On slender shells and related problems suggested by Torroja's structures

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Abstract

We study the rigidification phenomenon for several thin slender bodies or shells, with a small curvature in the transversal direction to the main length, for which the propagation of singularities through the characteristics is of parabolic type. The asymptotic behavior is obtained starting with the two-dimensional Love - Kirchoff theory of plates. We consider, in a progresive study, a starting basic geometry, we pass then to consider the "V-shaped" structure formed by two slender plates pasted together along two long edges forming a small angle between their planes and, finally, we analyze the periodic extension to a infinite slab. We introduce a scalar potential φ and prove that the equation and constrains satisfied by the limit displacements are equivalent to a parabolic higher order equation for φ. We get some global informations on φ, some on them easely associated to the different momenta and others of a different nature. Finally, we study the associate obstacle problem and obtain a global comparison result between the third component of the displacements with and without obstacle.

Key words: Thin shells, V-shaped structures, asymptotic behavior, scalar potential, parabolic higher order equations, one-side problems.

MSC2000: 74K,25, 74Q15, 35J55, 35K25, 35J85.

1 Introduction.

The experience shows that when considering a slender or shell a small curvature in the transversal direction to the main length supply an extra rigidification with respect to the planar case. Think, for instance, about the familiar flexible steel retractable meter tape measure, which enjoys rigidity from its special transversal curvature. Here we shall carry out the study of the asymptotic modelling of such kind of shell structures (see Figure 1 concerning the "basic problem" considered in Section 2). We also will consider more sophisticated structures formed by coupling two of such basic shells by means of an edge with slight folding (see Figure 2 in Section 3), as well as the case of an infinity set of shells obtained by the periodic repetition of the basic structure (see Figure 3 in Section 4). The consideration of this type of periodic structures is motivated by some of the structures designed by the outstanding engineer Eduardo Torroja (Madrid, 1899-1961). For instance, the shell roofs of the Madrid Racecourse (1935) are a brilliant result of the forms of the reinforced concrete consisting of a system of portal frames, spread at 5 m intervals and connected longitudinally by small reinforced concrete double curvature vaults. The cantilever roof, with a minimum thickness of 5 cm, overhangs to a distance of 12,8 m. Although Torroja also produced some theoretical works (see, for instance, [39]), the mentioned structure was calculated by trial and error so as to find out the directions and strengths of the stresses that would occur. Later on, tests were carried out on a full scale module of the roof (quite similar to the coupled shell considered here in Figure 2) and it was loaded to breaking point. The present paper try to carry out a mathematical study of such type structures which with difficulty would be available in the first half of the last century.

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Let us come back to the consideration of the mentioned "basic problem". As in many previous works on shell theory (see, for instance, [4], [32], [33]) we start here from "two-dimensional Love-Kirchhoff plate theory" sometimes called as "Koiter model" of shells (in contrast to studies which start from the "three-dimensional elasticity" as, e.g., [13]). In such approach of thin shells it is known (see [19], [10] as well as [34] and [35] for other related problems) that the propagation of singularities phenomena holds along the characteristics of the, so called, "limit problem" (also called as the associate "membrane problem"). In particular, in the case of developable middle surface (as we shall consider), such a phenomenon appears along the generators of the surface, which behave as "rigidity directions". If the small thickness of the shell is ε, and some loading is applied along a generator (either inside the surface or on its boundary), a special phenomenon of boundary layer appears: deformations and stresses are concentrated along the generator on a layer of thickness $0(\varepsilon^{1/4})$. One may wonder that this elongated region may "live alone" enjoying special rigidity properties. Moreover, that geometric structure may be modeled by a narrow rectangular plate of thickness $0(\varepsilon)$ and width $0(\varepsilon^{1/4})$ (whereas the length is $0(1)$). Such is the type of structure that we shall consider in Section 2. In fact, a slightly more general situation concerning orders of magnitude will be considered but the above mentioned one will be considered as the "typical example". It turns out that the geometric properties of the shell middle surface play a fundamental role separating the treatment in three different cases: elliptic, hyperbolic and parabolic. Here we shall only consider the parabolic case (other situations will be analyzed elsewhere [17]).

In particular, we shall assume that the characteristic directions of the principal curvatures of the middle surface coincide at any point. Following the analogy with layers in shells, we shall perform an asymptotic scaling of the variables and unknowns; the problem then appears in the general framework of singular - stiff problems (see [9]) with certain extra terms of classical singular perturbation. The limit solution is described in terms of an appropriate subspace of more or less classical spaces. We shall prove that, even if the starting system of equations is of elliptic type, the limit problem has rather parabolic character.

As indicated before, the variational formulation of the limit problem involves a constrained energy space for the admissible displacements $\mathbf{v} = (v_1, v_2, v_3)$, so that the (weak) variational formulation associated to the partial differential equation and boundary conditions should exhibits the corresponding Lagrange multipliers terms. This is the reason why we introduce the *scalar potential* φ generating the unknown triplet $\mathbf{u} = (u_1, u_2, u_3)$ for φ in a unconstrained energy space. It is clear that the theoretical variational formulation for φ and $\mathbf{u} = (u_1, u_2, u_3)$ are equivalent but the great advantage is that φ has no constraint and leads to a higher order problem which, if we assume a small normal loading of the type $\mathbf{f} = \varepsilon^3 F_3 \mathbf{e}_3$ vanishing on the boundary of the shell, can be formulated as

$$\frac{\partial^4 \varphi}{\partial y_1^4} + \lambda \frac{\partial^8 \varphi}{\partial y_2^8} = -\frac{\partial^2 F_3}{\partial y_2^2}, \tag{1.1}$$

for some $\lambda > 0$ given through the elastic coefficients of the shell (see (2.99)) and where (y_1, y_2) are the rescaled coordinates $y_1 = x_1$, $y_2 = \eta^{-1} x_2 = \varepsilon^{-1/4} x_2$. Some peculiar consequences arising when $F_3 \mathbf{e}_3$ does not vanish on the boundary of the shell are indicated in Section 2. A curious fact is that in spite of the parabolic nature of equation (1.1), the boundary conditions of the original problem allows to state a variational principle for its solutions (see Remark 2.10).

The introduction of the scalar potential φ (made also in [10]: a research started simultaneously to the present paper) seems to be new in the shell literature. Some closed, but different, ideas can be associated with the *stress function* introduced by G.B. Airy (1801-1892) (see, e.g. [26]) or with the stream function for incompressible planar flows (see, e.g. [5]). Some similar ideas in the context of distributions can be found in [37].

We also point out that in the finite element approximation of the limit problem, any direct algorithm in $\mathbf{u} = (u_1, u_2, u_3)$ should involve the Lagrange multiplier term corresponding to the numerical locking phenomena (see, e.g. [7] and [3]: considerations on adaptive meshes in this connection may be found in [14] and some related alternative methods may be seen in [29], [30]). This situation, which is obviously avoided by considering the potential formulation, classically appears in very many shell problems (see, for instance, [31]).

Section 3 is devoted to the study of a more sophisticated structure obtained by pasting together two shells by means of their longest edges forming an angle $0(\varepsilon^{1/4})$ with respect the same plane configuration for the adjacent components (see Figure 2). This problem incorporates a rigification effect which is already present in the case of a slight folding in a plate (see [36]), but the problem is somewhat different, as in the present case the adjacent shells are geometrically rigid, whereas the adjacent plates in [36] are not. Nevertheless, the mathematical treatment presents evident analogies. It should be pointed out in that context that there are two different ways of pasting together the two shells: we may or not allow angular deformations of the pasting (in other words, the adjacent shells may be either "fixed" or "clamped" to each other). The corresponding mathematical treatments are somewhat analogous, with the corresponding transmission conditions. In Section 3 we explicitly considered the case when the angular deformations are allowed, whereas the case of a "mutual clamping" was addressed in [36] (see also Remark 3.1 in this context). It should be noticed that the asymptotic structure of the coupled shells recalls the corresponding one for the basic problem. This is not very surprising, as the geometry for small ε is intermediate between a plate and a rod; there is some relation with slender and thin elastic structures (see [41] in this concern). This also explains in a heuristic way the rigidification phenomenon: the angle between the plates increases the inertia moment of the section with respect to the horizontal axis, which is classically responsible for flection rigidity.

In Section 4 we get some properties on the potential solution $\varphi(y_1, y_2)$ of the basic problem which can be understood as "global" ones since they are stated in terms of integrals of the potential (and its derivatives) $\varphi(y_1, y_2)$ with respect to the y_2 variable. They give some information on the y_1-dependence and some of them are connected with the resultant mechanical moments with respect to the sections $y_1 = $ constant. Nevertheless, we get also some formula which does not have an easy mechanical meaning and looks remarkably more unexpected than the previously mentioned formulae. We also obtain an abstract version of the above properties which is specially useful in order to study more sophisticated formulations as the case of the shell has an edge with slight folding considered in Section 3 for which the transmission conditions made very complex a direct analysis.

We end the paper by considering, in Section 5, a nonlinear variant of the basic problem stated in terms of some associate obstacle problem. We assume that the upper surface of a "well adapted" obstacle has the same cylindrical geometry than the shell in such a way that the possibility of contact between the shell and the obstacle is merely formulated in terms of the vertical displacement $u_3(y_1, y_2) \geq 0$ on the small obstacle $\mathcal{S}$ occupying a given part of the spatial domain. We pass to the limit in the associate variational inequality getting a limit variational inequality which, again, can be easier written in terms of the scalar potential φ. Finally, we get a global comparison result between the vertical displacements of the shells with and without obstacle.

2 The basic problem.

2.1 Setting of the basic problem

We consider a slender cylindrical shell as shown in Fig 1.
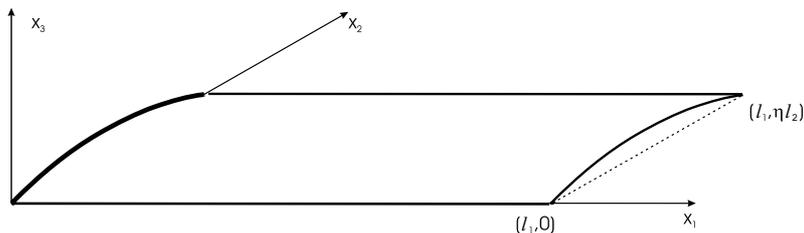
Figure 1

According to standard notations in cylindrical shell theory (see, e.g. [28], [13], [33]) the "plane of parameters x_1, x_2" is merely the middle surface (cylinder) of the shell developed into a plane. We chose

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x_1 in the direction of the generators and x_2 normal to them, so that the principal curvatures are zero in the direction x_1 and $b = 1/R$ (we assume positive curvature see Remark 2.13 on the case $b = -1/R$) in the direction x_2, where R denotes the radius of the cross section of the cylinder. Accordingly, the second fundamental form of the surface has components $b_{11} = b_{12} = 0$ and $b_{22} = b$, which is considered as a free parameter for the time being. Moreover, the Christoffel symbols of the surface vanish identically, so that covariant and classical differentiation coincide. Since $b_{12}^2 - b_{11}b_{22} = 0$ the surface is parabolic, i.e. the directions of the principal curvatures coincide (see, e.g. [33]).

Remark 2.1 *As a matter of fact, the Torroja's structure mentioned at the introduction was not composed by cylindrical elements but by slightly hyperbolic ones. Nevertheless, the curvature in the longitudinal direction was much smaller (and even it vanished in early projects by Torroja: see [38] Chapter 1) than in the transversal direction, so that our model with zero longitudinal curvature may be considered as a first approximation. The case of elliptic or hyperbolic middle surfaces shells case will be analyzed in [17]. Another example is the "pedestrian access shell in the southwestern side of the UNESCO building (Paris, 1953-58) due to Marcel Breuer and Bernard Zehrfuss with the collaboration of Antonio and Pier Luigi Nervi ([27], [23]).*

Let ε be a small parameter, the relative thickness of the plate. Let $\eta = \eta(\varepsilon)$ be a new small parameter satisfying

$$\varepsilon^{1/3} \leq \eta \leq 1 \tag{2.2}$$

(but the typical example will be $\eta = \varepsilon^{1/4}$, as announced in the introduction). Let us denote the shell domain by

$$\Omega_\varepsilon = (0, l_1) \times (0, \eta l_2), \tag{2.3}$$

with $\eta l_2 \leq 2R$. The corresponding tangential displacements are $\tilde{u}_1, \tilde{u}_2$, whereas $\tilde{u}_3$ is the displacement normal to the shell. Some times we shall use the notation $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^\varepsilon$ to indicate explicitly the ε-dependence.

We shall admit, in this section, that the shell is clamped by the "small curved boundary" ($\{0\} \times [0, \eta l_2]$ and free by the rest (see some comments on other cases in Remark 2.11). This implies the kinematic boundary conditions:

$$0 = \tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = \tilde{\partial}_1 \tilde{u}_3 \quad \text{on } \{0\} \times [0, \eta l_2], \tag{2.4}$$

where

$$\tilde{\partial}_\alpha = \frac{\partial}{\partial x_\alpha}. \tag{2.5}$$

The space of configuration will be denoted by V_ε. It is the subspace of

$$H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)$$

formed by the functions satisfying the kinematic boundary conditions (2.4).

Although it is possible to write the complete system of equations modeling the above elastic problem (the "strong formulation": see. e.g. [28]), here we shall follow a "variational or weak formulation" of the elasticity problem for this structure which takes the form

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \tag{2.6}$$

where the coefficients ε and ε^3 account for the fact that the membrane and flection rigidities are proportional to the thickness of the plate and to its cube, respectively. Moreover, the two bilinear forms $a(\mathbf{u}^\varepsilon, \mathbf{v})$ and $b(\mathbf{u}^\varepsilon, \mathbf{v})$ on the space $\mathbf{V}$ are defined thought the expressions (membrane strains in shell theory):

$$\begin{cases} \tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \tilde{\partial}_1 \tilde{v}_1 \\ \tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \tilde{\partial}_2 \tilde{v}_2 + b\tilde{v}_3 \\ \tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \frac{1}{2}(\tilde{\partial}_2 \tilde{v}_1 + \tilde{\partial}_1 \tilde{v}_2) \end{cases} \tag{2.7}$$

and

$$\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}) = \tilde{\partial}_{\alpha\beta}\tilde{v}_3 \tag{2.8}$$

for the triplets $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$.

Remark 2.2 *It should be noted that the very expression for $\tilde{\rho}_{22}$ in cylindrical shells is*

$$\tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \tilde{\partial}_2^2 \tilde{v}_3 + b\tilde{\partial}_2\tilde{v}_2$$

but, as we shall see in the sequel (2.21), (2.22), for instance, in the present framework the second term of the right hand side is always asymptotically small with respect to the first one. In order to avoid unnecessary cumbersome computations, we disregard it, according to (2.8).

The two bilinear forms on $\mathbf{V}$ are then defined by:

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} A^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}})\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}})d\mathbf{x} \tag{2.9}$$

$$b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} B^{\alpha\beta\lambda\mu}\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}})\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}})d\mathbf{x}, \tag{2.10}$$

where the coefficients $A^{\alpha\beta\lambda\mu}$ and $B^{\alpha\beta\lambda\mu}$ satisfy the symmetry and positivity conditions

$$A^{\alpha\beta\lambda\mu} = A^{\beta\alpha\lambda\mu} = A^{\lambda\mu\alpha\beta} \tag{2.11}$$

$$A^{\alpha\beta\lambda\mu}\theta_{\alpha\beta}\theta_{\lambda\mu} \geq c\theta_{\alpha\beta}\theta_{\alpha\beta} \quad \text{for} \quad \theta_{\alpha\beta} = \theta_{\beta\alpha} \tag{2.12}$$

with some $c > 0$. Analogous hypotheses will be assumed for the coefficients B; for technical reasons, we shall assume that

$$B^{1222} = B^{2122} = B^{2212} = B^{2221} = 0. \tag{2.13}$$

(in some results we shall require some additional conditions: see (2.93)). In shell theory they are the membrane and flection rigidities (see, e.g. [33]); their specific values are classical in the isotropic case (satisfying in particular (2.13)), but this also covers many anisotropic cases. This also allows us to define the membrane stresses:

$$\tilde{T}^{\alpha\beta}(\tilde{\mathbf{u}}) = \tilde{T}^{\beta\alpha}(\tilde{\mathbf{u}}) = A^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{u}}). \tag{2.14}$$

It will prove useful to define the entries $C_{\alpha\beta\lambda\mu}$ of the inverse matrix of A; they are the "membrane compliances" (see, e.g. [33]) and (2.14) may equivalently be written:

$$\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{u}}) = C_{\lambda\mu\alpha\beta}\tilde{T}^{\beta\alpha}(\tilde{\mathbf{u}}) \tag{2.15}$$

As applied forces, we shall give a normal loading depending on ε by the factor ε^3 (see Remark 2.3 hereafter), specifically

$$\langle \mathbf{f}, \mathbf{v} \rangle = \varepsilon^3 \int_{\Omega_\varepsilon} F_3(x_1, x_2/\eta)\tilde{v}_3(x_1, x_2)d\mathbf{x}, \tag{2.16}$$

(for other loading see Remark 2.7). We note that the shape of the profile of the applied loading in x_2 is independent of ε but applied to the points x_2/η. Defining $y_2 = x_2/\eta$ (see also the scaling (2.21) hereafter), the function $F_3(x_1, y_2)$ is independent of ε. We shall admit in the sequel that

$$F_3 \in L^2(\Omega) \tag{2.17}$$

where

$$\Omega = (0, l_1) \times (0, l_2). \tag{2.18}$$

Remark 2.3 *In the special case when the curvature b vanishes, there is uncoupling between the membrane and flection problems; the "normal" loading only produces flection. Moreover, as the width of the shell (the plate, in that case) is $0(\eta)$, the global rigidity is $0(\eta\varepsilon^3)$, of the same order as the total applied force in (2.16), so that, in that case, the solutions (u_3^ε in fact) have a non zero limit. This is almost evident and may rigorously be proved by the methods of [29]. We shall see that in our case (i. e., with non zero b) the displacements are very small and only converge to a non zero limit after an appropriate scaling. This amounts to a very high rigidity produced by the curvature as commented at the Introduction.*

The specific definition of the problem in variational formulation is

Problem P_ε. *Find* $\tilde{\mathbf{u}}^\varepsilon \in \mathbf{V}_\varepsilon$ *satisfying* (2.6) *with* (2.9), (2.10) *and* (2.16) $\forall \tilde{\mathbf{v}} \in \mathbf{V}_\epsilon$.

An easy application of the Lax-Milgram theorem allows to see that this problem has a unique solution depending on the parameter ε.

Remark 2.4 *Since the bilinear forms* $a(\mathbf{u}, \mathbf{v})$ *and* $b(\mathbf{u}, \mathbf{v})$ *are symmetric, from well known results we deduce that, in fact,* $\tilde{\mathbf{u}}^\varepsilon$ *is the unique solution of the minimization problem*

$$Min_{\mathbf{V}} \widetilde{J}_\varepsilon(\mathbf{v}) \tag{2.19}$$

where

$$\widetilde{J}_\varepsilon(\mathbf{v}) = \frac{\varepsilon}{2} a(\mathbf{v}, \mathbf{v}) + \frac{\varepsilon^3}{2} b(\mathbf{v}, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle. \tag{2.20}$$

The objective of the rest of the section is to study its asymptotic behavior as $\varepsilon \downarrow 0$.

2.2   Scaling and a priori estimates in the basic problem.

Let us perform the change of variables :

$$\begin{cases} \mathbf{x} = (x_1, x_2) \Rightarrow \mathbf{y} = (y_1, y_2), \\ y_1 = x_1, \quad y_2 = \eta^{-1} x_2 \end{cases} \tag{2.21}$$

so, the domain Ω_ε is transformed into Ω and

$$\partial_1 = \tilde{\partial}_1, \quad \partial_2 = \eta \tilde{\partial}_2; \quad \partial_\alpha = \frac{\partial}{\partial y_\alpha}. \tag{2.22}$$

Moreover, we shall perform the change of unknowns

$$\begin{cases} \tilde{u}_1(\mathbf{x}) = \eta^\theta u_1(\mathbf{y}), \\ \tilde{u}_2(\mathbf{x}) = \eta^{\theta-1} u_2(\mathbf{y}), \\ \tilde{u}_3(\mathbf{x}) = \eta^{\theta-2} b^{-1} u_3(\mathbf{y}), \end{cases} \tag{2.23}$$

(as before, some times we shall use the notation $\mathbf{u} = \mathbf{u}^\varepsilon$ to indicate explicitly the ε-dependence). The specific values of θ and $b(\varepsilon)$ will be found later (see (2.43) and (2.42)). Let us explain a little the meaning of (2.23). As θ is not defined, the total level of the scaling is not specified, only the mutual ratios of dilatation of the three components are fixed. They are chosen in analogy with layers in parabolic shells. Specifically, the ratio between the components 1 and 2 is fixed in order that the new form of the shear membrane strain $\tilde{e}_{12}$ be formed by two terms of the same order (which, on the other hand, are asymptotically large, forming a constraint for the limit problem). The ratio between the components 2 and 3 is also fixed in such a way that the new form of the membrane strain $\tilde{e}_{22}$ be formed by two terms of the same order.

We then perform the previous change for $\tilde{\mathbf{u}}^\varepsilon$ as well as for $\tilde{\mathbf{v}}$ in P_ε and we have

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^\theta \partial_1 v_1 \tag{2.24}$$

$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2), \tag{2.25}$$

$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-2}(\partial_2 v_2 + v_3), \tag{2.26}$$

$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^{\theta-2}b^{-1}\partial_1^2 v_3, \tag{2.27}$$

$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-3}b^{-1}\partial_1\partial_2 v_3, \tag{2.28}$$

$$\tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-4}b^{-1}\partial_2^2 v_3. \tag{2.29}$$

It will prove useful to define

$$\gamma_{11}^{\varepsilon}(\mathbf{v}) = \partial_1 v_1 \tag{2.30}$$

$$\gamma_{12}^{\varepsilon}(\mathbf{v}) = \gamma_{21}^{\varepsilon}(\mathbf{v}) = \eta^{-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2), \tag{2.31}$$

$$\gamma_{22}^{\varepsilon}(\mathbf{v}) = \eta^{-2}(\partial_2 v_2 + v_3); \tag{2.32}$$

$$\rho_{11}^{\varepsilon}(\mathbf{v}) = \eta^2 \partial_1^2 v_3, \tag{2.33}$$

$$\rho_{12}^{\varepsilon}(\mathbf{v}) = \rho_{21}(\mathbf{v}) = \eta\partial_1\partial_2 v_3, \tag{2.34}$$

$$\rho_{22}^{\varepsilon}(\tilde{\mathbf{v}}) = \partial_2^2 v_3. \tag{2.35}$$

so that:

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^{\theta}\gamma_{11}^{\varepsilon}(v)$$

$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta}\gamma_{12}^{\varepsilon}(\mathbf{v})$$

$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta}\gamma_{22}^{\varepsilon}(\mathbf{v})$$

$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^{\theta-4}b^{-1}\rho_{11}^{\varepsilon}(\mathbf{v})$$

$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-1}b^{-1}\rho_{12}^{\varepsilon}(\mathbf{v})$$

$$\tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-4}b^{-1}\rho_2^{\varepsilon}(\mathbf{v}).$$

We recall that the spatial domain is now $\Omega = (0, l_1) \times (0, l_2)$. The space of configuration, after scaling will be denoted by $\mathbf{V}$. It is the subspace of

$$H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$$

formed by the functions satisfying the kinematic boundary conditions

$$0 = u_1 = u_2 = u_3 = \partial_1 u_3 \text{ on } \{0\} \times [0, l_2]. \tag{2.36}$$

The expression (2.6) then becomes:

$$P \int_{\Omega} A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}^{\varepsilon}(\mathbf{u}^{\varepsilon})\gamma_{\lambda\mu}^{\varepsilon}(\mathbf{v})d\mathbf{y} + Q \int_{\Omega} B^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}^{\varepsilon}(\mathbf{u}^{\varepsilon})\rho_{\lambda\mu}^{\varepsilon}(\mathbf{v})d\mathbf{y} = R \int_{\Omega} F_3(y_1, y_2)v_3(y_1, y_2)d\mathbf{y}, \tag{2.37}$$

with

$$P = \varepsilon\eta^{2\theta+1} \tag{2.38}$$

$$Q = \varepsilon^3\eta^{2\theta-7}b^{-2} \tag{2.39}$$

$$R = \varepsilon^3\eta^{\theta-1}b^{-1}. \tag{2.40}$$

Inspecting this equation, we see that the integrals (up to the coefficients P, Q, R) are somewhat analogous to the corresponding ones in the problem of propagation of singularities along the characteristics in the parabolic case ([10]). We may then hope to go on with our problem provided that

$$P = Q = R. \tag{2.41}$$

Indeed, we shall determine the $b(\varepsilon)$ and θ as functions of ε and the function $\eta(\varepsilon)$ using the two equations (2.41). This gives

$$b = \varepsilon/\eta^4 \tag{2.42}$$

and

$$\eta^{\theta-2} = \varepsilon. \tag{2.43}$$

We note that, according to (2.2), b is always small with respect to η^{-1}, and equal to 1 (or rather $0(1)$) in the "typical example" $\eta = \varepsilon^{1/4}$. Moreover, the exponent θ which appears in the scaling (2.23) is well determined and satisfy $1 \leq \theta < \infty$. In the "typical example" $\eta = \varepsilon^{1/4}$ we have $\theta = 6$.

Once θ is determined, the scaling (2.23) is perfectly defined. We then observe that the factor $\eta^{\theta-2}b^{-1}$ takes the form: η^4 which is always small. It means that the scaling of the component u_3^ε is such that, after scaling, it is asymptotically large with respect to the case before scaling. As we shall prove in the sequel, the scaled unknown u_3^ε has a non zero limit; it follows that the initial unknown $\tilde{u}_3^\varepsilon$ tends to 0 at the ratio η^4. We shall come again on this property, which amounts to the rigidification of the plate with respect to the plane case.

Summing up, the problem P_ε becomes after scaling:

Problem Π_ε. *Find $\mathbf{u}^\varepsilon \in \mathbf{V}$ satisfying*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) = \int_\Omega F_3(y_1, y_2)v_3(y_1, y_2)d\mathbf{y} \tag{2.44}$$

$\forall v \in \mathbf{V}$, *where*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) \overset{def}{=} \int_\Omega A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon)\gamma_{\lambda\mu}^\varepsilon(\mathbf{v})d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon)\rho_{\lambda\mu}^\varepsilon(\mathbf{v})d\mathbf{y}.$$

It should be emphasized that, by virtue of the definitions (2.30) to (2.35), the coefficients in (2.44) involve various powers of η, running from -4 to $+4$. The terms in η^{-4} to η^{-1} are "penalty terms", whereas those in η^1 to η^4 are "singular perturbation terms". Only the terms of order 1 will remain in the limit expression.

Remark 2.5 *As in Remark 2.4, since the bilinear forms $a^\varepsilon(\mathbf{u}, \mathbf{v})$ is symmetric we conclude that $\mathbf{u}^\varepsilon$ is the unique solution of the minimization problem*

$$Min_{\mathbf{V}} J_\varepsilon(\mathbf{v}) \tag{2.45}$$

where

$$J_\varepsilon(\mathbf{v}) = \frac{1}{2}a^\varepsilon(\mathbf{v}, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle. \tag{2.46}$$

Let us proceed to the a priori estimates. We first estate a series of estimates in order to prove that the functional in the right hand side of the (2.44) remains bounded with respect to the energy norm of the left hand side. From the expression of $a^\varepsilon(\mathbf{v}, \mathbf{v})$ with $\mathbf{u}^\varepsilon = \mathbf{v}$, written under the form (2.44) and using the positivity of the coefficients $A^{\alpha\beta\lambda\mu}$ we see that each term in the left hand side is majorized by the right hand side. Specifically, using (2.11) - (2.12), we have:

Lemma 2.1 *The estimates:*

$$\|\partial_1 v_1\|_{L^2(\Omega)}^2 \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.47}$$

$$\|\eta^{-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2)\|_{L^2(\Omega)}^2 \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.48}$$

$$\|\eta^{-2}(\partial_2 v_2 + v_3)\|_{L^2(\Omega)}^2 \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.49}$$

$$\|\partial_2^2 v_3\|_{L^2(\Omega)}^2 \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.50}$$

$$\|\eta\partial_1\partial_2 v_3\|_{L^2(\Omega)}^2 \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.51}$$

$$\|\eta^2\partial_1^2 v_3\|_{L^2(\Omega)}^2 \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.52}$$

hold true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

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Now, in order to prove that the functional in the right hand side is bounded independently of ε, we need an estimate on u_3 itself.

Lemma 2.2 *The estimate:*

$$\|v_3\|^2_{L^2((0,l_1);H^2(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.53}$$

holds true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

PROOF. Discarding the factors in η in (2.48) and (2.49) and differentiating we have:

$$\|\partial_2^2 v_1 + \partial_2\partial_1 v_2)\|^2_{L^2((0,l_1);H^{-1}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.54}$$

$$\|\partial_1\partial_2 v_2 + \partial_1 v_3)\|^2_{H^{-1}((0,l_1);L^2(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}). \tag{2.55}$$

On the other hand, from (2.47), using the fact that v_1 vanishes on $\{0\} \times [0, l_2]$, by using the *generalized Poincaré inequality* (see Section 9 of [11]) we obtain:

$$\|v_1\|^2_{H^1((0,l_1);L^2(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v})$$

and differentiating,

$$\|\partial_2^2 v_1\|^2_{H^1((0,l_1);H^{-2}(0,l_2)))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}).$$

From the last estimate and (2.54), taking the weaker norm, it follows that

$$\|\partial_2\partial_1 v_2\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.56}$$

and using (2.55)

$$\|\partial_1 v_3\|^2_{H^{-1}((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}),$$

or even by applying the *generalized Poincaré inequality* of Section 9 of [11]) on account of the vanishing of the trace on $\{0\} \times [0, l_2]$):

$$\|v_3\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}). \tag{2.57}$$

We then use (2.50). Concerning the space $H^2(0, l_2)$, its norm up to affine functions is merely the norm in $L^2(0, l_2)$ of the second derivative; the kernel of affine functions is of finite dimension, so that in it the norms L^2 and H^{-2} are equivalent. The conclusion follows. $\qquad\square$

Then, provided that $F_3 \in L^2(\Omega)$ (this hypothesis is not optimal), we have

Lemma 2.3 *The estimate*

$$\left| \int_\Omega F_3 v_3 d\mathbf{y} \right| \leq ca^\varepsilon(\mathbf{v},\mathbf{v})^{1/2} \tag{2.58}$$

holds true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

Now, taking $\mathbf{v} = \mathbf{u}^\varepsilon$ in (2.44) and using (2.56) we get the energy estimate:

Lemma 2.4 *Let $\mathbf{u}^\varepsilon$ be the solution of problem Π_ε. The energy remains bounded independently of ε, i. e. the estimate*

$$a^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) \leq C \tag{2.59}$$

holds true for a certain $C > 0$ independent of ε.

From this, Lemma 2.2 gives the main estimates of the solutions

Lemma 2.5 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* Π_ε. *The estimates*

$$\|\gamma^\varepsilon_{\alpha\beta}(u^\varepsilon)\| \leq C \qquad \alpha,\beta = 1,2 \tag{2.60}$$

$$\|\partial_1 u_1^\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.61}$$

$$\|\eta^{-1}\frac{1}{2}(\partial_2 u_1^\varepsilon + \partial_1 u_2^\varepsilon)\|^2_{L^2(\Omega)} \leq C \tag{2.62}$$

$$\|\eta^{-2}(\partial_2 u_2^\varepsilon + u_3^\varepsilon)\|^2_{L^2(\Omega)} \leq C \tag{2.63}$$

$$\|\partial_2^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.64}$$

$$\|\eta\partial_1\partial_2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.65}$$

$$\|\eta^2\partial_1^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.66}$$

hold true for a certain $C > 0$ independent of ε.

We note that (2.60) is merely a new form of (2.61) - (2.66). We shall need an estimate on u_2^ε itself. We shall obtain it by differentiating with respect to y_2 and integrating in y_1.

Lemma 2.6 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* Π_ε. *The estimates*

$$\|u_1^\varepsilon\|_{H^1((0,l_1);L^2(0,l_2))} \leq C \tag{2.67}$$

$$\|u_2^\varepsilon\|_{\widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2))} \leq C \tag{2.68}$$

$$\|u_3\|^2_{L^2((0,l_1);H^2(0,l_2))} \leq C, \tag{2.69}$$

holds true for a certain $C > 0$ independent of ε, where

$$\widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2)) = \{w \in H^1((0,l_1);H^{-1}(0,l_2)) \text{ such that } w(0,\cdot) = 0\}. \tag{2.70}$$

PROOF. From (2.61), using the Poincaré inequality on account of the fact that the trace of u_1^ε vanishes on $\{0\} \times [0,l_2]$ we see that u_1^ε remains bounded in $H^1((0,l_1);L^2(0,l_2))$ (which proves (2.67)) and then $\partial_2 u_1^\varepsilon$ remains bounded in $H^1((0,l_1);H^{-1}(0,l_2))$. Using (2.62) we then see that $\partial_1 u_2^\varepsilon$ remains bounded in $L^2((0,l_1);H^{-1}(0,l_2))$. As the trace of u_2^ε vanishes on $\{0\} \times [0,l_2]$, integrating in y_1 we get the conclusion by applying the Poincaré inequality. Finally, (2.69) follows from (2.53) and (2.58). $\qquad\square$

A first result of convergence is

Lemma 2.7 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* Π_ε. *The following convergences (as $\varepsilon \to 0$) hold true (in the sense of subsequences, the limits being not necessarily unique):*

$$u_1^\varepsilon \to u_1^* \qquad weakly \ \ in \ \ \widetilde{H}_0^1((0,l_1);L^2(0,l_2)) \tag{2.71}$$

$$u_2^\varepsilon \to u_2^* \qquad weakly \ \ in \ \ \widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2)) \tag{2.72}$$

$$u_3^\varepsilon \to u_3^* \qquad weakly \ \ in \ \ L^2((0,l_1);H^2(0,l_2)) \tag{2.73}$$

where $\mathbf{u}^* = (u_1^*,u_2^*,u_3^*)$ are distributions on Ω, belonging to the spaces specified in (2.71) - (2.73). Moreover, they satisfy:

$$\partial_2 u_1^* + \partial_1 u_2^* = 0$$
$$\partial_2 u_2^* + u_3^* = 0.$$

Finally,

$$\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \to \gamma^*_{\alpha\beta} \qquad weakly \ \ in \ \ L^2(\Omega), \ \ \alpha,\beta = 1,2, \tag{2.74}$$

for some $\gamma^*_{\alpha\beta} \in L^2(\Omega)$.

PROOF. By weak compactness, the conclusions are obvious consequences of the estimates in lemmas 2.5 and 2.6. $\qquad\square$

2.3 Limit and convergence in the basic problem.

Let us define the space $\mathbf{G}$ for the definition of the limit problem:

$$\mathbf{G} = \{\mathbf{v} = (v_1, v_2, v_3) \in \widetilde{H}_0^1((0, l_1); L^2(0, l_2)) \times \widetilde{H}_0^1((0, l_1); H^{-1}(0, l_2)) \times L^2((0, l_1); H^2(0, l_2)),$$

$$\partial_2 v_1 + \partial_1 v_2 = 0, \quad \partial_2 v_2 + v_3 = 0\}, \tag{2.75}$$

where we observe that v_1 defines completely v_2 and then v_3. Clearly, $\mathbf{G}$ is a Hilbert space with the norm

$$\begin{cases} \|\mathbf{v}\|_{\mathbf{G}}^2 = \|v_1\|_{\widetilde{H}_0^1((0,l_1);L^2(0,l_2))}^2 + \|\partial_2^2 v_3\|_{L^2(\Omega)}^2 \\ \simeq \|\partial_1 v_1\|_{L^2(\Omega)}^2 + \|\partial_2^3 v_2\|_{L^2(\Omega)}^2 \end{cases}$$

Remark 2.6 *A straightforward comparison with the space* $\mathbf{V}$ *shows that the space* $\mathbf{G}$ *for the limit problem incorporates the two constraints corresponding to the "penalty terms" in* Π_ε *(2.44), whereas the boundary conditions for* u_3*, which are concerned with the "singular perturbation terms" in* Π_ε *(2.44) are lost.* $\square$

It is worthwhile to state an equivalent definition of the space $\mathbf{G}$ where the functions are defined in terms of a scalar "potential ψ":

Lemma 2.8 *The space* $\mathbf{G}$ *may equivalently be defined as the space of the triplets* $\mathbf{v} = (v_1, v_2, v_3)$ *such that:*

$$v_1 = \partial_1 \psi, \qquad v_2 = -\partial_2 \psi, \qquad v_3 = -\partial_2^2 \psi. \tag{2.76}$$

where ψ *is an element of*

$$\widetilde{G} = \tilde{H}_0^2((0, l_1); L^2(0, l_2)) \cap L^2((0, l_1); H^4(0, l_2)) \tag{2.77}$$

where

$$\tilde{H}_0^2((0, l_1); L^2(0, l_2)) = \{\psi \in H^2((0, l_1); L^2(0, l_2)); \psi(0, y_2) = \partial_1 \psi(0, y_2) = 0\}. \tag{2.78}$$

PROOF. Let $\mathbf{v} \in \mathbf{G}$. Because of the first constraint indicated in (2.75), there exist a distribution ψ, defined up to an additive constant, such that v_1 and v_2 are given by the two first relations in (2.76). The second constraint then shows that v_3 is then given by the last relation in (2.76). As the traces of v_1 and v_2 vanish for $y_1 = 0$, we see that

$$\psi(0, y_2) = C_1 \qquad \partial_1 \psi(0, y_2) = 0$$

with C_1 a constant. We then fix the arbitrary constant of ψ to have $C_1 = 0$. Using the Poincaré inequality, it follows from $\partial_1 v_1 \in L^2(\Omega)$ and the above boundary conditions for ψ that $\psi \in \tilde{H}_0^2((0, l_1); L^2(0, l_2))$, so that ψ is in the first one of the two spaces on the right hand side of (2.77). Belonging to the second space follows easily from the last relations in (2.75) and (2.76). Conversely, it is straightforward that $\psi \in \widetilde{G}$ implies $v \in \mathbf{G}$. $\square$

It should prove useful to prove a lemma on density in $\mathbf{G}$.

Lemma 2.9 *The subspace of* $\mathbf{G}$ *formed by the elements* $\mathbf{v} = (v_1, v_2, v_3)$ *which are smooth, vanish in a neighborhood of* $\{0\} \times [0, l_2]$ *and derive from a "potential"* ψ *according to (2.76) is dense in* $\mathbf{G}$*. In other words, the set of functions of* $\mathbf{G} \cap \mathbf{V}$ *which are smooth and vanish in a neighborhood of* $\{0\} \times [0, l_2]$ *is dense in* $\mathbf{G}$*.*

PROOF. Thanks to the equivalence of the spaces $\mathbf{G}$ and $\widetilde{G}$ given by lemma 2.8, the proof (in the unconstrained space $\widetilde{G}$) is almost classical (see, for instance, lemma 5.2 of [34] or lemma 8.1 of [10]). $\square$

We are now defining the limit problem. It involves the numerical coefficients $1/C_{1111}$, and B^{2222} where $C_{\alpha\beta\lambda\mu}$ is the matrix inverse of $A^{\alpha\beta\lambda\mu}$, i. e. the matrix of membrane compliances, and $\mathbf{B}$ is the matrix of flection rigidities. They are both strictly positive.

Problem Π_0. *Find* $\mathbf{u} \in \mathbf{G}$ *such that*

$$\int_\Omega \frac{1}{C_{1111}} \partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222} \partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}. \tag{2.79}$$

$\forall \mathbf{v} \in \mathbf{G}$, *or equivalently, in terms of the potential, find* $\varphi \in \widetilde{G}$ *such that*

$$\int_\Omega \frac{1}{C_{1111}} \partial_1^2 \varphi \partial_1^2 \psi d\mathbf{y} + \int_\Omega B^{2222} \partial_2^4 \varphi \partial_2^4 \psi d\mathbf{y} = -\int_\Omega F_3 \partial_2^2 \psi d\mathbf{y}, \tag{2.80}$$

$\forall \psi \in \widetilde{G}$.

Obviously, this problem is in the Lax - Milgram framework, as the right hand side of (2.79) is a continuous functional on $\mathbf{G}$. We then have

Theorem 2.1 *Under the assumption* $F_3 \in L^2(\Omega)$, *Problem* Π_0 *has a unique solution.*

Remark 2.7 *Clearly, the case considered here,* $F_1 = F_2 = 0$ *and* $F_3 \in L^2(\Omega)$ *is not the more general case we can deal with the above arguments. So, for instance, taking into account the previous a priori estimate we can consider other loadings* $\mathbf{F}$ *satisfying*

$$|\int_\Omega F_i v_i dy| \leq ca^\varepsilon(v,v)^{1/2}, \; i = 1,2,3,$$

as it is the special case of $F_2 = 0$ *and* $F_1, F_3 \in L^2(\Omega)$. *This is interesting for the special case in which* $\mathbf{F}$ *is the gravity and the middle surface* $x_3 = 0$ *makes an angle with respect to the horizontal, as it is the case of the Torroja's structure mentioned at the Introduction. Other possible choices are the concentrated loadings of the type* $F_1 = F_2 = 0$ *and* F_3 *given in terms of the Dirac delta and its derivatives as in [10].*

Our main convergence result is:

Theorem 2.2 *Let* $\mathbf{u}^\varepsilon$ *and* $\mathbf{u}$ *be the solutions of* Π_ε *and* Π_0 *respectively. Then, for* $\varepsilon \downarrow 0$, *we have:*

$$\mathbf{u}^\varepsilon \to \mathbf{u}$$

in the topologies indicated in (2.71) - (2.73) (lemma 2.7). In other words, the limit $\mathbf{u}^*$ *in lemma 2.7 is the solution of the limit problem (2.79).*

Before proving this theorem, let us define certain limits which will be useful in the sequel. We know by (2.74) that the $\gamma_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon)$ have limits $\gamma_{\alpha\beta}^*$. Correspondingly, we define:

$$T^{\alpha\beta\varepsilon}(\mathbf{u}^\varepsilon) = A^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}^\varepsilon(\mathbf{u}^\varepsilon)$$

and

$$T^{\alpha\beta*} = A^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}^*$$

so that

$$T^{\alpha\beta\varepsilon}(\mathbf{u}^\varepsilon) \to T^{\alpha\beta*} \qquad \text{weakly in } L^2(\Omega), \; \alpha, \beta = 1, 2.$$

Remark 2.8 *It seems important to point out that the a priori estimates (2.62) and (2.63) does not allow to conclude the identification $\gamma_{12}^* = 0$ and $\gamma_{22}^* = 0$ in spite to know that $\gamma_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon)$ weakly converge to $\gamma_{\alpha\beta}^*$ and that necessarily $\partial_2 u_1^* + \partial_1 u_2^* = 0$ and $\partial_2 u_2^* + u_3^* = 0$. The reason is due to the presence of the terms η^{-1} (respectively η^{-2}) in the definition of γ_{12}^ε (respectively γ_{22}^ε). Notice that, in fact, in most of the cases we must have that $\gamma_{12}^* \neq 0$ or $\gamma_{22}^* \neq 0$, since otherwise we could get that $T^{11\varepsilon}(\mathbf{u}^\varepsilon) = A^{1111}\gamma_{11}^\varepsilon(\mathbf{u}^\varepsilon) + 2A^{1112}\gamma_{12}^\varepsilon(\mathbf{u}^\varepsilon) + A^{1122}\gamma_{22}^\varepsilon(\mathbf{u}^\varepsilon)$ converges (weakly in $L^2(\Omega)$) to $T^{11*}(\mathbf{u}^*) = A^{1111}\gamma_{11}(\mathbf{u}^*) = A^{1111}\partial_1 u_1^*$ and this would imply (thanks to Theorem 2.2) that necessarily*

$$A^{1111} = \frac{1}{C_{1111}}, \tag{2.81}$$

which is not necessarily true since it depends of the constitutive assumptions made on the elastic medium.

PROOF OF THEOREM 2.2. That the limit $\mathbf{u}^*$ in lemma 2.7 belongs to $\mathbf{G}$ follows from the definition of this space. Let us now prove that $\mathbf{u}^*$ is the solution of (2.79). Let us take in (2.44) $\mathbf{v}$ in the dense set of $\mathbf{G}$ indicated in Lemma 2.9. From the definition of $\mathbf{G}$ and (2.30) - (2.32) we see that the only non vanishing $\gamma_{\alpha\beta}^\varepsilon$ is

$$\gamma_{11}^\varepsilon(\mathbf{v}) = \partial_1 v_1$$

and we have

$$\int_\Omega T^{11\varepsilon}(\mathbf{u}^\varepsilon)\gamma_{11}^\varepsilon(\mathbf{v})d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon)\rho_{\lambda\mu}^\varepsilon(\mathbf{v})d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}.$$

The term in the first integral obviously pass to the limit (note that $\gamma_{11}^\varepsilon(v)$ does not depend on ε: see (2.30)). Concerning the terms in ρ, we have, as we know, an estimate in the spaces involved in the definition of $\mathbf{G}$, so that the term

$$B^{2222}\partial_2^2 u_3^\varepsilon \partial_2^2 v_3$$

also pass to the limit. For the same reason, with the estimates (2.64) - (2.66) all the other terms tend to zero, with the exception of

$$\eta \int_\Omega B^{1222}\partial_1\partial_2 u_3^\varepsilon \partial_2^2 v_3 d\mathbf{y}$$

and

$$\eta^2 \int_\Omega B^{1122}\partial_1^2 u_3^\varepsilon \partial_2^2 v_3 d\mathbf{y}$$

which are not evident. The first one vanishes as, according to our hypotheses, B^{1222} is taken to be zero (see (2.13)). As for the second one, according to distribution theory (integration by parts) it is equal to

$$\eta^2 \int_\Omega B^{1122} u_3^\varepsilon \partial_1^2 \partial_2^2 v_3 d\mathbf{y}.$$

We note that u_3^ε remains bounded in $L^2(\Omega)$. Then, because of the factor η^2, the expression tends to 0. As a result, the limit is

$$\int_\Omega T^{11*}\partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3^* \partial_2^2 v_3 d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}. \tag{2.82}$$

We are now transforming the term in T^{11*} in the previous equation. To this end, let us take

$$w_1 = w_2 = 0, \qquad w_3 \in C_0^\infty(\Omega) \tag{2.83}$$

and let us take in (2.44) the test function

$$\mathbf{v} = \eta^2 \mathbf{w} \tag{2.84}$$

so that from (2.30) - (2.32) we see that the only non vanishing $\gamma_{\alpha\beta}^\varepsilon$ is

$$\gamma_{22}^\varepsilon(v) = -w_3$$

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and passing to the limit in (2.44) we have

$$\int_\Omega T^{22*}(-w_3)d\mathbf{y} = 0 \tag{2.85}$$

so that

$$T^{22*} = 0. \tag{2.86}$$

Let us now take

$$w_1 \in C_0^\infty((0,l_1); C^\infty(0,l_2)), \qquad w_2 = w_3 = 0, \tag{2.87}$$

and let us take in (2.44) the test function

$$\mathbf{v} = \eta\mathbf{w} \tag{2.88}$$

so that from (2.30) - (2.32) we see that the only non vanishing $\gamma_{\alpha\beta}^\varepsilon$ are

$$\gamma_{11}^\varepsilon(\mathbf{v}) = \eta\partial_1 w_1, \qquad \gamma_{12}^\varepsilon(\mathbf{v}) = \frac{1}{2}\partial_2 w_1 \tag{2.89}$$

and passing to the limit in (2.44) we have

$$\int_\Omega T^{12*}(\frac{1}{2}\partial_2 w_1)d\mathbf{y} = 0 \tag{2.90}$$

so that, as $\partial_2 w_1$ is "arbitrary",

$$T^{12*} = 0. \tag{2.91}$$

As the only non zero $T^{\alpha\beta*}$ is T^{11*}, the $\gamma_{\alpha\beta}^*$ are given by the expressions

$$\gamma_{\alpha\beta}^* = C_{\alpha\beta 11}T^{11*}$$

and in particular

$$\gamma_{11}^* = C_{1111}T^{11*}.$$

Moreover, it follows from (2.71) and (2.74) that

$$\gamma_{11}^\varepsilon(\mathbf{u}^\varepsilon) = \partial_1 u_1^\varepsilon \to \partial_1 u_1^* \quad \text{weakly in} \quad L^2(\Omega).$$

so that

$$T^{11*} = \frac{1}{C_{1111}}\partial_1 u_1^* \tag{2.92}$$

and replacing it in (2.82) we obtain (2.79). This expression holds true for $\mathbf{v}$ in a dense set in $\mathbf{G}$ (see lemma 2.9) so that $\mathbf{u}^*$ is the unique solution of (2.79). The proof is finished. $\square$

Our next result improves the convergence under some additional condition on the coefficients.

Theorem 2.3 *Assume that*

$$A^{11\lambda\mu} = 0 \text{ if } \lambda > 1 \text{ or } \mu > 1. \tag{2.93}$$

Let $\mathbf{u}^\varepsilon$ and $\mathbf{u}$ be the solutions of Π_ε and Π_0 respectively. Then

$$u_1^\varepsilon \to u_1^* \qquad \text{strongly in} \ \ \widetilde{H}_0^1((0,l_1); L^2(0,l_2)) \tag{2.94}$$

$$u_2^\varepsilon \to u_2^* \qquad \text{strongly in} \ \ \widetilde{H}_0^1((0,l_1); H^{-1}(0,l_2)) \tag{2.95}$$

$$u_3^\varepsilon \to u_3^* \qquad \text{strongly in} \ \ L^2((0,l_1); H^2(0,l_2)) \tag{2.96}$$

for $\varepsilon \downarrow 0$.

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PROOF. Our argument is inspired in some ideas contained in Lions [24] (see, e.g. Theorem 10.1 Chapter I). We reformulate the bilinear form as

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) = \int_\Omega A^{\alpha\beta\lambda\mu}\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\gamma^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\rho^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} \tag{2.97}$$

$$= \mathfrak{a}_0(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^{-1/4}\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^{-1/2}\mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon, \mathbf{v}),$$

for the (positive) symmetric bilinear forms $\mathfrak{a}_{1/2}$, $\mathfrak{a}_1$, $\mathfrak{a}_{-1/4}$, $\mathfrak{a}_{-1/2}$ given by

$$\mathfrak{a}_{1/2}(\mathbf{u}, \mathbf{v}) = \int_\Omega \partial_1\partial_2 u_3 \partial_1\partial_2 v_3 d\mathbf{y},$$

$$\mathfrak{a}_1(\mathbf{u}, \mathbf{v}) = \int_\Omega \partial_1^2 u_3 \partial_1^2 v_3 d\mathbf{y},$$

$$\mathfrak{a}_{-1/2}(\mathbf{u}, \mathbf{v}) = \int_\Omega (\partial_2 u_1 + \partial_1 u_2)(\partial_2 v_1 + \partial_1 v_2) d\mathbf{y},$$

$$\mathfrak{a}_{-1/4}(\mathbf{u}, \mathbf{v}) = \frac{1}{4}\int_\Omega (\partial_2 u_2 + u_3)(\partial_2 v_2 + v_3) d\mathbf{y},$$

(for the sake of simplicity, we assumed here that different coefficients are identically equal to 1 but the general case can be treated in the same way since which is relevant is the order of ε in the above expansion) and where, due to the assumption (2.93),

$$\mathfrak{a}_0(\mathbf{u}, \mathbf{v}) = \int_\Omega A^{1111}\partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y}. \tag{2.98}$$

We have that

$$\mathfrak{a}_0(\mathbf{u}^\varepsilon - \mathbf{u}^*, \mathbf{u}^\varepsilon - \mathbf{u}^*) + \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + \varepsilon^{-1/4}\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + \varepsilon^{-1/2}\mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon)$$

$$= \int_\Omega F_3(y_1, y_2)u_3^\varepsilon(y_1, y_2)d\mathbf{y} - 2\mathfrak{a}_0(\mathbf{u}^*, \mathbf{u}^\varepsilon) + \mathfrak{a}_0(\mathbf{u}^*, \mathbf{u}^*) \to$$

$$\to \int_\Omega F_3(y_1, y_2)u_3^*(y_1, y_2)d\mathbf{y} - \mathfrak{a}_0(\mathbf{u}^*, \mathbf{u}^*) = 0.$$

Then, by the above theorem (and since (2.93) implies (2.81))

$$\int_\Omega \frac{1}{C_{1111}}(\partial_1 u_1^* - \partial_1 u_1^\varepsilon)^2 d\mathbf{y} + \int_\Omega B^{2222}(\partial_2^2 u_3^* - \partial_2^2 u_3^\varepsilon)d\mathbf{y} \to 0,$$

which, by using the a priori estimates, leads to the result. $\qquad\square$

We emphasize that the limit problem (in terms of φ) is given by the variational formulation (2.80). The corresponding higher order partial differential equation for φ is obviously

$$(\frac{1}{C_{1111}}\partial_1^4 + B^{2222}\partial_2^8)\varphi = -\partial_2^2 F_3. \tag{2.99}$$

which may be a little misstating when considered without the corresponding boundary conditions (on $\Gamma_l := [0, l_1] \times \{0\} \cup [0, l_1] \times \{l_2\}$). Indeed, looking at (2.99) one may think that the data (and then the solution) vanishes when F_3 is affine with respect to y_2 (as in that case the right hand side of (2.99) vanishes). In fact, this is not the case as the natural boundary conditions are not homogeneous in general. This is a consequence of the very peculiar form of the right hand side of the variational formulation (2.80), which involves $\partial_2^2\psi$ instead of the test function ψ itself.

Let us write down the natural boundary conditions assuming, as usual, that F_3 and the solution are sufficiently smooth (e.g. $F_3, \partial_2^2 F_3 \in L^2(\Omega)$) then

$$\int_\Omega F_3 \partial_2^2 \psi d\mathbf{y} = \int_\Omega \partial_2 F_3 (\partial_2 \psi) d\mathbf{y} - \int_\Omega (\partial_2 F_3) \partial_2 \psi d\mathbf{y}$$

$$= \int_0^{l_1} dy_1 \left. (F_3 \partial_2 \psi) \right|_0^{l_2} - \int_\Omega \partial_2 [(\partial_2 F_3) \psi] d\mathbf{y} + \int_\Omega (\partial_2^2 F_3) \psi d\mathbf{y} \qquad (2.100)$$

$$= \int_0^{l_1} dy_1 \left. (F_3 \partial_2 \psi) \right|_0^{l_2} - \int_0^{l_1} dy_1 \left. [(\partial_2 F_3) \psi] \right|_0^{l_2} + \int_\Omega (\partial_2^2 F_3) \psi d\mathbf{y}.$$

Analogously, if we assume that $\varphi, \partial_2^8 \varphi \in L^2(\Omega)$ then

$$\int_\Omega \partial_2^4 \varphi \partial_2^4 \psi d\mathbf{y} = \int_0^{l_1} dy_1 \left. (\partial_2^4 \varphi \partial_2^3 \psi) \right|_0^{l_2}$$

$$- \int_0^{l_1} dy_1 \left. ((\partial_2^5 \varphi \partial_2^2 \psi) \right|_0^{l_2} + \int_0^{l_1} dy_1 \left. ((\partial_2^6 \varphi \partial_2 \psi) \right|_0^{l_2} \qquad (2.101)$$

$$- \int_0^{l_1} dy_1 \left. ((\partial_2^7 \varphi \psi) \right|_0^{l_2} + \int_\Omega (\partial_2^8 \varphi) \psi d\mathbf{y}.$$

Then, from (2.100) and (2.101), and as the *test* functions ψ, $\partial_2 \psi$, $\partial_2^2 \psi$ and $\partial_2^3 \psi$ are arbitrary, we deduce that the natural boundary conditions on $\Gamma_l := [0, l_1] \times \{0\} \cup [0, l_1] \times \{l_2\}$ are

$$\begin{cases} B^{2222} \partial_2^7 \varphi = -\partial_2 F_3, & B^{2222} \partial_2^6 \varphi = -F_3 & \text{on } \Gamma_l, \\ \\ \partial_2^5 \varphi = \partial_2^4 \varphi = 0 & \text{on } \Gamma_l, \end{cases} \qquad (2.102)$$

and so, the two first boundary conditions depend on the right hand side of the partial differential equation.

In fact, the previous Theorem 2.1 can be applied when, merely, $F_3 \in L^2(\Omega)$. Then, although $\widetilde{G} \subset \tilde{H}^2((0, l_1); L^2(0, l_2))$ the variational formulation is not enough as to formulate separately the partial differential equation (2.99) from the boundary conditions (2.102). For instance, let us consider the function $F_3(y_1, y_2) = (l_2 - y_2)^\alpha$ with $\alpha \in (-\frac{1}{2}, 0)$. Then, since $F_3 \in L^2(\Omega)$, the variational formulation makes sense whereas boundary conditions (2.102) do not, as the traces of F_3 and $\partial_2 F_3$ on Γ_l do not exist. It should be noticed that in the Lions-Magenes ([25]) theory (which is nevertheless only concerned with elliptic problems, and so out of our framework) singular right hand side terms are only allowed when their singularities are located at the interior of Ω and not when they are in a vicinity of the boundary. This is associated with the fact that the allowed right hand side should belongs to the space $\Xi^s(\Omega)$, $s < 0$, which are analogous to the space $H^s(\Omega)$, $s < 0$, inside of Ω but not near the boundary $\partial\Omega$ where $\Xi^s(\Omega)$ only contains smother functions (see [25], Section 6.3, Chapter 2).

Remark 2.9 *The problem (2.99) is parabolic according the theory of linear partial differential equations (see, e.g. [40]). Indeed, the characteristics are find as normal curves to the vectors (ξ_1, ξ_2) satisfying that $P^m(\xi_1, \xi_2) = 0$, where P^m is the "principal symbol" of the differential operator. In our case, $P^m(\xi_1, \xi_2) = B^{2222} \xi_2^8$, and so, $\xi_2 = 0$ is a multiple characteristic (of 8th-order). Thus, the passing to the limit arguments show that the parabolicity of the middle surface leads to a limit equation as (2.99) of parabolic type, with characteristic of multiplicity 8.*

Remark 2.10 *As in previous Remarks 2.4, and 2.5, if we define the bilinear form*

$$a^0(\mathbf{u}, \mathbf{v}) \stackrel{def}{=} \int_\Omega \frac{1}{C_{1111}} \partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222} \partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y}, \qquad (2.103)$$

then the symmetry of $a^0(\mathbf{u}, \mathbf{v})$ shows that the (unique) solution $\mathbf{u}$ of problem Π_0 can be characterized as the unique element of $\mathbf{G}$ solving the minimization problem

$$Min_{\mathbf{G}} J_0(\mathbf{v}) \qquad (2.104)$$

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where

$$J_0(\mathbf{v}) = \frac{1}{2}a^0(\mathbf{v}, \mathbf{v}) - \int_\Omega F_3 v_3 d\mathbf{y}. \tag{2.105}$$

We can formulate, equivalently, this property in terms of the potential φ (which is more relevant since, as mentioned at the introduction, (2.80) corresponds to a partial differential equation of parabolic type). So, the (unique) solution $\varphi \in \widetilde{G}$ of problem Π_0 can be characterized as the unique element of $\widetilde{G}$ solving the minimization problem

$$Min_{\widetilde{G}} \widetilde{J}_0(\psi) \tag{2.106}$$

where

$$\widetilde{J}_0(\psi) = \frac{1}{2C_{1111}} \int_\Omega \left|\partial_1^2 \psi\right|^2 d\mathbf{y} + \frac{B^{2222}}{2} \int_\Omega \left|\partial_2^4 \psi\right|^2 d\mathbf{y} + \int_\Omega F_3 \partial_2^2 \psi d\mathbf{y}, \tag{2.107}$$

or

$$\widetilde{J}_0(\psi) = \frac{1}{2}\widetilde{a}^0(\psi, \psi) + \int_\Omega F_3 \partial_2^2 \psi d\mathbf{y}, \tag{2.108}$$

with

$$\widetilde{a}^0(\varphi, \psi) = \frac{1}{C_{1111}} \int_\Omega \partial_1^2 \varphi \partial_1^2 \psi d\mathbf{y} + \frac{B^{2222}}{2} \int_\Omega \partial_2^4 \varphi \partial_2^4 \psi d\mathbf{y}. \tag{2.109}$$

We point out that very few variational principles hold for the case of parabolic problems (for some other exceptional variational principle for parabolic equations, of a very different nature, see [6]).

Remark 2.11 *The more realistic case in which the shell is clamped merely by a part of the "small curved boundary" ($\{0\} \times [0, \delta\eta l_2]$ for some $\delta \in (0, 1)$) is likely analogous but it may arise many technical difficulties in the definition of the functional spaces and in the proof of the density Lemma (2.9) due to the peculiar constraints of the formulation.*

Remark 2.12 *Obviously, to the implicit boundary condition on Γ_l given in (2.102) we must add the rest of boundary conditions. So, for instance, the fact that the boundary $\{l_1\} \times [0, l_2]$ is free leads to*

$$\begin{cases} T^{11} = 0 & \text{on } \{l_1\} \times [0, l_2], \\ \partial_1 T^{11} = 0 & \text{on } \{l_1\} \times [0, l_2]. \end{cases} \tag{2.110}$$

Remark 2.13 *The case $b = -1/R$ can be also considered with obvious modifications. For instance, in the rescaling change (2.23) we must assume now that $\tilde{u}_3(\mathbf{x}) = \eta^{\theta-2} |b|^{-1} u_3(\mathbf{y})$. We point out that corresponding sign changes at the different equations may justify the different behavior of solutions with respect the case of $b = 1/R$. Easy comparison experiences can be made by using a flexible steel retractable meter tape measure in its normal and reverse positions.*

3   The shell has an edge with slight folding.

In this section we consider a case slightly more complicated than the basic problem, when the section by $y_1 = const.$ is as sketched in Fig 2.

For reasons concerned with applications to homogenization problems (work in progress [16]), the tangent plane on $y_2 = -l_2/2$ and $y_2 = l_2/2$ is horizontal. This amounts to saying that the angle of the folding is 2ω, with $\omega = b\eta l_2/2$ (see Fig 2) where b always denote the (constant) curvature. Denoting by $\tilde{u}_i^-$ and $\tilde{u}_i^+$ the traces on $x_2 = 0$, the continuity of the displacement $\tilde{\mathbf{u}}$ at the folding gives in the projections along x_1, its normal in the "base plane" and the axis x_3 (see Fig. 2) respectively:

$$\begin{cases} \tilde{u}_1^+ = \tilde{u}_1^- \\ -sin\omega\tilde{u}_3^+ + cos\omega\tilde{u}_2^+ = sin\omega\tilde{u}_3^- + cos\omega\tilde{u}_2^- \\ cos\omega\tilde{u}_3^+ + sin\omega\tilde{u}_2^+ = cos\omega\tilde{u}_3^- - sin\omega\tilde{u}_2^- \end{cases} \tag{3.1}$$
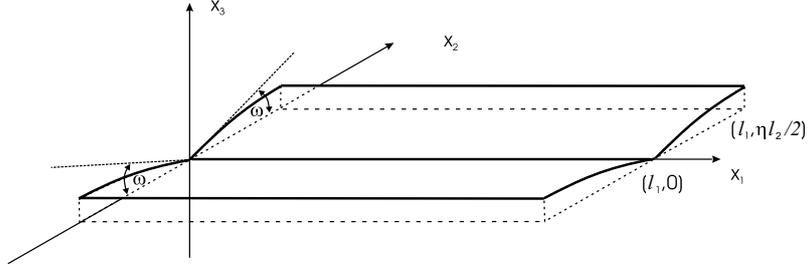
Figure 2

In order to avoid irrelevant and cumbersome expressions, as ω is small, we shall take $cos\omega = 1$, $sin\omega = \omega$. Moreover, we shall see in the sequel that the components u_3 are asymptotically larger than u_2, and we shall neglect $\omega^2 u_2$ with respect to u_3. Then we shall consider

$$\begin{cases} \tilde{u}_1^+ = \tilde{u}_1^- \\ -\omega\tilde{u}_3^+ + \tilde{u}_2^+ = \omega\tilde{u}_3^- + \tilde{u}_2^- \\ \tilde{u}_3^+ = \tilde{u}_3^- \end{cases} \tag{3.2}$$

so that we merely may keep in mind that $\tilde{u}_1$ and $\tilde{u}_3$ are continuous across $x_2 = 0$ and

$$\tilde{u}_2^+ - \tilde{u}_2^- = 2\omega\tilde{u}_3. \tag{3.3}$$

Remark 3.1 *It is possible to consider other possibilities as, for instance, the case in which the coupling is rigid (perhaps with a small jump in the tangent plane). Then, the condition that the angle ω is not modified by the displacement gives:*

$$\tilde{\partial}_2\tilde{u}_3^+ = \tilde{\partial}_2\tilde{u}_3^-. \tag{3.4}$$

We also mention that possible analogies with [21] and [22] are not obvious, as this author starts from three-dimensional elasticity, and with folding angle of order $0(1)$, which is in our problem very small.

The plate is considered to be clamped along a "small side" of $x_1 = 0$, which implies again the kinematic boundary conditions (2.4) (obviously along with the corresponding one for $x_2 < 0$).

Let us denote

$$\Omega_\varepsilon^+ = (0, l_1) \times (0, \eta l_2/2) \quad \text{and} \quad \Omega_\varepsilon^- = (0, l_1) \times (-\eta l_2/2, 0) \tag{3.5}$$

and we shall also denote by Ω_ε the union of Ω_ε^+ and Ω_ε^-. The space of configuration will be denoted by $\mathbf{V}_\varepsilon$. It is the subspace of

$$H^1(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^+) \times H^2(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^-) \times H^1(\Omega_\varepsilon^-) \times H^2(\Omega_\varepsilon^-) \tag{3.6}$$

formed by the functions satisfying the kinematic boundary conditions

$$\begin{aligned} 0 &= \tilde{u}_1^+ = \tilde{u}_2^+ = \tilde{u}_3^+ \quad \text{on } \{0\} \times [0, \eta l_2/2], \\ 0 &= \tilde{u}_1^- = \tilde{u}_2^- = \tilde{u}_3^- \quad \text{on } \{0\} \times [-\eta l_2/2, 0], \end{aligned} \tag{3.7}$$

and the transmission conditions (3.2). We observe that by virtue of the equalities of traces of $\tilde{u}_1$ and $\tilde{u}_3$, the functions $\tilde{u}_1$ and $\tilde{u}_3$ may be considered as elements of $H^1(\Omega_\varepsilon)$, whereas $\tilde{u}_2$ must be considered separately in $H^1(\Omega_\varepsilon^+)$ and $H^1(\Omega_\varepsilon^-)$ (note that, in fact, $\tilde{u}_3^+ \in H^2(\Omega_\varepsilon^+)$, $\tilde{u}_3^- \in H^2(\Omega_\varepsilon^-)$ and that, in the case mentioned in Remark 3.1, $\tilde{u}_3 \in H^2(\Omega_\varepsilon)$).

The "variational or weak formulation" of the elasticity problem for this structure takes again the form

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \tag{3.8}$$

with

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon^+} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}^+) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}^+) d\mathbf{x} + \int_{\Omega_\varepsilon^-} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}^-) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}^-) d\mathbf{x} \tag{3.9}$$

$$b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon^+} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}^+) \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}^+) d\mathbf{x} + \int_{\Omega_\varepsilon^-} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}^-) \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}^-) d\mathbf{x}, \tag{3.10}$$

where we are using the obvious decomposition $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}^+, \tilde{\mathbf{v}}^-)$ for any element of the energy space $\mathbf{V}_\varepsilon$. As in Subsection 2.1, an easy application of the Lax-Milgram theorem allows to see that this problem has a unique solution depending on the parameter ε.

The scaling and other developments are then analogous to those of the "basic problem". The space of configuration after scaling will be denoted by $\mathbf{V}$. It is the subspace of

$$H^1(\Omega^+) \times H^1(\Omega^+) \times H^2(\Omega^+) \times H^1(\Omega^-) \times H^1(\Omega^-) \times H^2(\Omega^-) \tag{3.11}$$

formed by the functions satisfying the transmission and kinematic boundary conditions

$$u_1^+ = u_1^-, \quad u_3^+ = u_3^- \text{ and } u_2^+ - u_2^- = l_2 u_3 \tag{3.12}$$

and

$$0 = u_1 = u_2 = u_3 \text{ on } \{0\} \times [-l_2/2, l_2/2], \tag{3.13}$$

(here we used again a notation corresponding to the decomposition $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}^+, \tilde{\mathbf{v}}^-)$ for any element of the energy space $\mathbf{V}$).

The scaled problem is stated as in Subsection 2.2 with obvious modifications. Those of the estimates (2.47) to (2.57) which involve u_2 should be considered separately in Ω^+ and Ω^-. For instance, (2.56) should be considered as two separate inequalities involving $H^{-2}(-l_2/2, 0)$ and $H^{-2}(0, +l_2/2)$. Obviously, Lemma 2.7 should be modified in the same way, as well as the definition of the space $\mathbf{G}$ for the limit problem.

The construction of the equivalent space $\widetilde{G}$ for the potential deserves of a comment. We use (2.76) in each one of the domains Ω^+ and Ω^-, and we always prescribe ψ to vanish on $y_1 = 0$ (obviously for $y_2 \in (-l_2/2, l_2/2)$). Then, as the traces of u_1 are equal on both sides of $y_2 = 0$, integrating the first equation (2.76), we see that ψ itself has equal traces on both sides of $y_2 = 0$. The other transmission conditions to be prescribed (for the traces of u_2, u_3 and $\partial_2 u_3$) are only concerned with the sections by $y_1 = const.$; as a result, we merely replace $H^4(0, l_2/2)$ by the (closed) subspace of $H^4(-l_2/2, 0)_{y_2} \times H^4(0, l_2/2)$ formed by the functions satisfying on $y_2 = 0$ the transmission conditions:

$$\begin{cases} \psi^+ = \psi^- \\ \partial_2^2 \psi^+ = \partial_2^2 \psi^- \\ 2l_2 \partial_2^2 \psi = \partial_2 \psi^+ - \partial_2 \psi^-. \end{cases} \tag{3.14}$$

The proof of the convergence (in the analogous topologies indicated in (2.71) - (2.74) (lemma 2.7) but now adapted to the coupled domain Ω, follows the same arguments except for the proof of the properties $T^{22*} = T^{12*} = 0$. For instance, to prove that $T^{22*} = 0$ we take a test function $v = \eta w$ with

$$w_1 \in C_0^\infty((0, l_1); C^\infty(0, l_2/2)), \qquad w_2 = w_3 = 0, \tag{3.15}$$

i.e., w_1 vanishes merely on the upper part and is arbitrary on the lower part. Taking later the reciprocal choice (w_1 vanishing merely on the lower part) we get that $T^{22*} = 0$. Analogously, to prove that $T^{12*} = 0$ we first take w_1 vanishing on the upper part and also in a small subinterval of the lower part $(-\beta_2, l_2/2)$ and we conclude that $T^{12*} = 0$ in Ω^-. By taking the reciprocal choice (w_1 vanishing on $(-l_2/2, \beta_2)$ with $\beta_2 > 0$ arbitrarily small) we get that $T^{12*} = 0$ in Ω^+.

In conclusion, we get to the following result

Theorem 3.1 *Let $\mathbf{u}^\varepsilon$ and $\mathbf{u}$ be the solutions of the above coupled problems respectively. Then, for $\varepsilon \downarrow 0$, we have:*

$$\mathbf{u}^\varepsilon \to \mathbf{u}$$

with convergence of each component $\mathbf{u}^\varepsilon = (\mathbf{u}^{\varepsilon+}, \mathbf{u}^{\varepsilon-})$ in the topologies indicated in (2.71) - (2.73) (lemma 2.7). Equivalently, the limit $\mathbf{u}$ can be obtained through its potential $\varphi = (\varphi^+, \varphi^-) \in \widetilde{G}$, solution of the problem

$$
\begin{aligned}
&\int_{\Omega^+} \frac{1}{C_{1111}} \partial_1^2 \varphi^+ \partial_1^2 \psi^+ \, d\mathbf{y} + \int_{\Omega^-} \frac{1}{C_{1111}} \partial_1^2 \varphi^+ \partial_1^2 \psi^+ \, d\mathbf{y} \\
&+ \int_{\Omega^+} B^{2222} \partial_2^4 \varphi^+ \partial_2^4 \psi^+ \, d\mathbf{y} + \int_{\Omega^-} B^{2222} \partial_2^4 \varphi^- \partial_2^4 \psi^- \, d\mathbf{y} \\
= \quad &- \int_{\Omega^+} F_3 \partial_2^2 \psi^+ \, d\mathbf{y} - \int_{\Omega^-} F_3 \partial_2^2 \psi^- \, d\mathbf{y}, \qquad\qquad\qquad (3.16)
\end{aligned}
$$

$\forall \psi = (\psi^+, \psi^-) \in \widetilde{G}.$

A strong convergence result (similar to Theorem 2.3) can be analogously proved.

Remark 3.2 *The "V" shape of the basic coupled structure has been used very often in many famous structures since it has many design advantages in comparison with other basic coupled structures. In the framework of its mathematical modeling we must point out, for instance that the dual energy space $\mathbf{V}'_\varepsilon$ contains elements which are not defined in the product of the dual of the energy spaces corresponding to each part $\mathbf{V}_\varepsilon(\Omega^+)' \times \mathbf{V}_\varepsilon(\Omega^-)'$ allowing to justify the presences of charges of the type $F_3(x_1)\delta_0(x_2)$ (with $\delta_0(x_2)$ the Dirac delta distribution concentrated on $x_2 = 0$) which are not justified in the space $\mathbf{V}_\varepsilon(\Omega^+)' \times \mathbf{V}_\varepsilon(\Omega^-)'$.*

4 Periodic shells

We consider now the case in which the shell is $2\eta l_2-$periodic with respect the section by $x_1 = const.$projected on the band $(0, l_1) \times (-\infty, +\infty)$ and having a slight folding at any section of the form $(0, l_1) \times \{k\eta l_2\}$ with $k \in \mathbb{Z}$, as sketched in Fig 3.
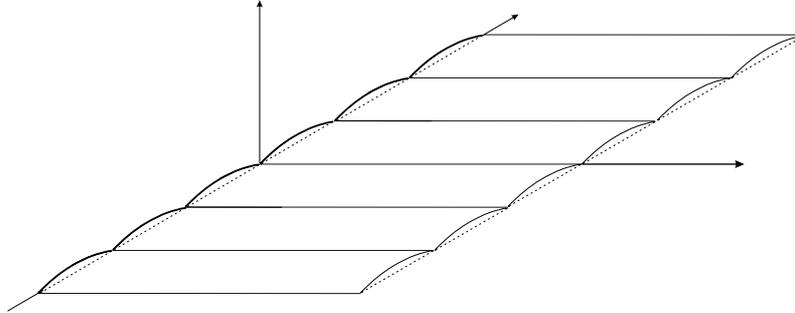
Figure 3

We can consider as "unit shell" the shell defied trough the rectangle $(0, l_1) \times (-\eta l_2/2, +\eta l_2/2)$, clamped along the "small sides" at $\{0\} \times [-\eta l_2/2, \eta l_2/2]$, which implies again kinematic boundary conditions similar to those indicated at (2.4). We then consider periodic loadings and search for periodic solutions. Moreover, for technical reasons of the proofs, we shall only consider the case when the loading on each period is symmetric with respect to its center (see (4.6) hereafter).

The formulation follows obvious modifications of the above section. We define again

$$\Omega_\varepsilon^+ = (0, l_1) \times (0, \eta l_2) \quad \text{and} \quad \Omega_\varepsilon^- = (0, l_1) \times (-\eta l_2, 0) \tag{4.1}$$

and we shall denote by Ω_ε^1 the union of Ω_ε^+ and Ω_ε^-, and, given $k \in \mathbb{Z}$, by $\Omega_\varepsilon^k = \{(x_1, x_2) \text{ such that } x_1 \in (0, l_1) \text{ and } x_2 - k \in (-\eta l_2, +\eta l_2)\}$. Finally we denote by $\Omega_\varepsilon = \cup_{k \in \mathbb{Z}} \Omega_\varepsilon^k = (0, l_1) \times (-\infty, +\infty)$.

The space of configuration will be denoted by $\mathbf{V}_\varepsilon$. It is the subspace of

$$H^1(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^+) \times H^2(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^-) \times H^1(\Omega_\varepsilon^-) \times H^2(\Omega_\varepsilon^-) \tag{4.2}$$

formed by the functions satisfying the kinematic boundary, transmission conditions (2.4), (3.2) and (3.5) as well as the periodicity conditions

$$\tilde{v}_1(x_1, -\eta l_2) = \tilde{v}_1(x_1, \eta l_2), \quad \tilde{v}_2(x_1, -\eta l_2) = \tilde{v}_2(x_1, \eta l_2), \quad \tilde{v}_3(x_1, -\eta l_2) = \tilde{v}_3(x_1, \eta l_2), \tag{4.3}$$

and

$$\tilde{\partial}_2 \tilde{v}_3(x_1, -\eta l_2) = \tilde{\partial}_2 \tilde{v}_3(x_1, \eta l_2). \tag{4.4}$$

The "variational or weak formulation" of the elasticity problem for this structure takes again the form $\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$ with $a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ and $b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ as in the previous Subsection and an easy application of the Lax-Milgram theorem shows that this problem has a unique solution depending on the parameter ε.

The convergence arguments (analogous to Theorem 2.2) follow as in previous Subsections with easy modifications. For instance, it is possible to show that the potential function ψ given in Lemma 2.8 becomes now a periodic function. Indeed, take any symmetric points $\mathbf{A} = (y_1, -l_2)$ and $\mathbf{B} = (y_1, l_2)$ with arbitrary $y_1 \in (0, l_1)$. Define the points $\mathbf{C} = (0, -l_2)$ and $\mathbf{D} = (0, l_2)$. Then, to show that $\psi(\mathbf{A}) = \psi(\mathbf{B})$ we take a piecewise straight path linking $\mathbf{A}, \mathbf{C}, \mathbf{D}$ and $\mathbf{B}$. By using the properties $v_1 = \partial_1 \psi$, $v_2 = -\partial_2 \psi$, $v_3 = -\partial_2^2 \psi$, and the boundary conditions $0 = v_1 = v_2 = v_3$ on $\{0\} \times [-l_2, l_2]$ we deduce that $\psi(\mathbf{C}) = \psi(\mathbf{D})$. It also follows that the increment of ψ from $\mathbf{A}$ to $\mathbf{C}$ is the opposite of the increment of ψ from $\mathbf{D}$ to $\mathbf{B}$ (due to the periodicity of v_1). So that $\psi(\mathbf{A}) = \psi(\mathbf{B})$ which proves the periodicity of ψ.

On the other hand, the periodicity conditions (4.3), (4.4) imply that

$$\partial_1 \psi(x_1, -\eta l_2) = \partial_1 \psi(x_1, \eta l_2), \quad \partial_2 \psi(x_1, -\eta l_2) = \partial_2 \psi(x_1, \eta l_2), \quad \partial_2^2 \psi(x_1, -\eta l_2) = \partial_2^2 \psi(x_1, \eta l_2),$$

and

$$\partial_2^3 \psi(x_1, -\eta l_2) = \partial_2^3 \psi(x_1, \eta l_2).$$

As in the previous section, the proof of the convergence (in the analogous topologies indicated in (2.71) - (2.73) (lemma 2.7) but now adapted to the periodic domain Ω, follows the same arguments except, again, for the proof of the properties $T^{22*} = T^{12*} = 0$. We first prove that $T^{22*} = 0$ by arguing as in (2.83)-(2.86) en each region Ω^- and Ω^+. Analogously, to prove that $T^{12*} = 0$ we proceed as in (2.88)-(2.44) over Ω^- and Ω^+. In our case $\partial_2 w_1$ is not arbitrary since $\int \partial_2 w_1 d\mathbf{y} = 0$. So,we merely get that

$$0 = \int_{\Omega^+} T^{12*} \partial_2 w_1 d\mathbf{y} = -\frac{1}{2} \int_{\Omega^+} \partial_2 T^{12*} w_1 d\mathbf{y}.$$

But this implies that $\partial_2 T^{12*} = 0$ and so

$$T^{12*} = T^{12*}(y_1). \tag{4.5}$$

Then, if the loading is symmetric

$$F_3(x_1, x_2) = F_3(x_1, -x_2) \tag{4.6}$$

we get that (from the uniqueness of solutions) T^{12*} is antisymmetric, $T^{12*}(x_1, x_2) = -T^{12*}(x_1, -x_2)$, which jointly with (4.5) implies that $T^{12*} = 0$.

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We can avoid the symmetry assumption (4.6) by using the fact that the boundary $x_1 = l_1$ is free. Indeed, we take $w_1 = 0$, $w_2 \in C_0^\infty(\Omega^+)$, $w_2 = -w_{3\alpha}$ and $v = \eta w$. Then, the only non vanishing $\gamma_{\alpha\beta}^\varepsilon$ is $\gamma_{12}^\varepsilon = \frac{1}{2}\partial_1 w_2$. Thus, passing to the limit $\frac{1}{2}\int_{\Omega^+} T^{12*}\partial_1 w_2 dy_2 = 0$ and as $\partial_1 w_2$ is arbitrary in Ω^+ we deduce that $T^{12*} = 0$ in Ω^+ (and similarly $T^{12*} = 0$ in Ω^-).

A strong convergence result (similar to Theorem 2.3) can be also analogously proved for the periodic case.

5 Global qualitative properties of solutions of the limit problems

Let us first consider the "basic problem" of section 2. In order to simply the exposition, we shall assume, additionally, that the charge satisfy the property

$$F_3 \text{ is sufficiently smooth and vanishes in the neighborhood of } y_2 = 0 \text{ and } y_2 = l_2. \tag{5.1}$$

The main goal of this section is to get some properties on the potential solution $\varphi(y_1, y_2)$ which we call as "global" since they will be stated in terms of integrals of the type $\int_0^1 \varphi(y_1, y_2)dy_2$ and give some information on the y_1-dependence of such terms. These properties can be connected with some information of the resultant mechanical moments with respect to the sections $y_1 = $ constant.

Note that, due to (5.1), by (2.102), the potential solution satisfies the natural boundary conditions

$$\begin{cases} \partial_2^7 \varphi = 0 & \text{on } \Gamma_l, \\ \partial_2^6 \varphi = 0 & \text{on } \Gamma_l, \\ \partial_2^5 \varphi = 0 & \text{on } \Gamma_l, \\ \partial_2^4 \varphi = 0 & \text{on } \Gamma_l, \end{cases} \tag{5.2}$$

where $\Gamma_l := [0, l_1] \times \{0\} \cup [0, l_1] \times \{l_2\}$. In particular,

$$\int_0^{l_2} \partial_2^8 \varphi(y_1, y_2)dy_2 = \partial_2^7 \varphi(y_1, y_2)\big|_0^{l_2} = 0.$$

Moreover, the resultant moments also vanishes since

$$\begin{aligned} \int_0^{l_2} \partial_2^8 \varphi(y_1, y_2)y_2 dy_2 &= \partial_2^7 \varphi(y_1, y_2)y_2\big|_0^{l_2} - \int_0^{l_2} \partial_2^7 \varphi(y_1, y_2)dy_2 \\ &= \partial_2^7 \varphi(y_1, y_2)y_2\big|_0^{l_2} - \partial_2^6 \varphi(y_1, y_2)\big|_0^{l_2} = 0, \end{aligned}$$

and analogously

$$\int_0^{l_2} \partial_2^8 \varphi(y_1, y_2)y_2^\alpha dy_2 = 0, \text{ for } \alpha = 0, 1, 2, 3. \tag{5.3}$$

In fact, we have

Theorem 5.1 *For $\alpha = 0, 1, 2, 3$ we have that*

$$\int_0^{l_2} T^{11}(y_1, y_2)y_2^\alpha dy_2 = -\int_{y_1}^{l_1} \int_0^{l_2} \alpha(\alpha - 1)(\lambda - y_1)F_3(\lambda, y_2)y_2^{\alpha-2}dy_2 d\lambda.$$

PROOF. From the partial differential equation (2.99) and the previous property (5.3) we get that

$$\frac{1}{C_{1111}}\partial_1^4 \int_0^{l_2} \varphi(y_1, y_2)y_2^\alpha dy_2 = -\int_0^{l_2} \partial_2^2 F_3(y_1, y_2)y_2^\alpha dy_2, \tag{5.4}$$

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which using

$$\partial_1 \varphi = u_1 \text{ and } \frac{1}{C_{1111}} \partial_1^2 \varphi = T^{11},$$

becomes

$$\partial_1^2 \int_0^{l_2} T^{11}(y_1, y_2) y_2^\alpha dy_2 = \int_0^{l_2} \partial_2^2 F_3(y_1, y_2) y_2^\alpha dy_2. \tag{5.5}$$

Integrating twice (5.5) in y_1 and on account of the boundary conditions on $\{l_1\} \times [0, l_2]$ (2.110), we obtain the expression

$$\int_0^{l_1} T^{11}(y_1, y_2) y_2^\alpha dy_2 = \int_{y_1}^{l_1} \int_\mu^{l_1} \int_0^{l_2} \partial_1^2 F_3(\lambda, y_2) y_2^\alpha d\mu d\lambda dy_1 = \int_{y_1}^{l_1} \int_0^{l_2} \alpha(\alpha - l_2)(\lambda - y_1) F_3(\lambda, y_2) y_2^{\alpha-2} d\lambda dy_2, \tag{5.6}$$

which proves the result. Indeed, we first introduce the notation $F(y_1) := \int_{y_1}^{l_1} f(\mu) d\mu$, for a given integrable $f \in L^1(0, l_1)$, and then we introduce $M(y_1) := \int_{y_1}^{l_1} F(\mu) d\mu = \int_{y_1}^{l_1} \int_\mu^{l_1} f(\lambda) d\mu d\lambda$. By Fubini's theorem, $M(y_1)$ represents the area in the triangular region (subset of $(0, l_1) \times (0, l_1)$) of vertices (y_1, y_1), (y_1, l_1) and (l_1, l_1). But, then, we can change the order of the integrations so that

$$M(y_1) = \int_{y_1}^{l_1} (\lambda - y_1) f(\lambda) d\mu. \tag{5.7}$$

Now, we integrate by parts twice in (5.5) with respect to y_1, and apply (5.7) with $f(\lambda) := \partial_2^2 F_3(\lambda, y_2) y_2^\alpha$. Finally, we integrate by parts again twice, now with respect to y_2, and, using that F_3 vanishes near the boundary of $(0, l_2)$, we obtain (5.6). $\qquad \square$

Note that, taking $\alpha = 0, 1$ we get that

$$\int_0^{l_2} T^{11}(y_1, y_2) dy_2 = \int_0^{l_2} T^{11}(y_1, y_2) y_2 dy_2 = 0 \tag{5.8}$$

which merely indicate that the resultant, on any section $y_1 = $constant, of the internal tensor stress component T^{11} and of its momentum with respect y_2 vanishes, which is due to the assumption that the loading has only a vertical component. The choice $\alpha = 2$ gives

$$\int_0^{l_2} T^{11}(y_1, y_2) y_2^2 dy_2 = -2 \int_{y_1}^{l_2} \int_0^{l_2} (\lambda - y_1) F_3(\lambda, y_2) d\lambda dy_2. \tag{5.9}$$

which can be understood as a balance between the moment with respect a horizontal plane (recall that the shape of the original section has the form $z = C y_2^2$). Finally, formula

$$\int_0^{l_2} T^{11}(y_1, y_2) y_2^3 dy_2 = -6 \int_{y_1}^{l_2} \int_0^{l_2} (\lambda - y_1) F_3(\lambda, y_2) y_2 d\lambda dy_2 \tag{5.10}$$

(which arise by taking $\alpha = 3$) does not seems to have a so easy mechanical meaning and looks remarkably more unexpected than the previous formulae.

It is possible to give an abstract version of the above properties. This is specially useful due to its application to more sophisticated formulations as the case of the shell has an edge with slight folding considered in Section 3 for which the transmission conditions made very complex the above direct analysis.

In order to search an abstract formulation of the key identities (5.3) and (5.4) we shall consider the solutions φ of (2.99) as a function of y_1 with values in a space of functions on the y_2 variable (as, for instance, $L^2(0, l_2)$). Let us introduce the notation

$$\partial_2^2 F_3 = \widehat{F_3}, \tag{5.11}$$

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and let C be the operator $B^{2222}\partial_2^8$ with natural boundary conditions

$$\partial_2^4 = \partial_2^5 = \partial_2^6 = \partial_2^7 = 0, \text{ on } y_2 \in \{0\} \cup \{l_2\}. \tag{5.12}$$

Obviously, C coincides with the unbounded self-adjoint operator of the space $L^2(0, l_2)$ defined by means of the bilinear form

$$c(u,v) = B^{2222} \int_0^{l_2} \partial_2^4 u \partial_2^4 v dy_2,$$

on $H^2(0, l_2)$, i.e.,

$$c(\varphi, \psi) = \langle C\varphi, \psi \rangle$$

Then, the partial differential equation (2.99) can be written as

$$\frac{1}{C_{1111}}\partial_1^4 \varphi + C\varphi = -\widehat{F_3} \tag{5.13}$$

in which the single variable is y_1 and φ is a function of y_1 with values in $L^2(0, l_2)$ or in $H^4(0, l_2)$ (the correct notation should replace the symbol ∂_1^4 by d_1^4 but we keep with the old one for the sake of simplicity in the comparison with previous analysis).

It is easy to see that C is self-adjoint with a compact resolvent (thanks to the compactness of the imbedding $H^4(0, l_2) \subset L^2(0, l_2)$) with a resolvent which is continuous from $(H^4)'$ in H^4 and compact from $L^2(0, l_2)$ in $L^2(0, l_2)$. In consequence, its spectrum is discrete and each eigenvalue has a finite multiplicity. Moreover, since $c(u,u) \geq 0$ then the eigenvalues are real nonnegative numbers. Additionally, it is easy to see that $\lambda_1 = 0$ with multiplicity equal to 4 and that y_2^α are eigenfunctions for any $\alpha = 0, 1, 2, 3$. Since C is self-adjoint

$$\frac{1}{C_{1111}}\partial_1^4(\varphi, w)_{L^2(0,1)} + \left\langle \widehat{F_3}, w \right\rangle = -(C\varphi, w) = -(\varphi, Cw).$$

As consequence, we have

Proposition 5.1 *If* $Cw = 0$ *then*

$$\frac{1}{C_{1111}}\partial_1^4(\varphi, w)_{L^2(0,1)} = -\left\langle \widehat{F_3}, w \right\rangle = -\int_0^{l_2} \partial_2^2 F_3 w dy_2. \tag{5.14}$$

Clearly, (5.14) is the abstract version of the key identities (5.3) and (5.4).

In the rest of this section we shall consider the case of the shell has an edge with slight folding as in Section 3. Inspired in the previous analysis, the main idea to apply such approach to this problem is to construct a similar selfadjoint operator C with compact resolvent and to find the eigenfunctions associated to the eigenvalue $\lambda_1 = 0$. Computations will be carried on with $l_2 = 1$. The kinematics boundary conditions, at $y_2 = 0$, are limited to the following ones:

$$\begin{cases} [\varphi] = 0, \\ [\partial_2^2 \varphi] = 0, \\ 2\partial_2^2 \varphi = [\partial_2 \varphi], \end{cases} \tag{5.15}$$

(note that $\partial_2^2 \varphi$ makes sense since $[\partial_2^2 \varphi] = 0$) where for a general function h, $[h]$ denotes the jump at $y_2 = 0$. The bilinear form is now

$$c(\varphi, \psi) = B^{2222} \int_{-l_2}^{l_2} \partial_2^4 \varphi \partial_2^4 \psi dy_2.$$

By taking a test function with support near the points $y_2 = \pm l_2$ we get the natural conditions

$$\partial_2^4 \varphi = \partial_2^5 \varphi = \partial_2^6 \varphi = \partial_2^7 \varphi = 0, \text{ on } y_2 \in \{l_2\} \cup \{-l_2\}. \tag{5.16}$$

Concerning the five remaining associated natural conditions we have:

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Lemma 5.1 *If φ is a solution of the limit problem for the shell with an edge with slight folding. the natural conditions, at $y_2 = 0$, associated to the kinematic conditions (5.15) are*

$$\begin{cases} \partial_2^4 \varphi^+ = 0 \ \text{and} \ \partial_2^4 \varphi^- = 0 \\ 2\partial_2^6 \varphi = [\partial_2^5 \varphi] \\ [\partial_2^6 \varphi] = 0 \\ [\partial_2^7 \varphi] = 0. \end{cases} \tag{5.17}$$

PROOF. Like in (2.101), using the bilinear form $c(\varphi, \psi)$, we get that

$$\begin{cases} -\partial_2^6 \varphi^+ \partial_2 \psi^+ + \partial_2^6 \varphi^- \partial_2 \psi^- - \partial_2^4 \varphi^+ \partial_2^3 \psi^+ + \partial_2^4 \varphi^- \partial_2^3 \psi^- + \partial_2^5 \varphi^+ \partial_2^2 \psi^+ - \partial_2^5 \varphi^- \partial_2^2 \psi^- \\ + (\partial_2^7 \varphi^+) \psi^+ - (\partial_2^7 \varphi^-) \psi^- = 0 \end{cases} \tag{5.18}$$

for any $\psi = (\psi^+, \psi^-)$ satisfying (5.15). In particular, we must have $\psi^+ = \psi^-$ (otherwise arbitrary), and thus $[\partial_2^7 \varphi] = 0$. Moreover from (5.15) we see that $\partial_2^3 \psi^+$ and $\partial_2^3 \psi^-$ are arbitrary, which implies that $\partial_2^4 \varphi^+ = 0$ and $\partial_2^4 \varphi^- = 0$. Analogously, $\partial_2 \psi^+$ and $\partial_2 \psi^-$ are arbitrary and $\partial_2^2 \psi^+ = \partial_2^2 \psi^- = \frac{1}{2}[\partial_2 \psi]$. Thus, (5.18) leads to

$$[\partial_2^5 \varphi] \frac{1}{2} [\partial_2 \psi] - \partial_2^6 \varphi^+ \partial_2 \psi^+ + \partial_2^6 \varphi^- \partial_2 \psi^- = 0 \tag{5.19}$$

and since $\partial_2 \psi^+$ and $\partial_2 \psi^-$ are arbitrary we get that $[\partial_2^6 \varphi] = 0$ and that $2\partial_2^6 \varphi = [\partial_2^5 \varphi]$. $\qquad\square$

Concerning the eigenfunctions associated to the eigenvalue $\lambda_1 = 0$, some easy computations allows to check the following result.

Proposition 5.2 *The function*

$$w(y_2) = \begin{cases} A^+ + B^+ y_2 + \frac{C^+}{2} y_2^2 + \frac{D^+}{6} y_2^3, & \text{if } y_2 > 0, \\ A^- + B^- y_2 + \frac{C^-}{2} y_2^2 + \frac{D^-}{6} y_2^3 & \text{if } y_2 < 0, \end{cases}$$

are eigenfunctions associated to the eigenvalue $\lambda_1 = 0$ once we assume

$$\begin{cases} A^+ = A^- \\ C^+ = C^- \\ 2C^+ = B^+ - B^-. \end{cases}$$

In particular, there are 5 different eigenfunctions linearly independent.

In conclusion, in we have shown that the above abstract arguments can be applied also to the case of the limit problem for the shell with an edge with slight folding, leading to five global properties. When considering the case of a "rigid pasting of the adjacent shells (see Remark 3.1), there is an additional transmission condition in (5.17) and (5.19), so that there are only four linearly independent eigenfunctions and only four global properties, as in the basic problem. One may think that the rigidity of the angle implies some kind of "solidification" of the set of two shells. We may guess that the corresponding global property is linked with torsional rigidity.

6 A nonlinear variant: the case with an obstacle

In this section,we shall consider the basic problem under the constrain of the eventual contact of the shell with an obstacle (or support) occupying a region which, for simplicity, we shall assume of the form

$$\mathcal{S}_\varepsilon := (\underline{s}_1, \overline{s}_1) \times (\eta \underline{s}_2, \eta \overline{s}_2) \subset \Omega_\varepsilon,$$

i.e. with

$$\begin{cases} 0 < \underline{s}_1, \overline{s}_1 \leq l_1, \\ 0 \leq \underline{s}_2, \overline{s}_2 \leq l_2. \end{cases}$$

We assume that the upper surface of the obstacle has the same cylindrical geometry than the shell in such a way that the possibility of contact between the shell and the obstacle is merely formulated in terms of

$$\tilde{u}_3(x_1, x_2) \geq 0 \text{ a.e. } (x_1, x_2) \in \mathcal{S}_\varepsilon.$$

By introducing the change of variables (2.21), the above constrain can be stated as

$$u_3(y_1, y_2) \geq 0 \text{ a.e. } (y_1, y_2) \in \mathcal{S},$$

where

$$\mathcal{S} := (\underline{s}_1, \overline{s}_1) \times (\underline{s}_2, \overline{s}_2) \subset \Omega.$$

As usual in such type of problems (see, for instance, [18]), the variational formulation (in the rescaled variables) is:

Problem OP_ε. *Find* $\mathbf{u}^\varepsilon \in \mathbf{K}$ *satisfying*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v} - \mathbf{u}^\varepsilon) \geq \int_\Omega F_3(y_1, y_2)(v_3(y_1, y_2) - u_3^\varepsilon(y_1, y_2)) d\mathbf{y} \text{ for any } v \in \mathbf{K}, \tag{6.1}$$

where F_3, a^ε *and* $\mathbf{V}$ *were given in Subsection 2.2, and*

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} \text{ such that } v_3(y_1, y_2) \geq 0 \text{ a.e. } (y_1, y_2) \in \mathcal{S}\}.$$

Since $\mathbf{K}$ is a convex closed set of $\mathbf{V}$ and a^ε the bilinear form is coercive on $\mathbf{V}$, by well-known results (see, for instance, [18]), given $F_3 \in L^2(\Omega)$ there exists a unique function $\mathbf{u}^\varepsilon \in \mathbf{K}$ solution of (6.1). Moreover, $\mathbf{u}^\varepsilon$ is characterized as the unique solution of the minimization problem

$$Min_{\mathbf{K}} J_\varepsilon(\mathbf{v}) \tag{6.2}$$

where

$$J_\varepsilon(\mathbf{v}) = \frac{1}{2} a^\varepsilon(\mathbf{v}, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle. \tag{6.3}$$

The passing to the limit, as $\varepsilon \to 0$, follows the same a priori estimates than in the proof of Theorem 2.2 since the convex set $\mathbf{K}$ is independent of ε and we also have that

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) = \int_\Omega F_3(y_1, y_2) u_3^\varepsilon(y_1, y_2) d\mathbf{y},$$

as it results from choosing $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}^\varepsilon$. In particular, we maintain, in the limit, the constraints obtained for the problem without obstacle, getting the following result.

Lemma 6.1 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* OP_ε. *The following convergences (as* $\varepsilon \to 0$*) hold true (in the sense of subsequences, the limits being not necessarily unique):*

$$u_1^\varepsilon \to u_1^* \qquad weakly \ in \ \widetilde{H}_0^1((0, l_1); L^2(0, l_2)) \tag{6.4}$$

$$u_2^\varepsilon \to u_2^* \qquad weakly \ in \ \widetilde{H}_0^1((0, l_1); H^{-1}(0, l_2)) \tag{6.5}$$

$$u_3^\varepsilon \to u_3^* \qquad weakly \ in \ L^2((0, l_1); H^2(0, l_2)) \tag{6.6}$$

where $\mathbf{u}^* = (u_1^*, u_2^*, u_3^*) \in \mathbf{K}$. *Moreover, they satisfy:*

$$\partial_2 u_1^* + \partial_1 u_2^* = 0$$

$$\partial_2 u_2^* + u_3^* = 0.$$

Finally,

$$\gamma_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon) \to \gamma_{\alpha\beta}^* \qquad weakly \ in \ L^2(\Omega), \ \alpha, \beta = 1, 2, \tag{6.7}$$

for some $\gamma_{\alpha\beta}^* \in L^2(\Omega)$.

We introduce the subspace $\mathbf{G}$, the potential φ and the subspace $\widetilde{G}$ as in Subsection 2.3. We define the convex and closed subset of $\mathbf{G}$ and $\widetilde{G}$ by

$$\mathbf{K_G} = \{\mathbf{v} \in \mathbf{G} \text{ such that } v_3(y_1, y_2) \geq 0 \text{ a.e. } (y_1, y_2) \in \mathcal{S}\},$$

$$K_{\widetilde{\mathbf{G}}} = \{\psi \in \widetilde{G} \text{ such that } \partial_2^2 \psi(y_1, y_2) \leq 0 \text{ a.e. } (y_1, y_2) \in \mathcal{S}\}.$$

Notice that, obviously $\mathbf{u}^* \in \mathbf{K_G}$. We define (which can be understood as the limit problem) as follows
Problem OP_0. *Find* $\mathbf{u} \in K_{\mathbf{G}}$ *such that*

$$a^0(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq \int_\Omega F_3(v_3 - u_3) d\mathbf{y} \text{ for any } \mathbf{v} \in \mathbf{K_G}, \tag{6.8}$$

or equivalently, in terms of the potential, find $\varphi \in K_{\widetilde{\mathbf{G}}}$ *such that*

$$\widetilde{a}^0(\varphi, \psi - \varphi) \geq - \int_\Omega F_3(\partial_2^2 \psi - \partial_2^2 \varphi) d\mathbf{y}, \text{ for any } \psi \in K_{\widetilde{\mathbf{G}}}, \tag{6.9}$$

with the bilinear forms a^0 and $\widetilde{a}^0$ given by (2.103) and (2.109).

This problem is in the Lions-Stampacchia framework (see, for instance, [18]), as the right hand side of (2.79) is a continuous functional on $\mathbf{G}$. We then have

Theorem 6.1 *Under the assumption $F_3 \in L^2(\Omega)$, Problem OP_0 has a unique solution.*

The proof of the identification of OP_0 as the limit problem of OP_ε is now more delicate than in Section 2. As in Theorem 2.3, we can obtain a positive result at least for a not too restrictive class of coefficients:

Theorem 6.2 *Assume (2.93). Let $\mathbf{u}^\varepsilon$ be the solutions of OP_ε and let $\mathbf{u}^*$ be the weak limit given in Lemma 6.1. Then $\mathbf{u}^*$ verifies problem OP_0. Moreover,*

$$u_1^\varepsilon \to u_1^* \qquad \text{strongly in } \widetilde{H}_0^1((0, l_1); L^2(0, l_2)) \tag{6.10}$$

$$u_2^\varepsilon \to u_2^* \qquad \text{strongly in } \widetilde{H}_0^1((0, l_1); H^{-1}(0, l_2)) \tag{6.11}$$

$$u_3^\varepsilon \to u_3^* \qquad \text{strongly in } L^2((0, l_1); H^2(0, l_2)) \tag{6.12}$$

for $\varepsilon \downarrow 0$. So, in particular, OP_0 can be identified as the limit problem of OP_ε for $\varepsilon \downarrow 0$.

PROOF. As in Theorem 2.3, our arguments are inspired in some ideas contained in Lions [24] (see now, Theorem 3.1 of Chapter II). We reformulate the bilinear form $a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v})$ as in the decomposition (2.97) and with $\mathfrak{a}_0$ given by (2.98). We introduce the sequence

$$X^\varepsilon := \mathfrak{a}_0(\mathbf{u}^\varepsilon - \mathbf{u}^*, \mathbf{u}^\varepsilon - \mathbf{u}^*) + \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + \varepsilon^{-1/4}\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + \varepsilon^{-1/2}\mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon).$$

Using the definition of OP_ε we get that, for any $\mathbf{v} \in \mathbf{K_G} \subset \mathbf{K}$,

$$\begin{aligned} X^\varepsilon \leq \ & \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon, \mathbf{v}) + a^0(\mathbf{u}^\varepsilon, \mathbf{v}) - \int_\Omega F_3(v_3 - u_3^\varepsilon) d\mathbf{y} \\ & - a^0(\mathbf{u}^\varepsilon, \mathbf{u}^*) - a^0(\mathbf{u}^*, \mathbf{u}^\varepsilon - \mathbf{u}^*), \end{aligned}$$

where we used that $\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon, \mathbf{v}) = \mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon, \mathbf{v}) = 0$ since $\mathbf{v} \in \mathbf{K_G}$. Thus, using Lemma 6.1 we get that, for any $\mathbf{v} \in \mathbf{K_G}$,

$$\limsup X^\varepsilon \leq a^0(\mathbf{u}^*, \mathbf{v} - \mathbf{u}^*) - \int_\Omega F_3(v_3 - u_3^*) d\mathbf{y}.$$

But, $X^\varepsilon \geq 0$ for any $\varepsilon > 0$ since all the bilinear forms are associated to norms or seminorms. Thus, by using the assumption (2.93) and recalling that it implies (2.81) we get that the weak limit function $\mathbf{u}^*$ satisfies OP_0 (notice that by Lemma 6.1 we already know that $\mathbf{u}^* \in \mathbf{K_G}$). Moreover, since

$$\inf_{\mathbf{v} \in \mathbf{K_G}} \{a^0(\mathbf{u}^*, \mathbf{v} - \mathbf{u}^*) - \int_\Omega F_3(v_3 - u_3^*)d\mathbf{y}\} = \mathbf{0}$$

(take $\mathbf{v} = \mathbf{u}^*$) we conclude that

$$\limsup X^\varepsilon = 0$$

which implies the strong convergence assertion since all the terms in X^ε are nonnegative. $\qquad\square$

Remark 6.1 *Some of the arguments of the proof of Theorem 2.2 (in order to identify the limit problem) remains still valid in a general framework, i.e. without the asummption (2.93). So, for instance, we have that $T^{12*} = 0$ since if we take as test function*

$$\mathbf{v} = \mathbf{u}^* \pm \eta \mathbf{w} \tag{6.13}$$

with

$$w_1 \in C_0^\infty((0,l_1); C^\infty(0,l_2)), \qquad w_2 = w_3 = 0, \tag{6.14}$$

(notice that $v \in \mathbf{K}$), passing to the limit in (6.1) we get that

$$\pm \int_\Omega T^{12*}\partial_2 w_1 d\mathbf{y} \geq 0, \tag{6.15}$$

so that, as $\partial_2 w_1$ is "arbitrary", we get that necessarily $T^{12*} = 0$. It can be proved also that $T^{22*} \geq 0$ but to conclude further information we need some additional assumption (as, for instance (2.93)).

Remark 6.2 *As in previous Remarks 2.4,and 2.5, the (unique) solution $\mathbf{u}$ of problem OP_0 can be characterized as the unique element of $\mathbf{G}$ solution of the minimization problem*

$$Min_{\mathbf{K_G}} J_0(\mathbf{v}) \tag{6.16}$$

where $J_0(\mathbf{v})$ was given by (2.105) and equivalently, So, the (unique) solution φ of problem OP_0 can be characterized as the unique element of $K_{\widetilde{\mathbf{G}}}$ solution of the minimization problem

$$Min_{K_{\widetilde{G}}} \widetilde{J}_0(\psi) \tag{6.17}$$

where $\widetilde{J}_0(\psi)$ was given in (2.107).

Remark 6.3 *Notice that the obstacle problem gives rise to a free boundary defined through the boundary of the "coincidence set" defined as $\{\mathbf{y} = (y_1, y_2) \in \mathcal{S}: u_3(y_1, y_2) = 0\}$ for which the "unique continuation property" fails (i.e. the Cauchy problem with datum on the free boundary has more than one solution). It seems possible to get some estimates on the location of the free boundary by means of suitable energy methods (see, for instance, [2], [1] for the consideration of other higher order equations and [15] for the application of such general approach to some variational inequalities).*

Our last result establishes some comparison results between the solution with and without obstacle. Since we can not apply the maximum principle, we shall consider only the special case in which F_3 represents the gravity (and so relevant in the applications). Our comparison have merely a global nature.

Theorem 6.3 *Let*

$$F_3(y_1, y_2) \equiv -g \text{ for any } \mathbf{y} \in \Omega,$$

and let $\mathbf{u} \in \mathbf{V}$ *and* $\mathbf{u}^* \in K_{\mathbf{G}}$ *be the solutions of problems* Π_0 *and* OP_0 *respectively. Then*

$$\int_\Omega u_3 d\mathbf{y} \leq \int_\Omega u_3^* d\mathbf{y} \tag{6.18}$$

and

$$a^0(\mathbf{u}, \mathbf{u}) \geq a^0(\mathbf{u}^*, \mathbf{u}^*).$$

PROOF. We know that

$$a^0(\mathbf{u}, \mathbf{u}) = -g \int_\Omega u_3 d\mathbf{y} \tag{6.19}$$

and that

$$a^0(\mathbf{u}^*, \mathbf{v} - \mathbf{u}^*) \geq -g \int_\Omega (v_3 - u_3^*) d\mathbf{y} \text{ for any } \mathbf{v} \in \mathbf{K_G}.$$

Since $\mathbf{0} \in \mathbf{K_G}$, taking $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}^*$ we get that

$$a^0(\mathbf{u}^*, \mathbf{u}^*) = -g \int_\Omega u_3^* d\mathbf{y}. \tag{6.20}$$

Obviously

$$Min_{\mathbf{G}} J_0(\mathbf{v}) \leq Min_{\mathbf{K_G}} J_0(\mathbf{v}).$$

Then, using (6.19) and (6.20) we have that

$$J_0(\mathbf{u}) = \frac{1}{2} g \int_\Omega u_3 d\mathbf{y} = -\frac{1}{2} a^0(\mathbf{u}, \mathbf{u})$$

and

$$J_0(\mathbf{u}^*) = \frac{1}{2} g \int_\Omega u_3^* d\mathbf{y} = -\frac{1}{2} a^0(\mathbf{u}^*, \mathbf{u}^*)$$

which leads to the conclusion. $\qquad\square$

Remark 6.4 *In some concrete experiences (think, for instance in the flexible steel retractable meter tape measure) the comparison (6.18) allows to understand how linear constitutive laws (as, for instance, the generalized Hooke law) remain applicable under the presence of obstacles in a better way than without them for which large displacements, as u_3, can not be justified without the assumption of some nonlinear constitutive law.*

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