On the Pseudo-Linearization and Quasi-Linearization Principles

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1. Introduction

In some recent papers [5] and [6] we introduced a quite general *pseudo-linearization principle* concerning the existence and stabilization, as $t \to \infty$, of the solutions of the nonlinear abstract Cauchy problem

$$(ACP)\begin{cases} \frac{du}{dt}(t) + Au(t) \ni F(u) & \text{in } X, \\ u(0) = u_0. \end{cases} \tag{1}$$

on a Banach space X in a neighborhood of some equilibrium point $w \in D(A) \cap D(F)$ such that $Aw \ni F(w)$. Such a principle generalize the *classical linearization principle* concerning the case in which both operators A and B are differentiable operators (it is required then that the first eigenvalue of the linear operator $y \to \mathrm{D}A(w)y - \mathrm{D}F(w)y$ have a negative real part). The generalization comes from the fact that quite often the nonlinear operator A is not differentiable near some equilibrium points and so the *classical linearization principle* is not applicable. Here $F : D(F) \subseteq X \longrightarrow X$ represents the operator associated to a real continuous function $f : D(f) \subset \mathsf{R} \longrightarrow \mathsf{R}$.

The main motivation to keep A nonlinear after the process of linearization in the above papers was the study the stabilization of the uniform oscillations for the *complex Ginzburg-Landau equation* by means of some global delayed feedback. In fact, due to the important role of a controlling term, in [5] and [6] we considered the more sophisticated case in which F depends also of some delayed term $F = F(u, u_t(.))$, where $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau, 0]$ for some $\tau > 0$, but we shall avoid the presence of such a term for the sake of the simplicity in the exposition. It is a curious fact that even if the *complex Ginzburg-Landau equation* is formulated in terms of a linear (vectorial) diffusion operator A the usual representation for the unknown as $\mathsf{z}(x,t) = \rho(x,t)e^{i\phi(x,t)}$ leads the original system to a coupled nonlinear system of equations for ρ and ϕ which can be formulated again in the form $\frac{dz}{dt}(t) + \widetilde{A}z \ni \widetilde{F}(z)$ but with a nonlinear (and not everywhere differentiable) operator $\widetilde{A}$.

Many other examples can be appealed to justify the philosophy of keeping A non-linear after linearizing the rest of the terms of the equation. For instance, this is the case when A is multivalued, or nondifferentiable or a degenerate quasilinear operator. We point out that some relevant examples of nonlinear equations of the type (ACP) arise in the most different contexts (see, for instance, Díaz and Hetzer [10] for one example in Climatology).

The main conclusion of the pseudolinearization principle was formulated in terms of the condition that the operator $y \to Ay - \mathrm{D}F(w)y$ belongs to $\mathcal{A}(\omega^* : X)$, for some $\omega^* \in \mathsf{C}$ with $\mathrm{Re}\,\omega^* = \gamma^* < 0$ where the class of operators $\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathcal{P}(X)$,such that $A + \omega I$ is a *m-accretive* operator$\}$ (see Brezis [3] for the case of $X = H$ a Hilbert space and Bénilan, Crandall and Pazy [2], Vrabie [14] for the case of a general Banach space).

The main goal of this communication is to present some connections between the above principle and the so called “method of quasi-linearization” introduced by R. Bellman and R. Kalaba in [1] in order to prove the existence of solutions of nonlinear parabolic *semilinear* problems with A a second order linear elliptic operator and f written as $f = f_{cv} + f_{cn}$ with f_{cv} convex and f_{cn} concave by means of some iteration schemes.

In our approach, in contrast with other results in the literature (see, for instance, Laksmikantham and Vatsala [12] and Carl and Laksmikantham [4]), we shall avoid any assumption on the second derivative of function f. To do that we shall combine an iterative scheme with some approximation arguments (we replace f by a regular approximation f^k) of the type

$$\begin{cases} \frac{d\overline{u}_{k+1}}{dt} + A\overline{u}_{k+1} \ni f^k(\overline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) & \text{in } X, \\ \frac{d\underline{u}_{k+1}}{dt} + A\underline{u}_{k+1} \ni f^k(\underline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) & \text{in } X, \\ \overline{u}_{k+1}(0) = u_0, \ \underline{u}_{k+1}(0) = u_0. \end{cases}$$

We prove that this method can be extended beyond the linear assumption on A and, which is perhaps more useful, we formulate and prove this principle in the abstract framework of *T-accretive* operators in the Banach lattice $X = L^p(\Omega)$ for some $p \in [1, +\infty]$ or $X = C(\overline{\Omega})$, where Ω is a regular open bounded set of $\mathbb{R}^N$ allowing to get, as applications the case of quasilinear or fully nonlinear parabolic equations. It is also applicable to some multivalued equations, as the obstacle problem (something proposed in Laksmikantham [11]).

Notice that, like in our pseudolinearization principle, the above system of uncoupled equations replace the nonlinear term $F(u(t))$ by linear (zero order terms) as $[(f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)]\overline{u}_{k+1}$ and $[(f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)]\underline{u}_{k+1}$.

As a matter of fact the quasilinearization method was introduced by Bellman in order to get the approximation of the solution of semilinear equations by means of a quadratic (sometimes called as *rapid*) convergence. We can prove something similar for concrete quasilinear equations. Details and further results will be given in Casal and Díaz [7].

2. Abstract results

Given Ω,a regular open bounded set of $\mathbb{R}^N$, we shall consider the abstract Cauchy problem (ACP) in the Banach lattice $X = L^p(\Omega)$ for some $p \in [1, +\infty]$ or $X = C(\overline{\Omega})$. The structural assumptions on the operators we shall assume in this section are the following

(H1): $A \in \mathcal{A}_+(\omega : X)$, for some $\omega \in \mathbb{C}$, with $\mathcal{A}_+(\omega : X) = \{A : D(A) \subset X \to \mathcal{P}(X)$,such that $A + \omega I$ is a m-T-accretive operator$\}$. Moreover A satisfies the *property (M)* of Bénilan [2].

(H2): The operator semigroup $T(t) : \overline{D(A)} \to X$, $t \geq 0$, generated by A, is compact (see Vrabie [14]).

(H3): $u_0 \in \overline{D(A)} \cap L^\infty(\Omega)$.

We shall assume that the nonlinear term $F(u)$ is generated through a continuous real function $f : \mathbb{R} \to \mathbb{R}$ satisfying that

(H4): $f = f_{cv} + f_{cn}$ with f_{cv} convex and f_{cn} concave.

Notice that, in contrast with previous works on the quasilinearization process, we do not require any assumption on the linearity of operator A neither on the differentiability of f (in the classical sense).

We define the notion of sub and supersolution of the original abstract Cauchy problem (ACP) : A couple of functions $\underline{u}, \overline{u} \in C([0,T] : X) \cap L^\infty((0,T) \times \Omega)$ are called sub (respect.) supersolutions of (ACP) if there exists $\underline{g}$ (respectively $\overline{g}$) in $L^1(0, T : L^\infty(\Omega))$ with $\underline{g} \leq 0$ (respectively $\overline{g} \geq 0$) such that $\underline{u}, \overline{u}$ are *mild solutions* of the problem

$$\begin{cases} \frac{d\underline{u}}{dt}(t) + A\underline{u}(t) \ni f(\underline{u}) + \underline{g} & \text{in } X, \\ \underline{u}(0) = u_0, \end{cases}$$

(respectively

$$\begin{cases} \frac{d\overline{u}}{dt}(t) + A\overline{u}(t) \ni f(\overline{u}) + \overline{g} & \text{in } X, \\ \overline{u}(0) = u_0, \end{cases}$$

in the case of $\overline{u}$). Notice that here we are identifying the operator $F(u)$ associated to f with the own function $f(u)$. We shall assume

(H5): there exists $\underline{u}, \overline{u}$ sub and super solutions of (ACP).

Finally, as we shall combine some ordering and some approximation arguments we shall need

(H6): the subdifferential operators ∂f_{cv} and $\partial(-f_{cv})$ are bounded on the set $I := [\inf ess_{t\in[0,T],x\in\Omega}\underline{u}(t,x),\ \sup ess_{t\in[0,T],x\in\Omega}\overline{u}(t,x)]$, i.e., $|b| \leq M$ for any $b \in \partial f_{cv}(r)$ or $b \in \partial(-f_{cn})(r)$, for any $r \in I$.

Remark. Since $\underline{u},\overline{u}$ are bounded functions then I is a compact interval of R. Moreover, by using some well known results (see, e.g., Brezis [3]) it shown that assumption (H6) implies the existence of a sequence of auxiliary functions $f^k \in C^2(\mathsf{R})$ such that $f^k = f^k_{cv} + f^k_{cn}$ with $f^k_{cv}, f^k_{cn} \in C^2(\mathsf{R})$, f^k_{cv} convex and f^k_{cn} concave for any $k \in \mathsf{N}$, such that

$$\begin{cases} f^k_{cv} \nearrow f_{cv}, \text{ as } k \to \infty, & \text{uniformly on any compact interval of } I, \\ f^k_{cn} \searrow f_{cn}, \text{ as } k \to \infty, & \text{uniformly on any compact interval of } I, \end{cases} \tag{2}$$

Moreover $\left\|(f^k_{cv})_u(\eta),(f^k_{cn})_u(\eta)\right\|_{L^\infty(0,T:X')} \leq M_k \leq M$, for the same $M > 0$ given in (H6), for any $\eta \in C([0,T]:X)$ such that $\underline{u} \leq \eta \leq \overline{u}$.

Finally, since the main goal is the approximation of the solution we can consider the uniqueness of solution question as an independent goal. So, we shall assume that

(H7): Problem (ACP) has at most one mild solution.

Remark. This can be proved once we assume (H1) and some extra condition on f such as, f is (globally) Lipschitz continuous (Casal and Díaz [7]).

In order to construct the iterative scheme we define the *pseudo linearized (approximated) abstract Cauchy system*

$$(PLACS)_k \begin{cases} \frac{d\overline{u}_{k+1}}{dt} + A\overline{u}_{k+1} \ni f^k(\overline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) & \text{in } X, \\ \frac{d\underline{u}_{k+1}}{dt} + A\underline{u}_{k+1} \ni f^k(\underline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) & \text{in } X, \\ \overline{u}_{k+1}(0) = u_0,\ \underline{u}_{k+1}(0) = u_0. \end{cases}$$

Theorem 1 *Assume (H1)-(H7). Then, for any $k \in \mathsf{N}$ there exists $(\underline{u}_k,\overline{u}_k) \in L^\infty((0,T)\times\Omega)^2$ mild solutions of the system $(PLACS)$ and with $(\underline{u}_1,\overline{u}_1) = (\underline{u},\overline{u})$. Moreover, the sequences $\{\underline{u}_k\},\{\overline{u}_k\}$ converge in $C([0,T]:X)$ to $u \in L^\infty((0,T)\times\Omega)$ (unique) mild solution of (ACP) and we have that $\underline{u} \leq u \leq \overline{u}$.*

We shall prove the result in several steps. i) *Existence of* $(\underline{u}_2,\overline{u}_2)$.The $(PLACS)_2$ is given by

$$(PLACS)_2 \begin{cases} \frac{d\overline{u}_2}{dt} + A\overline{u}_2 \ni f^1(\overline{u}) + (f^1_{cv})_u(\underline{u})(\overline{u}_2 - \overline{u}) + (f^1_{cn})_u(\overline{u})(\overline{u}_2 - \overline{u}) & \text{in } X, \\ \frac{d\underline{u}_2}{dt} + A\underline{u}_2 \ni f^1(\underline{u}) + (f^1_{cv})_u(\underline{u})(\underline{u}_2 - \underline{u}) + (f^1_{cn})_u(\overline{u})(\underline{u}_2 - \underline{u}) & \text{in } X, \\ \overline{u}_2(0) = u_0,\ \underline{u}_2(0) = u_0. \end{cases}$$

The existence (and uniqueness) of solution of this uncoupled system comes from the fact that $A \in \mathcal{A}_+(\omega : X)$, and that $f^1(\overline{u}) - (f^1_{cv})_u(\underline{u})\overline{u} - (f^1_{cn})_u(\overline{u})\overline{u},\ f^1(\underline{u}) - (f^1_{cv})_u(\underline{u})\underline{u} - (f^1_{cn})_u(\overline{u})\underline{u} \in L^1(0,T:X)$ (recall that $\overline{u},\underline{u}$ are bounded and that f^1 is continuous in R).

ii) *Estimates on* $[\overline{u}_2 - \overline{u}]_+$ *and* $[\underline{u} - \underline{u}_2]_+$. By construction we get that

$$\frac{d(\overline{u}_2 - \overline{u})}{dt} + A\overline{u}_2 - A\overline{u} \ni a_1(t,x)(\overline{u}_2 - \overline{u}) - \overline{g} + f^1(\overline{u}) - f(\overline{u}),$$

where $a_1(t,x) = (f^1_{cv})_u(\underline{u}(t,x)) + (f^1_{cn})_u(\overline{u}(t,x))$ and so, $\|a_1\|_{L^\infty((0,T)\times\Omega)} \leq M_1$. Since $\overline{g} \geq 0$ and $A + \omega I$ is a T-accretive operator we get the estimates

$$\text{máx}_{t\in[0,T]} \left\|[\overline{u}_2(t,.) - \overline{u}(t,.)]_+\right\|_{L^\infty(\Omega)} \leq e^{(\omega+M_1)T} T\,|\Omega| \left\|\left[f^1 - f\right]_+\right\|_{C(I)},$$
$$\text{máx}_{t\in[0,T]} \left\|[\overline{u}_2(t,.) - \overline{u}(t,.)]_-\right\|_{L^\infty(\Omega)} \leq e^{(\omega+M_1)T} (T\,|\Omega| \left\|\left[f^1 - f\right]_-\right\|_{C(I)} + \|\overline{g}\|_{L^1(0,T:L^\infty(\Omega))}).$$

The proof of the existence of $\underline{u}_2$ is analogous. In that case we get the estimates

$$\text{máx}_{t\in[0,T]} \left\|[\underline{u}(t,.) - \underline{u}_2(t,.)]_+\right\|_{L^\infty(\Omega)} \leq e^{(\omega+M_1)T} T\,|\Omega| \left\|\left[f - f^1\right]_+\right\|_{C(I)},$$
$$\text{máx}_{t\in[0,T]} \left\|[\underline{u}(t,.) - \underline{u}_2(t,.)]_-\right\|_{L^\infty(\Omega)} \leq e^{(\omega+M_1)T} (T\,|\Omega| \left\|\left[f^1 - f\right]_-\right\|_{C(I)} + \|\underline{g}\|_{L^1(0,T:L^\infty(\Omega))}).$$

Remark. If no regularization is needed, and so $f^k = f$, then we get that $\underline{u} \leq \underline{u}_2$ and $\overline{u}_2 \leq \overline{u}$ (Casal and Díaz [7]).

iii) *Proof of the inequality* $\underline{u}_2 \leq \overline{u}_2$. We have that

$$\frac{d(\overline{u}_2 - \underline{u}_2)}{dt} + A\overline{u}_2 - A\underline{u}_2 \ni [(f^1_{cv})_u(\underline{u}) + (f^1_{cn})_u(\overline{u})](\overline{u}_2 - \underline{u}_2) - F_1$$

with $F_1 = f^1(\overline{u}) - f^1(\underline{u}) + (f^1_{cv})_u(\underline{u})(\underline{u} - \overline{u}) + (f^1_{cn})_u(\overline{u})(\underline{u} - \overline{u})$. But, from the convexity of f^1_{cv} we get that for any $u, v \in I$, $f^1_{cv}(u) \geq f^1_{cv}(v) + (f^1_{cv})_u(v)(u - v)$. Analogously, the concavity of f^1_{cn} implies, for any $u, v \in I$, that $f^1_{cn}(u) \geq f^1_{cn}(v) + (f^1_{cn})_u(u)(u - v)$. Both properties imply that $F_1 \geq 0$ and so, by the T-accretiveness we get the conclusion.
iv) *Existence of* $(\underline{u}_k, \overline{u}_k)$ for $k \in \mathsf{N}$, $k > 1$. It is analogous to the above step. For instance, the forcing term (independent on $\overline{u}_{k+1}$) is now $f^k(\overline{u}_k) - (f^k_{cv})_u(\underline{u}_k)\overline{u}_k - (f^k_{cn})_u(\overline{u}_k)\overline{u}_k$ which, again, is in $L^\infty((0,T) \times \Omega)$.
v) *Estimates on* $[\overline{u}_{k+1} - \overline{u}_k]_+$ *and* $\left[\underline{u}_k - \underline{u}_{k+1}\right]_+$. By construction and (H6) we get that

$$\frac{d(\overline{u}_{k+1} - \overline{u}_k)}{dt} + A\overline{u}_{k+1} - A\overline{u}_k \ni a_k(t,x)(\overline{u}_{k+1} - \overline{u}_k) + F_k$$

with $a_k(t,x) = (f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)$ (and so $\|a_k\|_{L^\infty((0,T)\times\Omega)} \leq M_k$) and $F_k = f^k(\overline{u}_k) - f^{k-1}(\overline{u}_{k-1}) + a_{k-1}(t,x)(\overline{u}_k - \overline{u}_{k-1})$. So, using the convexity of f^k_{cv} and the concavity of f^k_{cn} we get that $F_k \geq f^k(\overline{u}_k) - f^{k-1}(\overline{u}_k)$. Thus, by the T-accretiveness of A, we get that

$$\begin{aligned} \text{máx}_{t\in[0,T]} \left\| [\overline{u}_{k+1}(t,.) - \overline{u}_k(t,.)]_+ \right\|_{L^\infty(\Omega)} &\leq e^{(\omega+M_1)T} T\,|\Omega| \left\| \left[f^k - f^{k-1}\right]_+ \right\|_{C(I)}, \\ \text{máx}_{t\in[0,T]} \left\| \left[\underline{u}_k(t,.) - \underline{u}_{k+1}(t,.)\right]_+ \right\|_{L^\infty(\Omega)} &\leq e^{(\omega+M_1)T} T\,|\Omega| \left\| \left[f^{k-1} - f^k\right]_+ \right\|_{C(I)}. \end{aligned}$$

Remark. If no regularization is needed and so $f^k = f^{k-1}$ then we get that $\underline{u}_k \leq \underline{u}_{k+1}$ and that $\overline{u}_k \leq \overline{u}_{k+1}$ (Casal and Díaz [7]).
vi) *Proof of the inequality* $\underline{u}_{k+1} \leq \overline{u}_{k+1}$. As in step iii) we have

$$\begin{gathered} \tfrac{d(\overline{u}_{k+1} - \underline{u}_{k+1})}{dt} + A\overline{u}_{k+1} - A\underline{u}_{k+1} \ni [(f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)](\overline{u}_{k+1} - \underline{u}_{k+1}) - F_k, \\ F_k = f^k(\overline{u}_k) - f^k(\underline{u}_k) + [(f^k_{cv})_u(\underline{u}_{k-1}) + (f^1_{cn})_u(\overline{u}_{k-1})](\underline{u}_k - \overline{u}_{k-1}). \end{gathered}$$

Using the convexity of f^k_{cv} and the concavity of f^k_{cn} we have that $F_k \geq 0$ and so, by the T-accretiveness, we get the wanted comparison.
End of the proof of Theorem 1. The sequences $\{\underline{u}_k\}$ and $\{\overline{u}_k\}$ are uniformly bounded in $L^\infty((0,T)\times\Omega)$. In consequence, using assumption (H6), the sequences $\{f^k(\overline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1} - \overline{u}_k)\}$ and $\{f^k(\underline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1} - \underline{u}_k)\}$ are also uniformly bounded in $L^\infty((0,T)\times\Omega)$. Thus, by the assumption (H2) there exists $\underline{U}, \overline{U}$ such that $\{\underline{u}_k\} \to \underline{U}$ and $\{\overline{u}_k\} \to \overline{U}$ (strongly) in $C([0,T] : X)$ (and, at least, weakly in $L^\infty((0,T)\times\Omega)$). Finally, $\{f^k(\overline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1} - \overline{u}_k)\} \to f(\overline{U})$ and $\{f^k(\underline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1} - \underline{u}_k)\} \to f(\underline{U})$ and so $\underline{U}, \overline{U} \in L^\infty((0,T)\times\Omega)^2$ are mild solutions of (ACP) which must coincide due the assumption (H7). From steps i)-vi), passing to the limit, we get that $\underline{u} \leq u \leq \overline{u}$.¥
Remark. If no regularization is needed, and so $f^k = f$, then it is easy to see (Casal and Díaz [7]) that $\underline{u} \leq \underline{u}_2 \leq ... \leq \underline{u}_k \leq ... \leq u \leq ... \leq \overline{u}_k \leq \overline{u}_2 \leq \overline{u}$.
Remark. Some abstract results of a difference nature (in which A is a regular function, and so, of difficult application to PDEs) can be found in Section 4.6 of the book [12] (see also the references cited there).
Remark. More general functions $f(t,x,u)$ can be also considered (Casal and Díaz [7]). For instance, it is possible to improve Theorem 1 by assuming the existence of ϕ convex and ψ concave such that $f = f_{cv} + f_{cn}$ with $f_{cv} + \phi$ convex and $f_{cn} + \psi$ concave. In that case, we assume merely that $A + \phi + \psi$ is a m-T-accretive operator on X. The method can be applied to systems of nonlinear PDEs as well as to PDEs with some delayed terms (see some iterative schemes in Pao [13], Casal, Díaz and Stich [8] and Casal and Díaz [7]).

Many different examples are possible ([7]). For instance, we can consider

Example 1. $A : D(A) \to \mathcal{P}(L^1(\Omega))$ given by $Au = -div(|\nabla\phi(u)|^{p-2}\nabla\phi(u)) + \beta(u)$ with $D(A) = \{\phi(u) \in W^{1,1}(\Omega),\ u(x) \in D(\beta)$, a.e. $x \in \Omega$, $Au \in L^1(\Omega)$, $-\left|\frac{\partial\phi(u)}{\partial n}\right|^{p-2}\frac{\partial\phi(u)}{\partial n} \in \gamma(\phi(u))$ on $\partial\Omega\}$ where $p > 1$, ϕ is continuous and increasing, and β, γ are maximal monotone graphs of R^2 (not necessarily associated to differentiable functions). We recall that the obstacle problem can be formulated in this framework by taking as β the maximal monotone graph of R^2 given by $\beta(r) = \phi$ (the empty set) if $r < 0$, $\beta(0) = (-\infty, 0]$ and $\beta(r) = \{0\}$ if $r > 0$ (see, e.g. Díaz [9]).

Example 2. Many different operators A, satisfying the above conditions, can be taken on the space $X = C(\overline{\Omega})$. A first class of operators concerns the operator in divergence form given in Example 1 when we assume that $\phi(u) = u$. Another class of operators concerns some fully non linear operators of the type $A : D(A) \to C(\overline{\Omega})$ with $Au = \sigma(-\Delta u)$ with $D(A) = \{u \in C^2(\overline{\Omega}) : \sigma(-\Delta u) \in C^2(\overline{\Omega}),\ u = 0 \text{ on } \partial\Omega\}$ where σ is a continuous strictly increasing function (see, e.g. Díaz [9]).

One of the reasons argued by Bellman to introducing this method for semilinear equations is the quadratic (sometimes called as *rapid*) convergence. We can prove something similar for concrete quasilinear equations:

Theorem 2 *([7]). Let A be as in the Example 1 with $p \geq 2$, $\phi(u) = u$, $\beta = 0$ and γ corresponding to Dirichlet boundary conditions and assume (H3)-(H7). Then*

$$\begin{cases} \text{máx}_{t\in[0.T]} \|u(t) - \overline{u}_k(t)\|^2_{L^2(\Omega)} + \|u - \overline{u}_k\|^p_{L^p(0,T:W_0^{1,p}(\Omega)} \leq \\ \quad C(\|u - \overline{u}_{k-1}\|^2_{L^2((0,T)\times\Omega)} + \|u - \underline{u}_{k-1}\|^2_{L^2((0,T)\times\Omega)}), \\ \text{máx}_{t\in[0.T]} \|u(t) - \underline{u}_k(t)\|^2_{L^2(\Omega)} + \|u - \underline{u}_k\|^p_{L^p(0,T:W_0^{1,p}(\Omega)} \leq \\ \quad C(\|u - \overline{u}_{k-1}\|^2_{L^2((0,T)\times\Omega)} + \|u - \underline{u}_{k-1}\|^2_{L^2((0,T)\times\Omega)}). \end{cases}$$

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