\dot{x}$, the existence and uniqueness of solutions of the associate Cauchy problem

$$(P_\alpha)\begin{cases} \ddot{x}+|\dot{x}|^{\alpha-1}\dot{x}+x=0 & t>0, \\ x(0)=x_0,\ \dot{x}(0)=v_0 \end{cases}$$

is well known in the literature: see, e.g., Brezis [39]. The asymptotic behavior, for $t \to \infty$, of solutions of the Coulomb and linear problems (P_0) and (P_1) (limit cases when $\alpha \to 0$ or $\alpha \to 1$) is well known. In the second case the decay is exponential. In the first one, as mentioned in Section 2, given x_0 and v_0 there exist a finite time $T = T(x_0, v_0)$ and $\zeta \in [-1, 1]$ such that $x(t) \equiv \zeta$ for any $t \geq T(x_0, v_0)$. When $\alpha \in (0, 1)$ it is well known that the solutions of (P_α) verify that $(x(t), \dot{x}(t)) \to (0, 0)$ as $t \to \infty$ (see, e.g., Haraux [70]).

In a series of papers ([61], [62] and [25]) it was shown that the generic asymptotic behavior above described for the limit case (P_0) is only exceptional for the sublinear case $\alpha \in (0, 1)$ since the generic orbits $(x(t), \dot{x}(t))$ decay to $(0, 0)$ in a infinite time and only two one-parameter families of them decay to $(0, 0)$ in a finite time: in other words, when $\alpha \to 0$ the exceptional behavior becomes generic.

In order to present the main arguments of those papers, we can start with some formal results via asymptotic arguments. We can rewrite the equation (23) as the planar system

$$\begin{cases} \dot{x}=y \\ \dot{y}=-x-|y|^{\alpha-1}y \end{cases} \tag{24}$$

which, by eliminating the time variable, for $y \neq 0$, leads to the differential equation of the orbits in the phase plane

$$y_x = \frac{-x-|y|^{\alpha-1}y}{y} \tag{25}$$

and that allows us to carry out a phase plane description of the dynamics.

The plane phase is antisymmetric since if $y = \sigma(x)$ is a solution of (25) then the function $y = -\sigma(-x)$ is also solution. So, it is enough to describe a semiplane (for instance $x \geq 0$). By multiplying by x and y, respectively, we get that $(x^2+y^2)_t = 2|y|^{\alpha+1}$ and $(1/x, 1/y)$ satisfy a system which has the point $(0, 0)$ as a spiral unstable point. For values of $x^2+y^2 >> 1$ the orbits of the system are given, in first approximation, by $x^2+y^2 = C$ because $|y|^{\alpha-1}y$ is small compared with x. The effect of this term is to decrease slowly C with time giving the trajectory a spiral character. Notice that for $\alpha = 1$ the character of the trajectories close to the origin depends on the parameter β: for $\beta > \beta_c := 2$ the origin is a stable mode and for $\beta < \beta_c$ is a stable spiral (underdamped oscillations).

As we shall see, there are two modes of approach to the origin and so that the origin $(0, 0)$ is a node for the system (24). The lines of zero slope are given by

$$-x = |y|^{\alpha-1}y. \tag{26}$$

So the convergence to $(0,0)$ is only possible through the regions $\{(x,y) : x > 0, y < -x^{1/\alpha}\} \cup \{(x,y) : x < 0, y > (-x)^{1/\alpha}\}$. Let us see (formally) that the "ordinary" mode corresponds to orbits that are very close to the ones corresponding to small effects of the inertia. Due to the symmetry it is enough to describe this behavior for the orbits approaching the origin with values of $x > 0$ and $y < 0$. Let $-y = \widetilde{y} > 0$. Equation (25) takes the form

$$\widetilde{y}\widetilde{y}_x = -x + \widetilde{y}^\alpha. \tag{27}$$

The line of zero slope is $\widetilde{y} = x^{1/\alpha}$ and we search for orbits obeying, for $0 < x << 1$, to the expression $\widetilde{y} = x^{1/\alpha} + z(x)$ for some function $z(x)$. If we anticipate the condition $0 < z(x) << x^{1/\alpha}$, equation (25) takes the "linearized form" $\frac{1}{\alpha}x^{(\frac{1}{\alpha}-1)}z + x^{\frac{1}{\alpha}}z_x - \alpha x^{(1-\frac{1}{\alpha})}z = 0$. Thus the first term can be neglected, compared with the last one, and then the solution can be written as $z(x) \sim C\exp\{-[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\}$ with C an arbitrary constant (which explain the name of "ordinary" orbits). This type of orbits are given, close to the origin, by the approximate equation (26), which for the orbits that reach the origin from below implies that $\widetilde{y} \sim x^{1/\alpha} \sim -\frac{dx}{dt}$ and so, integrating the simplified equation

$$\frac{dx}{dt} = -x^{1/\alpha} \tag{28}$$

we get that

$$x(t) \sim [\frac{\alpha}{(1-\alpha)(t+t_1)}]^{\alpha/(1-\alpha)} \tag{29}$$

and so that it takes an infinite time to reach the origin.

Some different orbits approaching the origin can be found by searching among solutions with large values of $|y|$ compared with $|x|^{1/\alpha}$. Thus, close to the origin, the orbits with negative y are "very close" to the solutions of the equation found by replacing (27) by the simplified the equation

$$\widetilde{y}\widetilde{y}_x = \widetilde{y}^\alpha \tag{30}$$

corresponding to a balance of inertia and damping. The solution ending at the origin ($\widetilde{y}(0) = 0$) is given by

$$\widetilde{y}(x) = -\{(2-\alpha)x\}^{1/(2-\alpha)}. \tag{31}$$

Notice that it involves no arbitrary constant. So this curve is unique (a symmetric curve arises for $y > 0$ and $x < 0$) which justifies the term of "extraordinary" orbit. The time evolution of this orbit is given, for $x << 1$, by integrating the equation

$$-\frac{dx}{dt} = [(2-\alpha)x]^{1/(2-\alpha)} \tag{32}$$

and so

$$x(t) = \frac{1}{(2-\alpha)}[\frac{(2-\alpha)(1-\alpha)}{2\alpha}(t_0 - t)_+]^{(2-\alpha)/(1-\alpha)},$$

where, in general, $h(t)_+ = \max\{0, h(t)\}$. This indicates that the motion (of this approximated solution) ends at a finite time, t_0, determined by the initial conditions which, by (32) must satisfy that $v_0 \sim \pm[(2-\alpha)\,|x_0|]^{1/(2-\alpha)}$. We point out that the two exceptional orbits emanating from the origin spiral around the origin when $x^2 + y^2$ grows toward infinity and so each of them is a separatrix curve in the phase plane. Notice that due to the autonomous nature of the equation, if $x(t)$ is the solution of the Cauchy problem (P_α) of initial data (x_0, v_0) then for any parameter $\tau \geq 0$ the function $\widetilde{x}(t) := x(t+\tau)$ coincides with the solution of (P_α) of initial data $(x(\tau), \dot{x}(\tau))$. In this way, the above extraordinary orbits give rise to two curves of initial data for which the corresponding solutions of (P_α) vanish after a finite time.

In order to make more rigorous the arguments on the extraordinary orbits we can use a fixed point argument to show that there exists two curves Γ_+ and Γ_- of initial data (x_0, v_0) for which the solutions $x(t)$ of the corresponding Cauchy problem (P_α) vanish after a finite time.(see [61]). Moreover, we can obtain some additional results on these two curves ([60]):

(i) Near the origin the curves Γ_+ and Γ_- can be represented by two functions, $y = \psi_+(x)$ and $y = \psi_-(x)$, solutions of the equation (25), where $\psi_+ : [0, \varepsilon] \to (-\infty, 0]$ and $\psi_- : [-\varepsilon, 0] \to [0, +\infty)$, for some $\varepsilon > 0$.

(ii) Functions ψ_+ and ψ_-, satisfy that $\psi_\pm(0) = 0$

$$-\infty < \int_0^\varepsilon \frac{ds}{\psi_+(s)} \text{ and } \int_{-\varepsilon}^0 \frac{ds}{\psi_-(s)} < +\infty. \tag{33}$$

In particular, $\psi_+'(x) \downarrow -\infty$ when $x \downarrow 0$ and $\psi_-'(x) \uparrow +\infty$ when $x \uparrow 0$.

(iii) We have

$$\begin{aligned} -Cx^{\frac{1}{2-\alpha}} &\leq \psi_+(x) \leq -x^{\frac{1}{\alpha}} \text{ for } x \in [0, \varepsilon] \text{ and} \\ (-x)^{\frac{1}{\alpha}} &\leq \psi_-(x) \leq C(-x)^{\frac{1}{2-\alpha}} \text{ for } x \in [-\varepsilon, 0], \end{aligned}$$

for some $C > 0$.

(iv) There exists a $x_s \in (0, \varepsilon]$ such that $\psi_+(x_s) = -(x_s)^{\frac{1}{\alpha}}$ and $(-x_s)^{\frac{1}{\alpha}} = \psi_-(x_s)$. Moreover the regions $D_+ := \{(x, y)\colon x \in [0, x_s] \text{ and } \psi_+(x) \leq y \leq -x^{\frac{1}{\alpha}}\}$, $D_- := \{(x, y)\colon x \in [-x_s, 0] \text{ and } (-x)^{\frac{1}{\alpha}} \leq y \leq \psi_-(x)\}$ are time invariants for equation (25).

Curiously, the rigorous proof of the other decay to zero (now in an infinite time) is harder than for the exceptional case. Although an invariant subset M (such that each orbit meeting M needs infinite time to reach the rest) was given in [60], a more careful study of this case was carried out in [25] for an equation slightly more general

$$\ddot{x} + \varphi(\dot{x}) + x = 0, \tag{34}$$

where the friction function φ is assumed now to be continuous on $\mathbb{R}$ and locally Lipschitz continuous on $\mathbb{R}\diagdown\{0\}$, odd, positive for positive arguments and such that

$\varphi(\eta) = a\eta^\alpha$ for $0 < \eta \le \varepsilon$, for some $a, \varepsilon > 0$ and $\alpha \in (0,1)$. In addition we assume that

$$\lim_{\eta\to\infty} \varphi(\eta)/\eta < \frac{1}{2}. \tag{35}$$

It is clear that the above case $\varphi(\eta) = |\eta|^{\alpha-1}\eta$ is included as well as some nonmonotone functions such as

$$\varphi(\eta) = \begin{cases} a\eta^\alpha, & 0 \le \eta \le \varepsilon, \\ 1 + 1/\eta, & \varepsilon < \eta < \infty, \\ -\varphi(-\eta), & -\infty < \eta < 0, \end{cases}$$

where $a = \varepsilon^{-1-\alpha} + \varepsilon^{-\alpha}$. This nonlinearity appears in the literature since the pioneering work of Lord Rayleigh (see [73], [80]). The continuous approximations of the Coulomb equation also leads often to such type of formulations.

As in the previous case, is easy to see that for any $R > 0$, the disc

$$\bar{\mathbb{B}}_R := \left\{ (x,y) \in \mathbb{R}^2 \; ; \; x^2 + y^2 \le R^2 \right\}$$

is positively invariant and the origin is globally asymptotically stable. In [25] we show, by elementary techniques from the theory of ordinary differential equations, that problem (34) possesses also at least two orbits converging to the rest state in finite time. Furthermore, we give a rather detailed analysis of the flow in the neighborhood of the critical point, showing that most orbits reach it in infinite time only. In fact, on the basis of our results we conjecture that there are precisely two orbits converging to zero in finite time.

By the oddness of φ, the phase portrait of

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varphi(y) \end{cases} \tag{36}$$

is, again, invariant under the reflection $(x,y) \mapsto (-x,-y)$. Thus it suffices to study the semiflow on the closure of the half-plane $\mathbb{H} := \left\{ (x,y) \in \mathbb{R}^2 \; ; \; y > 0 \right\}$, induced by restriction from the flow generated by 36. Consequently, in the rest of this section all assertions on invariant regions etc. pertain to this semiflow. We refer to [6] for the elementary facts about semiflows and invariant regions.

From (36) we see that every orbit meets the real axis off zero vertically. For $0 < s \le 1$ we denote by Γ_s the curve

$$\Gamma_s := \left\{ (x,y) \in \mathbb{H} \; ; \; -x = s\varphi(y) \right\},$$

oriented so that y is decreasing. Then every orbit in $\mathbb{H}$ meets $\Gamma := \Gamma_1$ horizontally, and the direction field has a positive x-component and a negative y-component on the right of Γ. Thus no orbit approaches the origin through the first quadrant. Furthermore, given any $\eta > 0$,

$$\left\{ (x,y) \in \mathbb{H} \; ; \; -x \le \varphi(y), \; y \le \eta \right\}$$

is positively invariant.

We denote by $C(x, y)$ the solution curve of (36) passing through $(x, y) \in \mathbb{H}$ and being oriented in positive time direction. It is not difficult to prove $C(x, y)$ meets the positive y-axis whenever $(x, y) \in \Gamma_s$ for some $s \in (0, 1]$.

In order to built a sharper invariant region, for small values of $s > 0$, we define the functions

$$k_1(s) := -s\varphi(k_2(s)) \text{ and } k_2(s) := \lambda(s)^\beta$$

with

$$\lambda(s) := \alpha a^2 s(1 - s) \text{ and } \beta := \frac{1}{2}(1 - \alpha).$$

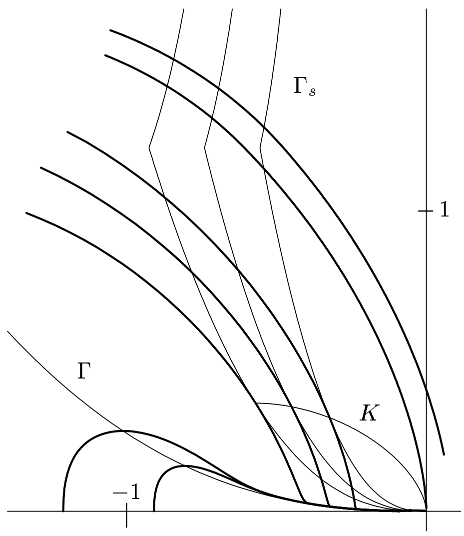
Then we define the oriented curve $K := K_\varepsilon$ in $\mathbb{H}^-$ by means of the parametrization

$$(0, s_\varepsilon] \to \mathbb{H}^-, \quad s \mapsto k(s) := \bigl(k_1(s), k_2(s)\bigr),$$

for some suitable $s_\varepsilon > 0$. Then, by construction

$$k(s) \in \Gamma_s, \qquad 0 < s \le s_\varepsilon,$$

and the positive unit tangent vectors to $C(x, y)$ and Γ_s at $(x, y) = k(s)$ coincide for $0 < s \le s_\varepsilon$. In the following picture we depict $\mathbb{H}^-$ in the neighborhood of the origin together with K, Γ, and Γ_s for three values of s, as well as seven orbits for problem (36) with $\alpha = 0.4$ and $\varepsilon = 1.2$.



Then we prove in [4] that

a) the region M in $\mathbb{H}^-$, bounded by $\{\,\bigl(k_1(s_\varepsilon), \eta\bigr)\ ;\ 0 < \eta \le k_2(s_\varepsilon)\,\}$ and K, is positively invariant,
b) there exists a positive semiorbit in $\mathbb{H}^-$ reaching the origin in finite time and
c) every orbit meeting M needs infinite time to reach the origin.

The main theorem of [4] gives a rather precise picture of the flow on $\mathbb{R}^2$ generated by (36)

Theorem 3. *The origin is the only critical point of the phase flow of* (36)*, and it is globally asymptotically stable. More precisely, given any* $R > 0$*, the disc* $\bar{\mathbb{B}}_R$ *is positively invariant and every orbit converges to the rest state. The phase portrait is symmetric with respect to the origin. There exist real numbers satisfying* $0 < \overline{x} \leq \widehat{x}$ *with the following:*

(i) *Every orbit meets* $[-\widehat{x}, \widehat{x}\,] \times \{0\}$.

(ii) *If an orbit meets* $[-\widehat{x}, 0) \times \{0\}$*, resp.* $(0, \widehat{x}\,] \times \{0\}$*, then it stays in* $\mathbb{H}^-$*, resp.*

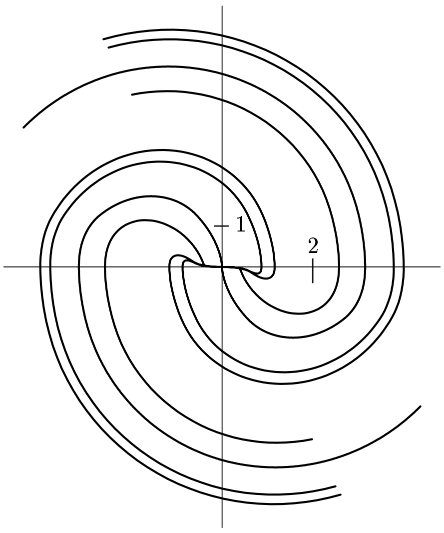
$$(-\mathbb{H})^+ := \left\{ (x, y) \in \mathbb{R}^2 \ ; \ x > 0, \ y < 0 \right\}.$$

If an orbit meets $(-\infty, -\widehat{x}\,) \times \{0\}$*, resp.* $(\widehat{x}, \infty) \times \{0\}$*, then it leaves* $\mathbb{H}^-$*, resp.* $(-\mathbb{H})^+$*, after finite time.*

(iii) *The orbits through* $(-\widehat{x}, 0)$ *and* $(0, \widehat{x}\,)$ *reach the rest point in finite time.*

(iv) *Every orbit meeting* $\left[(-\overline{x}, 0) \cup (0, \overline{x})\right] \times \{0\}$ *needs infinite time to reach the critical point.*

In the next picture a portion of the phase plane of problem (**??**) is depicted where φ is given by (**??**) with $\alpha = 0.4$ and $\varepsilon = 1.2$.



In the mentioned paper it was conjectured that there are precisely two orbits reaching the origin in finite time. This was proved in the paper Vázquez [82] where other related equations were also considered.

5. Final conclusions

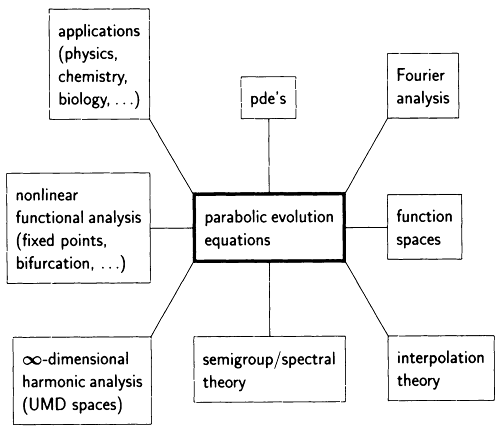
The methods and results of most of the results presented here have many common points with the exposition made by H. Amann ([24]) in occasion of his nomination as Academico Extranjero de la Real Academia de Ciencias de España. At the Introduction of his lecture he says:

In the following, I shall try to give an idea of some of my research interests of the last twenty year. During that period I was predominantly concerned with a functional-analytical approach to parabolic evolution equations. In my opinion, functional analysis, combined with so-called "hard analysis" and many other mathematical subjects, is particularly well suited for providing an abstract, powerful, and sufficient general framework for the study of nonlinear partial differential equations.

Of course, it is well known that there are intimate connections between functional analysis and partial differential equations. In fact, large parts of linear functional analysis have been developed in order provide the abstract tools for an efficient and unified study of linear partial differential equations. The point I want to make is that functional analysis is also very useful for the investigation of nonlinear different equations.

The central exposition was concerning some nonlinear systems for which he explained some of his many works (he mentioned, in particular, [7], [8], [10], [11], [11], [13], [14], [15], [9], [12], [16], [17], [19], [20], [21], [22], [23] among them. At the end of his paper he included some final remarks: a splendid sample of the beauty and depth reached by the Mathematical Analysis and the Applied Mathematics of the last third of the last century and beginnings of the present one thanks to his immense work.

In order to summarize and to give a somewhat broader view I discuss now some of the interrelationships of the theory of parabolic evolution equations with other fields of analysis, as indicated in the following diagram.



I have put parabolic evolution equations in the middle, since they are in the center of my present interest, and have grouped around them several other subjects. I did not put arrows on the connecting lines since in many cases the interaction is bilateral.

Let us start at the left upper corner. It is well known – and I have taken reaction-diffusion systems as an example – that many concrete models for the understanding of phenomena in science lead to parabolic evolution equations and, vice versa, results on parabolic evolution equations have immediate interpretations and consequences for those models.

Partial differential equations are, of course, intimately connected with parabolic evolution equations. However, neither forms a subfield of the other. For example, parabolic evolution equations encompass also other systems like integro-differential equations or infinite systems of reaction-diffusion equations involving even uncountably many unknowns, as they occur in statistical physics (see [15], [23]).

The connection between parabolic evolution equations and Fourier analysis lies on a more technical level and can be described adequately by more detailed explanations only.

As pointed out earlier, the choice of the correct state space is fundamental when studying partial differential equations, parabolic evolution equations in particular. The well-developed theory of function spaces provides us with a wide variety of possibilities. Spaces more refined than integer order Sobolev spaces like Besov and Bessel potential spaces have become increasingly important during the last years. This is true, in particular, in the study of the Navier-Stokes equations (cf. [16], [19], [20], and the references therein).

Interpolation theory provides us, on a more abstract level, with the right tool for measuring very precisely regularity properties which are the key to a successful approach to nonlinear equations.

As I have explained, semigroup theory is precisely what is needed – on the abstract level – to derive the most general local existence theory for parabolic evolution equations. Spectral theory comes in when one starts to study stability questions and the long-time behavior.

I could not into detail on the relation between parabolic evolution equations and infinite-dimensional harmonic analysis. Among other things, it has to do with "maximal regularity" questions and has, in particular during the last few years, stimulated much research in Banach space theory. I only want to mention recent results of Weis [83], Kalton [74], and others on Fourier multiplier theorems with operator-valued symbols in vector-valued Lp-spaces. Those results are tied to the theory of UMD spaces, for example. (Also see [48], [49] and, for results valid for arbitrary Banach spaces, [16], [17].)

Finally, methods from nonlinear functional analysis, fixed point theorems, bifurcation theory, etc., play an important role in the difficult and fascinating investigation of qualitative properties of the semiflows generated by parabolic evolution equations.

I hope that this enumeration of subjects, which is far from being complete, shows that the field of parabolic evolution equations is a fascinating one, invoking a lot of deep and beautiful mathematics.

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