ON A PARABOLIC PROBLEM WITH DIFFUSION ON THE BOUNDARY ARISING IN CLIMATOLOGY.*

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We consider the mathematical treatment of a two dimensional climate model (latitude – deep) which models the coupling of the mean surface temperature of the Earth with the ocean temperature. The model incorporates a dynamic and diffusive boundary condition. Our results concern the existence of a bounded weak solution. In the case of multivalued coalbedo functions the uniqueness of solutions may fail for initial datum flat enough.

1. Introduction

This work is concerned with the mathematical treatment of a two dimensional climate model (latitude – deep) which models the coupling of the mean surface temperature of the Earth with the ocean temperature. The model was proposed by Watts and Morantine (1990). The boundary condition at the top layer is given by an energy balance. This balance incorporates a second order diffusion operator, the derivative of the temperature with respect the time and a multivalued term (the coalbedo function).

Our results concern the existence of a bounded weak solution. Moreover we find positive and negative answers for the uniqueness of solution

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depending on the initial data and the assumptions made on the absorbed radiation on the boundary.

2. Existence of solution

The model represents the evolution of the temperature in a global ocean Ω with depth H. The unknown are (U, u), ocean temperature and surface temperature and the spatial variables are parametrized as $(x, z) \in \Omega = (-1, 1) \times (-H, 0)$. We call (P) to the problem,

$$\begin{cases} \dfrac{\partial U}{\partial t} - \dfrac{K_H}{R^2}\dfrac{\partial}{\partial x}((1-x^2)\dfrac{\partial U}{\partial x}) - K_V\dfrac{\partial^2 U}{\partial z^2} + w\dfrac{\partial U}{\partial z} = 0 & (0,T)\times\Omega, \\ wx\dfrac{\partial U}{\partial x} + K_V\dfrac{\partial U}{\partial z} = 0 & (0,T)\times\Gamma_H \\ D\frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}) + G(U) + & \\ +f(x) + K_V\frac{\partial U}{\partial n} + wx\frac{\partial U}{\partial x} = QS(x)\beta(x,U) & (0,T)\times\Gamma_0 \\ (1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} = 0 & (0,T)\times\Gamma_1\cup\Gamma_{-1} \\ U(0,x,z) = U_0(x,z) & \Omega, \\ U(0,x,0) = u_0(x) & (-1,1), \end{cases}$$

where the boundary of Ω is the union of the sets $\Gamma_0 = \{z = 0\}$, $\Gamma_H = \{z = -H\}$, $\Gamma_1 = \{x = 1\}$ and $\Gamma_{-1} = \{x = -1\}$.

By using some fixed point arguments we proved the existence of solutions under the hypotheses,

- (H_β) β is a bounded maximal monotone graph, that is, $|v| \le M$ $\forall v \in \beta(s)$, $\forall s \in D(\beta) = \mathbb{R}$,
- (H_G) $G{:}\mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function such that $G(0) = 0$ and $|G(\sigma)| \ge C|\sigma|^r$ for some $r > 0$,
- (H_S) $S : (-1, 1) \to \mathbb{R}$, $s_1 \ge S(x) \ge s_0 > 0$ a.e. $x \in (-1, 1)$,
- (H_f) $f \in L^\infty(\Omega \times (0, T))$, $w \in C^1(\overline{\Omega})$,
- (H_K) K_{H_0}, R, D and Q are positive constants and $p \ge 2$.

Theorem 2.1. *Given $U_0 \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\Gamma_0)$, there exists at least a weak global bounded solution of (P).*

3. On the uniqueness of solution

In spite the parabolic nature of problem (P) it turns out that under some additional assumptions it is possible to find more than one solution.

3.1. *A nonuniqueness result*

In the spirit of [2], a counterexample to the uniqueness of solutions for (P) can be built, by assuming:

(H_m) The coalbedo function is piecewise constant

$$\beta(u) = \begin{cases} [m, M] \text{ if } u = -10, \\ m \text{ if } u < -10, \\ \beta(u) = M \text{ if } u > -10, \text{ with } 0 < m < M, \end{cases} \tag{1}$$

(H_C) $G(U) = BU$ and $f(x) = C$, where B and C are positive constants verifying $-10B + C > \frac{Qs_1 m}{\rho c}$,

(H_0) $(U_0, u_0) \in C^\infty(\Omega) \times C^\infty(\Gamma_0), u_0(x) = u_0(-x)$ for all $x \in [-1, 1]$,
$\frac{du_0}{dx}(0) = \frac{d^2u_0}{dx^2}(0) = 0, \; u_0(0) = -10,$
$\frac{du_0}{dx}(x) < 0$ if $x \in (0,1), \frac{du_0}{dx}(1) = 0,$
$\frac{\partial U_0}{\partial z}(x, 0) > 0, \; U_0(x, 0) = u_0(x), \text{ if } x \in (0, 1).$

We also assume $w(x) \leq 0$ for all $x \in (-1, 1)$.

Theorem 3.1. *Under the above conditions, problem (P) has at least two bounded weak solutions.*

3.2. *Uniqueness of solutions in a class of non degenerate functions*

We define the class of nondegenerate functions (as in Diaz [1993] and Diaz–Tello[1999]). We say that $w \in L^\infty(\Gamma_0)$ satisfies the strong nondegeneracy property if there exist $C > 0$ and $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$

$$|\{x \in \Gamma_0 : |w(x) + 10| \leq \epsilon\}| \leq C\epsilon.$$

Theorem 3.2. *Assume that there exist a solution (U, u) of (P) such that $u(t)$ verifies the strong nondegeneracy property for every $t \in [0, T]$ then (U, u) is the unique bounded weak solution of (P).*

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