ON THE EULER BEST COLUMN: A SINGULAR NON LOCAL QUASILINEAR EQUATION WITH A BOUNDARY BLOWING UP FLUX CONDITION

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1. Introduction

In 1757 L. Euler [2] started the study of the best column: i.e., the shape of a stable column with symmetry of revolution such that it attains a maximum of height once the total mass is prescribed. He used the Bernoulli-Euler theory connecting the bending moment of the column with its curvature. Since then many mathematicians have contributed to this subject which in fact is not completely solved.

In this communication we follow the formulation posed, in 1966, by J. B. Keller and F. I. Niordson [3], in which, after suitable arguments (we send to this paper for details on the modelling) the problem is reduced to the existence of a function $u(x)$, $x \in (0,1)$, and a positive constant Λ satisfying

$$P(A,B)\begin{cases} -\big(A(x)\phi(u_x)\big)_x + (B(x)+\Lambda)u = 0 & \text{in } (0,1), \\ u(0)=0 & \\ u_x(x) \to +\infty & \text{as } x \to 1, \end{cases}$$

with, for a given $\mu > 0$,

$$A(x) = \big(\mu + \int_0^x u(s)^2 ds\big)^2, \quad B(x) = 2\int_x^1 u_x(\tau)^{-2}\big(\mu + \int_0^\tau u(s)^2 ds\big)d\tau \text{ and } \phi(q) = -\frac{1}{q^3}.$$

Notice that problem $P(A,B)$ can be written as the integro-differential equation

$$\begin{cases} \left[\dfrac{A(u)}{u_x^3}\right]_x + (B(u)+\Lambda)u = 0 & \text{in } (0,1), \\ u(0)=0, \ u'(1) = +\infty, \end{cases} \tag{1}$$

where

$$A(u)(x) = \left(\mu + \int_0^x u(t)^2 dt\right)^2 \tag{2}$$

$$B(u)(x) = \int_x^1 \frac{1}{u_x(t)^2}\left(\mu + \int_0^x u(t)^2 dt\right)dt\,. \tag{3}$$

The main goal of our study is to provide a rigorous approach to this problem, solved by the mentioned authors in [3] merely by means of asymptotic arguments. Our main result is the following:

Theorem 1. *Assume that*

$$\Lambda > \frac{9\pi^2}{128}. \tag{4}$$

Then problem $P(A,B)$ admits a solution u with $u \in W^{1,p}(0,1)$ for any $p \in [1,3)$ and such that,

$$u_x(x) = 0\left(\frac{1}{(1-x)^{1/3}}\right) \text{ near } x = 1. \tag{5}$$

In particular, $u \notin W^{1,3}(0,1)$.

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We shall prove the above result by iteration. For given u_{n-1}, we define $A_n = A(u_{n-1})$, $B_n = B(u_{n-1})$ and then u_n is constructed as the (unique) solution of

$$\text{(6)} \qquad \begin{cases} \left[\dfrac{A_n}{u_x^3}\right]_x + (B_n + \Lambda)u = 0 \quad \text{in } (0,1), \\ u(0) = 0, \ u'(1) = +\infty, \end{cases}$$

(we start, for instance, with $u_0(x) = x$). In Section 2, we shall prove that the *singular* problem (6) has a unique bounded variational solution (for prescribed A_n and B_n satisfying some suitable assumptions). In fact, u is obtained by minimization of the functional J_n given by

$$\text{(7)} \qquad J_n(u) = \int_0^1 \frac{A_n(x)}{u_x(x)^2}dx + \int_0^1 (B_n(x) + \Lambda)u(x)^2 dx$$

on the set $\{u \in C([0,1]) \cap W^{1,1}_{loc}(0,1),\ u \geq 0,\ u_x \geq 0,\ u(0) = 0,\ \frac{\sqrt{A_n}}{u_x} \in L^2(0,1)\}$.

Finally, in Section 3, we show that under condition (4) the iterated sequences $\{A_n\}$ and $\{B_n\}$ are equicontinuous and that $\{u_n\}$ converge (in suitable functional spaces) to a solution of $P(A,B)$.

A detailed version containing other complementary results will be given in [1].

2. On the minimizing of $J(u)$ and its Euler-Lagrange equation.

The minimization of functional $J_n(u)$ can be considered as a particular case of the problem of to minimize a functional $J : \mathcal{K} \to]0, +\infty[$ of the type

$$J(u) = \int_0^1 \frac{a(x)}{u_x(x)^2}dx + \int_0^1 b(x)u(x)^2 dx$$

with $a(x) > \mu > 0$ and $b(x) > 0$ a.e. $x \in (0,1)$, on the convex cone

$$\mathcal{K} = \{u \in C([0,1]) \cap W^{1,1}_{loc}(0,1),\ u \geq 0,\ u_x \geq 0,\ u(0) = 0,\ \frac{\sqrt{a}}{u_x} \in L^2(0,1)\}$$

Remark 2. *We do not suppose a priori that u is convex ($u_{xx} \geq 0$) but from the equation (11) below we can show that any minimizer u is such that u_x increases, whenever $a(x)$ is assumed to be nondecreasing.*

Let us denote $\eta = \inf_{u \in \mathcal{K}} J(u)$. It is easy to construct special functions $v \in \mathcal{K}$ such that $J(v) < \infty$. So, we have that $\eta < +\infty$.

Proposition 3. *There exists a unique minimizer v of J on $\mathcal{K}$.*

Proof. The uniqueness of the minimizer comes from the strict convexity of J. The proof of the existence of the minimizer will be divided in several steps.

Step 1. Let $\{u_n\}$ be a minimizing sequence in $\mathcal{K}$ such that $J(u_n) \to \eta$. Then $\{u_n\}$ is bounded in the weighted space $L^2_b(0,1) = \{u : (0,1) \to \mathbb{R}\text{: } \int_0^1 b(x)u(x)^2 dx < \infty\}$. Thus, replacing it by a subsequence, one can suppose that it has a weak limit $u \in L^2_b(0,1)$. On the other hand, the sequence $\{\frac{1}{u_{nx}}\}$ is bounded in $L^2_a(0,1)$ and so, there exists $\psi \in L^2_a(0,1)$ such that $\{\frac{1}{u_{nx}}\} \rightharpoonup \psi$ weakly in $L^2_a(0,1)$ and

$$\text{(8)} \qquad \int_0^1 a(x)\psi(x)^2 dx + \int_0^1 b(x)u(x)^2 dx \leq \eta\,.$$

Step 2. Define $\varepsilon_N = \sup_{n \geq N} J(u_n) - \eta$, so that $\varepsilon_N \to 0$ as $N \to \infty$. Since ψ belongs to the weak closure of the set $\{\frac{1}{u_{nx}},\ n \geq N\}$, for each N there exists a convex combination $\psi_N = \sum_{k=1}^K \lambda_k \frac{1}{u'_{n_k}}$ (with $n_k \geq N$ $\forall k$) such that $||\psi_N - \psi||_{L^2_a(0,1)} \leq \varepsilon_N$. Let $v_N = \sum_{k=1}^K \lambda_k u_{n_k}$ be the corresponding convex combination of the u_n. Then $v_N \to u$ strongly in $L^2_b(0,1)$. In fact, as $x \to 1/x$ is a convex function on $\mathbb{R}^*_+$, then $\{u_{nx}$, $n \geq N\}$ converges weakly in $L^2_a(0,1)$, and so, by replacing $\{u_n\}$ by a sibsequence, one can suppose that $\{u_n\} \to u$ strongly in $L^2_b(0,1)$. Moreover $\frac{1}{v_{Nx}} \leq \psi_N$ and consequently

$$\text{(9)} \qquad \limsup_{N \to \infty} \int_0^1 \frac{a(x)}{v_{Nx}(x)^2}dx \leq \lim_N \int_0^1 a(x)\psi_N^2(x)dx = \int_0^1 a(x)\psi(x)^2 dx\,.$$

Step 3. Since $J(v_N) \geq \eta$, using (8), we get

$$\liminf_N \int_0^1 a(x)\frac{1}{v_{Nx}(x)^2}dx + \int_0^1 b(x)u(x)^2dx$$
$$= \liminf_N \left(\int_0^1 a(x)\frac{1}{v_{Nx}(x)^2}dx + \int_0^1 b(x)v_N(x)^2dx \right) = \liminf_N J(v_N)$$
$$\geq \eta \geq \int_0^1 a(x)\psi(x)^2dx + \int_0^1 b(x)u(x)^2dx.$$

Hence

(10) $$\lim \int_0^1 a(x)\frac{1}{v_{Nx}(x)^2}dx = \int_0^1 a(x)\psi(x)^2dx\,.$$

Step 4. We have that $0 \leq \dfrac{1}{v_{Nx}} \leq \psi_N$ and $\lim_N \int_0^1 \left(\psi_N^2 - \dfrac{1}{{v'_N}^2} \right) a(x)dx = 0$. Then $\lim_N \dfrac{1}{v_{Nx}} = \psi$ strongly in $L^2_a(0,1)$. Substituting the v_N by a suitable subsequence, one can suppose that $v_N \to u$ and $\dfrac{1}{v_{Nx}} \to \psi$ a.e. in $(0,1)$. Then $v_{Nx} \to \frac{1}{\psi}$ a.e. and by Fatou's Lemma $\int_0^x \frac{1}{\psi(t)}dt \leq \liminf_N \int_0^x v_{Nx}(t)dt = \liminf_N v_N(x) = u(x)$. Defining $v(x) = \int_0^x \frac{1}{\psi(t)}dt$, then $v \in \mathcal{K}$, $J(v) \leq \int_0^1 a(x)\psi(x)^2dx + \int_0^1 b(x)u(x)^2dx \leq \eta$ and so v is a minimizer for J.■

Proposition 4. *Let v be the minimizer of J on $\mathcal{K}$. Then*

(11) $$\frac{a(x)}{v_x(x)^3} = \int_x^1 b(t)v(t)dt$$

and hence the boundary flux blowing-up condition $\lim_{x\to 1} v'(x) = +\infty$ holds. Moreover v is a weak solution of

(12) $$\left(\frac{a(x)}{{v_x}^3} \right)_x + b(x)v = 0.$$

Proof. Choose $\sigma \neq 0$ in $L^\infty(0,1)$ and set $w(x) = \int_0^x \sigma(t)v_x(t)dt$ and $v_s = v + sw$ for $s \in \mathbb{R}$. For s in $\mathbb{R}$, $|s| < \dfrac{1}{||\sigma||_\infty}$, one has $v_{sx}(x) = u_x(x)(1+s\sigma(x)) \geq (1 - s||\sigma||_\infty)u_x(x) \geq 0$ and $v_s(x) \leq (1+s||\sigma||_\infty)u(x)$. So $v_s \in \mathcal{K}$ with

$$J(v_s) = \int_0^1 a(x)\frac{1}{(1+s\sigma(x))^2 v_x(x)^2}dx + \int_0^1 b(x)v_s(x)^2dx$$
$$\leq \frac{1}{(1-s||\sigma||_\infty)^2}\int_0^1 a(x)\frac{1}{v_x(x)^2}dx + (1+s||\sigma||_\infty)^2 \int_0^1 b(x)v(x)^2dx$$
$$\leq \max\left(\frac{1}{(1-s||\sigma||_\infty)^2}, (1+s||\sigma||_\infty)^2 \right) J(v) < +\infty.$$

Then, $\dfrac{d}{ds}J(v_s) = 0$ at $s = 0$ provides

(13) $$-2\int_0^x a(x)\frac{1}{v_x(x)^2}\sigma(x)dx + 2\int_0^1 b(x)v(x)w(x)dx = 0\,.$$

The second term in this equation is

$$\int_0^1 b(x)v(x)w(x)dx = \int_0^1 b(x)v(x)\left(\int_0^x \sigma(t)v_x(t)dt \right) dx = \int_0^1 \sigma(t)v_x(t)\left(\int_t^1 b(x)v(x)dx \right) dt$$
$$= \int_0^1 \sigma(x)v_x(x)\left(\int_x^1 b(t)v(t)dt \right) dx,$$

so, (13) becomes

$$\int_0^1 \left(-a(x)\frac{1}{v_x(x)^2} + v_x(x)\left(\int_x^1 b(t)v(t)dt \right) \right) \sigma(x)dx = 0.$$

This holds for any σ in $L^\infty(0,1)$, and so we get $a(x)\frac{1}{v_x(x)^2} = v_x(x)\left(\int_x^1 b(t)v(t)dt)\right)$, i.e.

$$a(x)\frac{1}{v_x(x)^3} = \int_x^1 b(t)v(t)dt. \tag{14}$$

This provides also the boundary condition

$$\lim_{x\to 1} a(x)\frac{1}{v_x(x)^3} = 0\ . \tag{15}$$

and, by derivation

$$\left(a(x)\frac{1}{v_x(x)^3}\right)_x = -b(x)v(x).$$

3. Proof of Theorem 1.

Let u_n be are defined as at Section 1. It is clear that $\{B_n(x)\} \to 0$ as $x \to 1$. Moreover we have

Lemma 5. *Assume (4). Then $\{u_n\}$ is bounded in $L^2(0,1)$.*

Proof. Since A_n and u_n are nonnegative and increasing functions we have

$$\frac{A_n(1)}{u_{nx}(x)^3} \geq \frac{A_n(x)}{u_{nx}(x)^3} = \int_x^1 (B_n(t)+\Lambda)u_n(t)dt \geq \Lambda(1-x)u_n(x),$$

which provides

$$u_n(x)^{1/3}u_{nx}(x) \leq \frac{A_n(1)^{1/3}}{\Lambda^{1/3}}(1-x)^{-1/3}. \tag{16}$$

Integrating between 0 and x

$$u_n(x)^{4/3} \leq 2\frac{A_n(1)^{1/3}}{\Lambda^{1/3}}\left[1-(1-x)^{2/3}\right]. \tag{17}$$

Notice that $A_n(1) = (\mu + ||u_{n-1}||_2^2)^2$. Then, since $\int_0^1 \left[1-(1-x)^{2/3}\right]^{3/2}dx = \frac{3\pi}{32}$, integrating between 0 and 1, we get that

$$||u_n||_2^2 \leq \frac{3\pi\sqrt{2}}{16\sqrt{\Lambda}}\left(\mu + ||u_{n-1}||_2^2\right). \tag{18}$$

Consequently, due to (4) the sequence $||u_n||_2^2$ is bounded.■

We point out that $||A_n||_\infty \leq \left(\mu + C_2^2\right)^2$ with $C_2 = \sup_n ||u_n||_2$. More precisely we have

Proposition 6. *The sequence $\{u_n\}$ is bounded and equicontinuous in $C([0,1])$.*

Proof. By (16) and (17), the $u_n^{4/3}$ are bounded and their derivative are equiintegrable, so that the sequence $u_n^{4/3}$ is bounded and equicontinuous. The same property holds then for the u_n.■

In order to prove the equicontinuity of the B_n we shall prove previously the following result:

Lemma 7. *Assume (4). Then there exists $\gamma > 0$ such that $u_{nx}(x) \geq \gamma(1-x)^{-1/3}$ for any n and a.e. $x \in (0,1)$. In particular $u_n(x) \geq \frac{3\gamma}{2}[1-(1-x)^{2/3}]$ for any n and any $x \in [0,1)$.*

Proof. For x close to 1 we have $u_{nx}(x) \sim c_n(1-x)^{-1/3}$ with $c_n^3 = A_1(1)/\Lambda u_n(1) > 0$. In particular, $(1-x)^{1/3}u_1(x) > 0$ on $[0,1[$ and has a strictly positive limit as $x \to 1$. So, there exists $\gamma_1 > 0$ such that $(1-x)^{1/3}u_{1x}(x) \geq \gamma_1$ a.e. $x \in (0,1)$. By induction, suppose $(1-x)^{1/3}u_{(n-1)x}(x) \geq \gamma > 0$ with γ small enough. In particular, $\gamma \leq 1$ and $\gamma \leq \gamma_1$. Setting $C_\infty = \sup_m ||u_m||_\infty$ we have

$$\begin{aligned}
\frac{\mu^2}{u_{nx}(x)^3} \leq \frac{A_n(x)}{u_{nx}(x)^3} &= \int_x^1 (B_n(x)+\Lambda)u_n(x)dx \\
&\leq C_\infty\left[(1-x)\Lambda + 3\gamma^{-2}(\mu + C_\infty^2)\int_x^1 (1-t)^{2/3}dt\right] \\
&\leq C_\infty\left[(1-x)\Lambda + \frac{9}{5}\gamma^{-2}(\mu + C_\infty^2)(1-x)^{5/3}\right] \\
&\leq C_\infty\,\gamma^{-2}(1-x)\left[\Lambda + 2(\mu + C_\infty^2)\right].
\end{aligned}$$

So, if we take γ small enough, and in particular such that

$$\mu^2 \geq \gamma\, C_\infty \left[\Lambda + 2(\mu + C_\infty^2)\right]$$

we get that $u_{nx}(x)^3 \geq \gamma^3(1-x)^{-1}$.■

Proposition 8. *Assume (4). Then the sequences $\{A_n\}$ and $\{B_n\}$ are bounded and equicontinuous in $C([0,1])$.*

Proof. The uniform boundedness of $\{A_n\}$ and $\{A_{nx}\}$ results from previous results. The uniform boudnedness of $\{B_n\}$ and $\{B_n'\}$ results from the boundedness of $\{u_n\}$ and the previous lemma.■

To end the proof of Theorem 1 we apply Ascoli-Arzela lemma to get the uniform convergence of $\{u_n\}, \{A_n\}$ and $\{B_n\}$ to some functions u, A and B in $C([0,1])$. Obviously $A(x) = A(u)(x) = \left(\mu + \int_0^x u(t)^2 dt\right)^2$. Moreover, from (16) and Lemma 7 we get that

$$\frac{d_1}{(1-x)^{1/3}} \leq u_{nx}(x) \leq \frac{d_2}{(1-x)^{1/3}}, \text{ a.e. } x \in (0,1) \tag{19}$$

for some uniform constants d_1,$d_2 > 0$ (use the strong maximum principle near $x = 0$). Then $\{u_{nx}\}$ converges weakly to u_x in $W^{1,p}(0,1)$ for any $p \in [1,3)$ and the estimate (5) holds. Moreover, from (19) we get that $\left\{\frac{A_n(x)}{u_{nx}(x)^3}\right\}$ is uniformly bounded in any $L^p(0,1)$ for any $p \in [1,3)$ and so, there exists a function $\xi \in L^p(0,1)$ for any $p \in [1,3)$ such that $\left\{\frac{A_n}{u_{nx}{}^3}\right\} \rightharpoonup \xi$ weakly in $L^p(0,1)$. In particular $\left\{\frac{1}{u_{nx}{}^3}\right\} \rightharpoonup \frac{\xi}{A(u)}$ and as the function $r \to -r^{-3}$ generates a maximal monotone graph in $L^2(0,1)$ we get that necessarily $\frac{\xi}{A(u)} = \frac{1}{u_x{}^3}$. Then, multiplying by a test function in the corresponding equation (12) for u_n we can pass to the limit and so u is a weak solution of $P(A,B)$.■

Remark 9. *Further regularity results can be found in* [1].

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