On the Burgers' equation in unbounded domains without condition at infinity

J.I. Díaz, S. González

Depto. de Matemática Aplicada, Universidad Complutense de Madrid

ildefonso.diaz@ucm.es, sogonzal@mat.ucm.es

1. Introduction

It is well known that Burgers' equation plays a relevant role in many different areas of the mathematical physics, specially in Fluid Mechanics. Moreover the simplicity of its formulation, in contrast with the Navier-Stokes system, make of the Burgers' equation a suitable model equation to test different numerical algorithms and results of a varied nature.

The main goal of this communication is to put in evidence a property which seems to have been unobserved until now: *there is no limitation on the growth of the nonnegative initial datum* $u_0(x)$ *at infinity when the problem is formulated on unbounded intervals, as, e.g.* $(0+\infty)$, *and the solution is unique without prescribing its behaviour at infinity.* A related property was used in [7] for the study of the controllability question for this equation. This property contrast with the pioneering results by A. N. Tychonov (1935) for the linear heat equation and its more recent generalizations by many authors. We prove that the property requires the nonnegativeness of the initial datum u_0. This contrasts also with the results on existence of solutions without growth conditions at infinity in the literature dealing with other classes of nonlinear parabolic and elliptic equations ([11], [5], [2], [9], [6], [3], [12],...).

More precisely, in this first presentation of a longer work ([8]), given $u_0 \in L^1_{loc}(0,+\infty)$, $u_0(x) \geq 0$ a.e. $x \in (0,+\infty)$, we consider the parabolic Burgers' problem

$$(PBP)\begin{cases} u_t - u_{xx} + uu_x = 0 & x \in (0,+\infty),\ t>0, \\ u(0,t)=0,\ \liminf_{x\to\infty} u(t,x) \geq 0 & t>0, \\ u(x,0)=u_0(x) & \text{on } (0,+\infty), \end{cases} \tag{1}$$

as well as the associated (for instance via implicit time discretization) elliptic Burger problem

$$(EBP)\begin{cases} -u_{xx} + uu_x + \lambda u = f(x) & x \in (0,+\infty), \\ u(0)=0,\ \liminf_{x\to\infty} u(x) \geq 0 \end{cases}$$

with $\lambda \geq 0$ and $f \in L^1_{loc}(0,+\infty)$ ($f(x) \geq 0$ a.e. $x \in (0,+\infty)$).

Several generalizations of different nature are possible. Finally, our results lead to some new (as far as we know) properties for the linear heat equation $v_t - v_{xx} = 0$ with Robin type boundary conditions at $x=0$: starting without any limitation on the growth rate v_x/v at $t=0$ a growth estimate (in terms of x/t) holds for any $t>0$.

2. The elliptic problem

The main goal of this Section is to prove the existence and uniqueness of the solution of the elliptic boundary value problem

$$(EBP)\begin{cases} -u_{xx} + uu_x + \lambda u = f(x) & x \in (0,+\infty), \\ u(0)=0,\ \liminf_{x\to\infty} u(x) \geq 0 \end{cases}$$

with $\lambda \geq 0$ and

$$f \in L^1_{loc}(0,+\infty) \text{ and } f(x) \geq 0 \ \text{ a.e. } x \in (0,+\infty). \tag{2}$$

This problem is sometimes named as the elliptic Burgers-Sivashinsky problem (see Brauner [4]). Notice that we do not make any limitation on the possible growth of the solution at the infinity since, as we shall show, it is determined by the behaviour of the datum $f(x)$ when $x \to +\infty$.

We start by defining the notion of weak solution.

Definition 1 *A function* $u \in L^2_{loc}(0,+\infty)$ *is a "very weak solution" of problem EBP if* $\liminf_{x\to\infty} u(x) \geq 0$ *and*

$$\int_0^\infty (-u\zeta_{xx} - \frac{u^2}{2}\zeta_x + \lambda u\zeta)dx = \int_0^\infty f\zeta dx \quad \forall \zeta \in W^{2,\infty}(0,+\infty) \text{ with compact support.}$$

It is easy to see that any very weak solution u must satisfy some additional regularity. For instance, necessarily $u \in C[0,+\infty)$, $u(0) = 0$ and u is a *strong solution* in the sense that $(u_x - (\frac{u^2}{2}))_x \in L^1_{loc}(0,+\infty)$. Moreover, since $f(x) \geq 0$ a.e. $x \in (0,+\infty)$ (and $\liminf_{x\to\infty} u(x) \geq 0$) we get $u(x) \geq 0$ for any $x \in (0,+\infty)$.

The main result of this section is the following

Theorem 1. *Assume* (2). *Then for any* $\lambda \geq 0$ *there exists a unique very weak solution* u *of* (EBP).

The proof of the theorem will be divided in different steps. Let us start by proving the existence of a very weak solution. This can be carried out in several different ways. The common idea is to consider, as a previous problem, the boundary value problem on a bounded domain $(0,n)$ or to approximate f by f_n (for instance with an expanding sequence of compact supports) and then to prove that the sequence of the associate solutions $\{u_n\}$ converges, as $n \to +\infty$, to a very weak solution u of (EBP). Here we shall follow a methodology which seems to be quite general and allows to connect two qualitative properties apparently disconnected: the existence of the so called *large solutions* on bounded domains and the existence of solutions on unbounded domains *without prescribing the behaviour at infinity.* To be more precise, as we shall see later (and as in Díaz and Oleinik [9]), it will be useful to work with a slightly more general framework (in particular to get an easy proof of the uniqueness of solutions of (EBP)). Let $n > 0$. Given

$$A \in L^\infty_{loc}(0,n),\ A(x) \geq A_0 \ \text{ a.e. } x \in (0,n), \text{ for some } A_0 > 0, \tag{3}$$

we shall prove a *localizing property* which contain some similitudes with the one used as key idea in the pioneering paper by Brezis [5] on the study of semilinear equations in $\mathbb{R}^N$ but with an entirely different proof.

Lemma 1. *Let* $n > 0$ *and* $f \in L^1(0,n)$, $f \geq 0$, *a.e. on* $(0,n)$. *Let* $u \in L^2_{loc}(0,n)$, $u \geq 0$, *satisfying*

$$(EBP)_n \begin{cases} -u_{xx} + (A(x)u^2)_x + \lambda u = f(x) & x \in (0,n), \\ u(0) = 0 \end{cases}$$

in the very weak sense that

$$\int_0^n (-u\zeta_{xx} - A(x)u^2\zeta_x + \lambda u\zeta)dx = \int_0^n f\zeta dx \quad \forall \zeta \in W^{2,\infty}(0,n) \text{ with } \zeta(n) = 0.$$

For any $k \geq 0$, *let* $\psi_k : (k,+\infty) \to (0,+\infty)$ *the function defined by* $\psi_k(s) = \int_s^{+\infty} \frac{dr}{A_0 r^2 - k}$. *Then, if* $k = \|f\|_{L^1(0,n)}$, *we get*

$$u(x) \leq \frac{1}{n}\Big(n\sqrt{\frac{k}{A_0}}\chi_{\{x\in(0,n):u(x)\leq n\sqrt{\frac{k}{A_0}}\}} + (\psi_{n^2k})^{-1}(1-x)\chi_{\{x\in(0,n):u(x)>n\sqrt{\frac{k}{A_0}}\}}\Big) \text{ a.e. } x \in (0,n). \tag{4}$$

Moreover, if $\liminf_{x\to n} u(x) \geq 0$ *then* $u(x) \geq 0$ *a.e.* $x \in (0,n)$

Proof. Assume for the moment that $n = 1$. Integrating from 0 to x at the equation we get that $-u_x(x) + A_0u(x)^2 \leq k$, where we use the fact that $u(x) \geq 0$, that $u_x(0) \geq 0$ and (3). Then, if v satisfies

$$\begin{cases} -v_x(x) + A_0v(x)^2 = k, \\ v(1) = +\infty, \end{cases} \tag{5}$$

we deduce that $u(x) \leq v(x)$ at least on the set where $v(x) \geq \sqrt{\frac{k}{A_0}}$. Since the above equation has separated variables, the (unique) solution of (5) is given by $v(x) = (\psi_k)^{-1}(1-x)$ and we get the estimate on the set $(x_k, 1)$ where $x_k \in [0,1)$ is such that $(\psi_{k,1})^{-1}(x_k) \geq \sqrt{\frac{k}{A_0}}$. Since, necessarily $\{x \in (0,n) : u(x) > \sqrt{\frac{k}{A_0}}\} \subset (x_k, 1)$ we get the conclusion. Once proved (4) for $n = 1$, we introduce the change of variable $x = nx'$ and the change of unknown $u(x) = hw(nx')$. Then, if u satisfies $(EBP)_n$ and we take $h = 1/n$ we get that w satisfies $(EBP)_1$ but replacing λ by λn^2 and f by

$n^3 f(nx')$. Since $\int_0^1 f(nx')dx' = \frac{1}{n}\int_0^n f(x)dx$ we get the conclusion through the proof of the case $n = 1$. The nonnegativeness of u, once we assume that $\liminf_{x\to n} u(x) \geq 0$ is consequence of the maximu principle. ■

Remark 1. Notice that the first estimate does not require any information on the (nonnegative) boundary value $u(n)$ and that the dependence on f is merely through the global information given by $f \geq 0$ and $\|f\|_{L^1(0,n)}$. Moreover, as in [9], the estimate allow to get some result on the asymptotic behaviour of solutions when $x \to +\infty$ (see [8] for more details).
Proof of the existence of solutions of Theorem 1. The previous a priori estimate can be used in different ways to get the existence of a weak solution of the statement. Here we shall consider the auxiliary problem

$$(EBP_\infty^n)\begin{cases} -u_{xx} + (A(x)u^2)_x + \lambda u = f(x) & \text{in } (0,n), \\ u(0) = 0,\ u(n) = +\infty. \end{cases}$$

In general, (weak) solutions of quasilinear elliptic equations satisfying the boundary condition $u(n) = +\infty$ (in some appropriate sense) are usually called as "large solutions". Some modifications of the proof of the above lemma lead to

Lemma 2. ([8]). *Assume* (2) *and* (3). *Then for any* $\lambda \geq 0$ *there exists at least a weak solution* u *of* (EBP_∞^n).■

Then, if we consider the sequence $\{u_n\}$ formed by solutions u_n of (EBP_∞^n) we get that $\{u_n\}$ is decreasing with n (in the sense that $u_{n-1}(x) \leq u_n(x)$ a.e. $x \in (0,n)$). It is now an easy task to prove that the function $u(x)$ defined through the pointwise limit of $\{u_n(x)\}$ is a weak solution of (EBP) (see [8] for details).
Proof of the uniqueness of solutions of Theorem 1. We follow some arguments introduced in [9] for other superlinear problem. Let u_1, u_2 be two possible (nonnegative) weak solutions of (EBP). Let $v = u_1 - u_2$. Then $v(0) = 0$ and v satisfies $-v_{xx} + (A(x)v^2)_x + \lambda v = 0$ on $(0, +\infty)$ with

$$A(x) = \frac{u_1(x)^2 - u_2(x)^2}{(u_1(x) - u_2(x))^2}.$$

Notice that such a function satisfies (3) for any $A_0 \in (0,1)$. Then we can apply estimate (4) with $k = 0$. Since $\psi_0(s) = \int_s^{+\infty} \frac{dr}{A_0 r^2} = \frac{1}{A_0 s}$ for any $s > 0$, we get that

$$0 \leq v(x) \leq \frac{1}{n}\left(\frac{1}{A_0(1-x)}\right) \text{ a.e. } x \in (0,n).$$

Finally, since n is here arbitrary, we get that $v \equiv 0$.■

3. On the parabolic problem and applications to the linear heat equation

The arguments for the elliptic problem can be adapted (in different ways) to be applied to the parabolic problem. Nevertheless other points of view are also possible. We start with some technical results which show (by some easy computations) the existence of some *universal solutions* (see Bandle, G. Díaz and J. I. Díaz [1] for other superlinear problems).

Lemma 3. *The function* $U^*(x,t) = \frac{x}{t}$ *is an universal solution of* (PBP) *in the sense that*

$$\begin{cases} U_t^* - U_{xx}^* + U^* U_x^* = 0 & x \in (0,\infty),\ t > 0, \\ U^*(0,t) = 0,\ U^*(x,t) \to +\infty \text{ as } x \to +\infty & t > 0, \\ U^*(x,0) = +\infty & \text{on } (0,+\infty). \end{cases}$$

In particular, given $T > 0$ *and* $n > 0$ *arbitrarily and given* $u_0 \in L^1_{loc}(0,n)$ *and* $q \in C(0,T)$ *with* $u_0 \geq 0$ *a.e. on* $(0,n)$ *and* $0 \leq q(t) \leq \frac{t}{n}$ *for any* $t \in (0,T)$, *then any weak solution* u *of the problem*

$$(PBP)_{n,q}\begin{cases} u_t - u_{xx} + uu_x = 0 & x \in (0,n),\ t \in (0,T), \\ u(0,t) = 0,\ u(t,n) = q(t), & t \in (0,T), \\ u(x,0) = u_0(x) & on\ (0,n), \end{cases}$$

satisfies

$$0 \leq u(x,t) \leq \frac{x}{t} \text{ on } (0,n)\times(0,T).■ \tag{6}$$

Lemma 4. *Given $n > 0$ arbitrarily, the function $U^*(x,t) = \frac{x}{t} + \frac{2}{n-x}$ satisfies*

$$\begin{cases} U_t^* - U_{xx}^* + U^* U_x^* \geq 0 & x \in (0,n), \ t > 0, \\ U^*(0,t) = \frac{2}{n}, \ U^*(n,t) = +\infty & t > 0, \\ U^*(x,0) = +\infty & on \ (0,n). \end{cases}$$

In particular, given $T > 0$ and $n > 0$ arbitrarily and given $u_0 \in L^1_{loc}(0,n)$ and $q \in C(0,T)$ with $u_0 \geq 0$, a.e. on $(0,n)$ and $q(t) \geq 0$ for any $t \in (0,T)$ then any weak solution u of the problem $(PBP)_{n,q}$ satisfies

$$0 \leq u(x,t) \leq \frac{x}{t} + \frac{2}{n-x} \quad \text{on } (0,n) \times (0,T). \blacksquare \tag{7}$$

Remark 2. In a recent paper ([14]) K. Yamada studies the Cauchy problem on $\mathbb{R}^N$ associated to the equation $u_t - \Delta u + div\mathbf{G}(u) = 0$ for some vectorial functions such that $|\mathbf{G}'(u)| \leq C$ and under the growth $|u_0(x)| \leq c\,|x|$ for $|x| \to \infty$ (the case of u_0 without any limitation on the growth is not considered there).

By using some not difficult arguments it is possible to construct a monotone sequence of approximate solutions $\{u_n\}$ which, passing to the limit, leads to the following result:

Theorem 2 ([8]). *Assume $u_0 \in L^1_{loc}(0,+\infty)$ and $u_0(x) \geq 0$ a.e. $x \in (0,+\infty)$. Then there exists a very weak solution u of (PBP). Moreover, if $u_0 \in L^2_{loc}(0,+\infty)$ the solution u belongs to $C([0,T] : L^2_{loc}(0,+\infty))$, for any $T > 0$, and it is the unique solution in this class of functions.*

Idea of the proof of the uniqueness. We can prove that, for given $T > 0$, $k \geq 0$ and $N > 2 + k$, there exist a positive constant K such that

$$\frac{d}{dt} \int_0^n (n-x)^N u(x,t)^{k+1} dx \leq K n^{N+2-k}. \tag{8}$$

This show the equicontinuity in the approximating arguments and, by Ascoli-Arzela theorem, it leads to that the limit function u belons to $C([0,T] : L^2_{loc}(0,+\infty))$. Now, due to the estimate (6) we can argue as in the proof of Theorem 4.3 of ([14]) and using the equicontinuity $C([0,T] : L^2_{loc}(0,+\infty))$ we get the result.$\blacksquare$

Remark 3. It is easy to construct counterexamples showing that the condition $u_0(x) \geq 0$ and/or the fact that the spatial domain is bounded from above, as it is the case of $(0,+\infty)$, are, in some sense, necessary conditions to get the conclusion of Theorem 1 (for instance, $u(x,t) = (-x)/(1-t)$ is a solution of the equation). Similar counterexamples can be contructed for the elliptic problem (use a time implicit discretization of the above counterexample).

Remark 4. Although estimates (6) and (7) are universal (i.e. independent of any $u_0 \in L^1_{loc}(0,+\infty)$), it is possible to get some of localizing estimates (in the spirit of the ones obtained for the ellliptic problem). For instance, from (8), for given $T > 0$ there exists two positive constants C_1, C_2 such that any solution u of the problem $(PBP)_{n,q}$ must satisfy the estimate

$$\int_0^{n/2} u(x,t)^{k+1} dx \leq C_1 \int_0^n u_0(x)^{k+1} dx + \frac{C_2}{n^{k-2}} t, \text{ for any } t \in (0,T), \tag{9}$$

for any $n > 0$, any $u_0 \in L^{k+1}_{loc}(0,+\infty)$ and any $q \in C(0,T)$ with $u_0 \geq 0$ a.e. on $(0,n)$ and $q(t) \geq 0$ for any $t \in (0,T)$.

Remark 5. Since, for any $k > 0$ the function $U^{\#}(x,t) = \frac{x}{t} + k$ is a supersolution of the non homogeneous problem

$$(PBP:k) \begin{cases} u_t - u_{xx} + u u_x = 0 & x \in (0,+\infty), \ t \in (0,T), \\ u(0,t) = k, \ \text{lím inf}_{x\to\infty} u(t,x) \geq 0 & t \in (0,T), \\ u(x,0) = u_0(x) & on \ (0,+\infty), \end{cases}$$

Theorem 2 can be extended to this problem.

Finally, the above theorem leads to some new (as far as we know) properties for the linear heat equation with Robin boundary conditions

$$(LHE:m) \begin{cases} v_t - v_{xx} = 0 & x \in (0,+\infty), \ t > 0, \\ v_x(t,0) + m v(t,0) = 0, & t > 0, \\ v(x,0) = v_0(x) & \text{on } (0,+\infty). \end{cases} \tag{10}$$

Indeed, by the Hopf-Cole transformation (see, e.g., Witham [13]) we know that the solutions of (1) and (10) can be connected by the expression

$$u(t,x) = -\frac{2v_x(t,x)}{v(t,x)}.$$

So, we get as corollary a result proving that there is not any limitation on the growth rate v_x/v at the infinity.

Corollary 1. *Let $m \geq 0$ and let $v_0 \in W^{1,1}_{loc}(0,+\infty)$ such that $\frac{v_{0x}}{v_0} \in L^1_{loc}(0,+\infty)$, $|v_0(x)| \leq Ce^{x^2}$ for some $C > 0$ when $x \to \infty$, $\frac{v_{0x}(x)}{v_0(x)} \leq 0$ a.e. $x \in (0,+\infty)$ and such that $\liminf_{x\to\infty} \frac{v_x(t,x)}{v(t,x)} \leq 0$ for any $t \geq 0$, where v is the solution of (10).Then necessarily $\frac{2v_x(t,x)}{v(t,x)} \geq -\frac{x}{t} - 2m$ for any $t > 0$ and any $x \in (0,+\infty)$.*$\blacksquare$

Remark 6. It seems possible to generalize some of the results of this communication to more general equations as, for instance, $u_t - (u^m)_{xx} + (u^\lambda)_x = 0$, once we assume $\lambda > m \geq 0$.

Acknowledgements: Partially supported by the project MTM2004–07590–C03–01 of the DGIS-GPI (Spain). J.I. Díaz is member of the RTN HPRN–CT–2002–00274 of the EC.

Palabras clave: Burgers' equation, unbounded domains, no growth conditions
Clasificación para el CEDYA 2005: EDPs

Bibliografía

[1] C. Bandle, G. Díaz and J. I. Díaz, Solutions d'equations de reaction diffusion nonlineaires, exploxant au bord parabolic.*Comptes Rendus Acad. Sci. Paris*, **318**, Série I (1994), 455 460.

[2] F. Bernis, Elliptic and parabolic semilinear problems without conditions at infinity, *Archive for Rational Mechanics and Analysis,* Vol. **106**, n^{o}3, 1989, 217-241.

[3] L. Boccardo, T. Gallouët and J. L. Vázquez, Solutions of nonlinear parabolic equations without growth restrictions on the data, *Electronic Journal Differential Equations,* Vol. 2001, **2001**. (URL http://ejde.math.swt.edu o http://ejde.math.unt.edu)

[4] C.-M. Brauner, Some models of cellular flames: Lecture at the Summer School "Fronts and singularities: mathematics for other sciences" El Escorial, Madrid. July 11-15, 2005

[5] H. Brézis, Semilinear equations in $\mathbb{R}^N$ without condition at infinity, *Appl. Math. Optim.* ***12**, n^{o}3,* 1984, 271-282

[6] G. Díaz and R. Letelier, Explosive solutions of quasilinear elliptic equations: Existence and uniqueness, *Nonlinear Anal. T.M.A.*, **20** (1993), 97-125.

[7] J. I. Díaz, Sobre la controlabilidad aproximada de problemas no lineales disipativos. In *Jornadas Hispano Francesas sobre Control de Sistemas Distribuidos*. Univ. de Málaga, 1991, 41-49.

[8] J. I. Díaz and S. González. Paper in preparation.

[9] J. I. Díaz and O. A. Oleinik, Nonlinear elliptic boundary value problems in unbounded domains and the asymptotic behaviour of its solutions. *Comptes Rendus Acad. Sci. Paris*,t. **315**, Série I, 1992, 787-792.

[10] J. I. Díaz, Obstruction and some approximate controllability results for the Burgers equation and related problems. In *Control of partial differential equations and applications* (E. Casas ed.), Lecture Notes in Pure and Applied Mathematics, Vol. 174, Marcel Dekker, Inc., New York , 1995, 63-76.

[11] M. Á. Herrero and M. Pierre, Some results for the Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$, *Trans. A. M. S.* ***291**,* 1985, 145-158.

[12] A. Porreta, Uniqueness of solutions of some elliptic equations without condition at infinity, *C. R. Acad. Sci. Paris,* Ser. I **335**, 2002, 739-744.

[13] G. B. Witham, *Linear and nonlinear waves*, Wiley Interscience, New York, 1974.

[14] K. Yamada, On viscous conservation laws growing initial data, *Differential and Integral Equations*, **18,** 8 (2005), 841-854.