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An extinction delay mechanism for some evolution equations

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Resumen

We study the "finite extinction phenomenon"(there exists $t_0 \geq 0$ such that $u(t, \mathbf{x}) \equiv 0 \quad \forall t \geq t_0$, *a.e.* $x \in \Omega$) for solutions of parabolic reaction-diffusion equations of the type $\frac{\partial u}{\partial t} - k\Delta u + \lambda b(t) f(u(t-\tau, \mathbf{x})) = 0$ and ordinary delayed differential equations ($k = 0$) with a delay term $\tau > 0$.

1. Introduction.

In the last years the "finite extinction phenomenon" (there exists $t_e \geq 0$ such that $u(t,x) \equiv 0 \quad \forall t \geq t_e$, and *a.e.* $x \in \Omega$) has been proved for solutions of suitable parabolic reaction-diffusion equations (usually involving some non-Lipschitz nonlinear terms) on an open bounded set Ω in $\mathbb{R}^N$: see, e.g., [2].

The main goal of this work is to show how the finite extinction phenomenon may be the result of the mere presence of a suitable time-delayed reaction term. We consider some special evolution equations on the space $X = L^p(\Omega)$

$$(P_{A,B}) \begin{cases} \frac{\partial u}{\partial t} + Au + \lambda b(t) f(u(t-\tau, \mathbf{x})) = 0 & (0, +\infty) \times \Omega, \\ Bu = 0 & (0, +\infty) \times \partial\Omega, \\ u(s, \mathbf{x}) = u_0(s, \mathbf{x}) & (-\tau, 0) \times \Omega, \end{cases}$$

where $\lambda > 0, f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(0) = 0$, $b \in L^1_{loc}(0, +\infty)$, $\quad b \geq 0$, $A : D(A) \to X$ is a partial differential operator (in most of the cases assumed to be linear) and $Bu = 0$ represents the boundary conditions on $(0, +\infty) \times \partial\Omega$. We want to show that, for many given f, A and B, it is possible to select some u_0, λ and b for which the associated solution becomes extinct after a finite time. We anticipate that function b will extinct also after a finite time and so that problem $(P_{A,B})$ recall some problems formulated in the framework of Control Theory.

If $b(t) \equiv 0$ and A is linear the "finite extinction phenomenon" cannot hold because of well-known properties such as the unique continuation property or the strong maximum principle. In the case of zero delay $\tau = 0$, extinction in finite time is typical of equations containing a strong absorption term. For instance, in the case of reaction-diffusion equations of the type

$$\frac{\partial u}{\partial t} - \Delta u + \lambda |u|^{m-1} u = 0 \tag{1}$$

for some $\lambda, m > 0$ it is well-known (see e.g. Antontsev, Díaz and Shmarev[2] and its references) that the finite extinction phenomenon takes place if and only if $m \in (0,1)$.

We point out that a sistematic study about under which non local terms of the general form $G(t, u_t)$ (here $u_t(s,\cdot) := u(t+s,\cdot)$ for $s \in [-\tau, 0]$) the solutions of $\frac{\partial u}{\partial t} - \Delta u + G(t, u_t) + \lambda |u|^{m-1} u = 0$ becomes extinct after a finite time was made in Redheffer and Redlinger[7] but always under condition $m \in (0,1)$. Our point of view is different since we are interested in the pure memory effects, and no condition of the type $m \in (0,1)$ will be required here.

Since in the case of Neumann boundary conditions and spatially constant initial data, $u(s, \mathbf{x}) = u_0(s)$, we can produce solutions by solving an ordinary delayed differential equation (ODDE) it is natural to start our study by the consideration of this simpler type of problems. We start by considering, in Section 2, linear ODDEs of the type $A = 0$ and $f(s) = \lambda s$ for some $\lambda > 0$. We show that if $b(t)$ becomes extinct after a "small" time $t_b = 2\tau$, $b(t)$ being inactive (i.e. zero) on $[0, \tau]$, then the solution $u(t,x)$ becomes extinct after the finite time t_b. Nonlinear ODDE requires a separated treatment which is presented in Section 3. Finally, the application to the study of the finite extinction time for some delayed partial differential equations is the object of Section 4.

It seems interesting to point out that, in contrast with what happens for parabolic second order equations without any delay, the comparison principle is not useful in our context. Indeed, it is well known (see, e.g. Pao [6] Chapter 1, Theorem 8.1) that if f is nonincreasing the following general comparison principle holds: given $T > 0$, if $\underline{u}$, $\overline{u} \in C([-\tau, T] : L^p(\Omega))$ are sub- and supersolutions of the Dirichlet problem

$$\begin{array}{l} \frac{\partial \overline{u}}{\partial t} - \Delta \overline{u} + \lambda b(t) f(\overline{u}(t-\tau, \mathbf{x})) + g(\overline{u}(t, \mathbf{x})) \geq 0, (0,T) \times \Omega, \\ \overline{u}(t, \mathbf{x}) \geq 0 \;, (0,T) \times \partial\Omega, \overline{u}(s, \mathbf{x}) \geq u_0(s, \mathbf{x}) \;\; (-\tau, 0) \times \Omega, \end{array}$$

(replacing $\geq$ by $\leq$ for the case of $\underline{u}$) then $\underline{u} \leq u \leq \overline{u}$ on $[-\tau, T] \times \Omega$. But here we are interested in the opposite case in which f is increasing (otherwise the extinction property would require the additional conditions already considered in [7]). It is not difficult to contruct counterexamples showing that, in that case, the comparison principle fails. Our main method of proof will rest on the *constants variation formula*, and so, independently of the comparison principle.

2. The linear ODDE case.

We begin by considering the case of constant initial data:

Lemma 1. *Consider the initial value problem*

$$\begin{cases} W'(t) + b(t)f\left(W(t-\tau)\right) = 0, \\ W(s) = W_0(s) \quad for \quad s \in (-\tau, 0]. \end{cases} \tag{2}$$

where f is an increasing function and $b(t)$ is such that

$$b(t) \equiv 0 \text{ for a.a. } t \in [t_b, +\infty) \text{ for some } t_b \in (0, \tau]. \tag{3}$$

Assume that $W_0(s) = W_o \quad$ for all $s \in (-\tau, 0]$, with

$$W_o = f\left(W_o\right) \int_0^{t_b} b(s)ds. \tag{4}$$

Then, the unique solution W of *(2) verifies that $W(t) \equiv 0 \quad \forall t \geq t_b$.*
Proof. The existence and uniqueness of a strong solution to (*2*) can be found in classical books as, for instance, Hale [5]. Integrating on $(0,t)$ for $t \in (0, \tau]$ we get

$$\int_0^t W'(t)dt = W(t) - W_o = -f\left(W_o\right) \int_0^t b(s)ds.$$

But from (4) we get that $W(t_b) = 0$ and as $W'(t) = 0$ for a.e. $t \in [t_b, +\infty)$ we conclude.■

To get some other applications to partial differential delayed equations it would be useful the following:

Lemma 2. *Let $W(t)$ be a solution of the delay-differential equation*

$$\begin{cases} W'(t) + \lambda_n W(t) + \lambda b(t) W(t-\tau) = 0, \\ W(s) = \mu(s), \quad for \quad s \in (-\tau, 0)\,. \end{cases}.$$

Then $W(t) \equiv 0 \quad \forall t \in [2\tau, +\infty)$ assumed that

$$b(t) = 0, \quad for \quad t \in [0, \tau] \cup [2\tau, \infty]\,, \tag{5}$$

$$1 = \lambda \int_\tau^{2\tau} b(s) e^{\lambda_n s} ds. \tag{6}$$

Proof. Let us integrate the equation by the method of steps. In the interval $t \in [0, \tau]$, the equation is $W'(t) + \lambda_n W(t) = 0$ and its solution is $W(t) = W(0)e^{-\lambda_n t}$, and so $W(\tau) = W(0)e^{-\lambda_n \tau} \Longrightarrow W(0) = W(\tau)e^{\lambda_n \tau}$. In the interval $t \in [\tau, 2\tau]$, we have $W(t-\tau) = W(0)e^{-\lambda_n (t-\tau)}, \quad for \quad t \in [\tau, 2\tau]$, so the equation is $W'(t) + \lambda_n W(t) = -\lambda b(t) W(0) e^{-\lambda_n (t-\tau)}$ and the solution is

$$W(t) = e^{-\lambda_n t} \left\{ c + \int_\tau^t e^{\lambda_n s} \left(-\lambda b(s) W(0) e^{-\lambda_n (s-\tau)} \right) ds \right\}.$$

For $t = \tau$, we obtain the value of $W(t) = W(0)e^{-\lambda_n t} \left\{ 1 - \lambda e^{\lambda_n \tau} \int_\tau^t b(s)ds \right\}, \quad t \in [\tau, 2\tau]$. So, the condition $\lambda e^{\lambda_n \tau} \int_\tau^{2\tau} b(s)ds = 1$ implies $W(2\tau) = 0$. Using that $b(t) = 0$ for $t \in [2\tau, +\infty)$ we conclude that $W(t) = 0$ for any $t \in [2\tau, +\infty)$.■

We shall show, in Section 4,that the above result is of interest in semilinear parabolic delayed equations. As a matter of fact, the arguments can be applied also to some homogeneous nonlinear problems such as $\frac{\partial u}{\partial t} - \Delta_p u + \lambda b(t) |u(t-\tau,\mathbf{x})|^{p-2} u(t-\tau,\mathbf{x}) = 0$, giving rise to the ordinary delayed differential equation $W'(t) + \lambda_n |\mathbf{W}|^{p-2} \mathbf{W} + \lambda b(t) |W(t-\tau)|^{p-2} W(t-\tau) = 0$.This, and other different motivations, justify the relevance of the study of the finite extinction time for nonlinear delayed differential equations.

3. The nonlinear ODDE case.

Consider the nonlinear delayed differential equations.

$$\begin{cases} u'(t) + F(u(t)) + \lambda b(t) f(u(t-\tau)) = 0, \\ u(s) = u_0(s) \quad for \quad s \in (-\tau, 0], \end{cases} \tag{7}$$

with our standing hypothesis $b(t) \equiv 0$ for a.e. $t \in [0,\tau] \cup [2\tau, +\infty)$.We will show the existence of a "branch.ºf nonzero solutions which vanish in finite time and *bifurcate* from the zero solution. Since we have no explicit expression for the solutions in this case, we have to assume some extra regularity on F and f. We use a simple version of a well-known result:

Theorem 1 (Crandall-Rabinowitz bifurcation theorem [3]). *Let $H = H(\xi,\lambda) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be smooth, $H(0,\lambda) = 0$ for all λ, $\partial_\xi H(0,\lambda_0) = 0$ and $\partial_\lambda \partial_\xi H(0,\lambda_0) \neq 0$. Then there exists a smooth "bifurcating curve"$(\xi^*(\varepsilon), \lambda^*(\varepsilon))$ for ε near 0 such that $\xi^*(0) = 0$, $\lambda^*(0) = \lambda_0$ and $\xi^*(\varepsilon) \neq 0$ if $\varepsilon \neq 0$. This curve satisfies $H(\xi^*(\varepsilon), \lambda^*(\varepsilon)) = 0$.*

We shall also need the, so called, *Alekesev's nonlinear variation of constants formula.* In 1965 Alekseev [1] proved the following: consider the problems

$$u' = -F(u), \ u(s) = \xi, \tag{8}$$

$$v' = -F(u) + g(t,v), \ v(s) = \xi, \tag{9}$$

where F and g are assumed to be C^1. The corresponding solutions will be denoted by $u(t,s,\xi)$ and $v(t,s,\xi)$, respectively. Let $U(t,s,\xi)$ denote $\partial_\xi u(t,s,\xi)$. Then $U(t,s,\xi)$ sastisfies the *linear variational equation* $U' = -F'(u(t,s,\xi))U$, $U(s) = 1$. Then the solution $v(t,\xi)$ of (9) satisfies Alekseev's nonlinear integral equation

$$v(t,t_0,\xi) = u(t,t_0,\xi) + \int_{t_0}^{t} U(t,s,u(s,t_0,\xi)) g\left(s, v(s,t_0,\xi)\right) ds \tag{10}$$

In our case, the perturbative case $g(t,v)$ is just $-\lambda b(t) f(u(t-\tau))$.

Theorem 2. *Assume $F(0) = f(0) = 0$, $F'(0) = 0$, $f'(0) = 1$, let $\beta := \int_\tau^{2\tau} b(s)ds \neq 0$ and let $\lambda^* = 1/\beta$. Then, there exists a family of initial data $u_0(0) = u_0^\lambda(0)$ such that*

$$u'(t) + F(u(t)) + \lambda b(t) f(u(t-\tau)) = 0, \tag{11}$$

has a branch of nonzero solutions (λ, u) which bifurcate from $(\lambda^*, 0)$ and vanish for $t \geq 2\tau$ with $\left\| u_0^\lambda \right\| \neq 0$, if $\lambda > \lambda^*$.

Proof. Since $F'(0) = 0$, is is clear that $u(t,s,0) = 0$, $U(t,s,0) = 1$ for all t,s.. Alekseev's formula gives for $t \geq \tau$:

$$v(t,0,\xi,\lambda) = u(t,0,\xi) - \lambda \int_0^t U(t,s,u(s,0,\xi)) b(s) f(v(s-\tau,0,\xi,\lambda)) ds$$

Now, since $b(t) = 0$ for $t \notin [\tau, 2\tau]$, $v(s-\tau, 0, \xi, \lambda) = u(s-\tau, 0, \xi) \quad \text{for } \tau \leq s \leq 2\tau$. So,

$$v(t,0,\xi,\lambda) = u(t,0,\xi) - \lambda \int_0^t U(t,s,u(s,0,\xi))b(s)f(u(s-\tau,0,\xi))ds \tag{12}$$

Our goal is to show that the equation

$$v(2\tau, 0, \xi, \lambda) = 0 \tag{13}$$

satisfies the hypotheses the Crandall-Rabinowitz bifurcation theorem. By direct differentiation, the relevant derivatives to be computed satisfy the following identities:

$$\partial_\xi v(t,0,\xi,\lambda) = U(t,0,\xi) - \lambda \int_\tau^t \partial_\xi U(t,s,u(s,0,\xi))U(s,0,\xi)b(s)f(u(s-\tau,0,\xi))ds -$$
$$- \lambda \int_\tau^t U(t,s,u(s,0,\xi))b(s)f'(u(s-\tau,0,\xi))U(s-\tau,0,\xi)ds$$

When $\xi = 0$, $u(s-\tau, 0, 0) = 0$ and then $f(u(s-\tau, 0, 0) = 0$, so the first integral vanishes. The result is

$$\partial_\xi v(t,0,0,\lambda) = U(t,0,0) - \lambda \int_\tau^t U(t,s,0)b(s)f'(0)U(s-\tau,0,0)ds =$$
$$= 1 - \lambda \int_\tau^t U(t,s,0)b(s)U(s-\tau,0,0)ds$$

since $f'(0) = 1$. Now, $U(t,s,\xi)$ is a solution of a linear homogeneous equation with variable coefficients, so it satisfies the property $U(t,s,\xi)U(s,z,\xi) = U(t,z,\xi) \quad \text{for } z \leq s \leq t$.On the other hand, since $u(t,s,\xi)$ satisfies an autonomous equation, we have $u(t,s,\xi) = u(t-s,0,\xi)$, and this property is inherited by its derivative $U(t,s,\xi) = \partial_\xi u(t,s,\xi)$. Therefore, $U(t,s,0)U(s-\tau,0,0) = U(t,s,0)U(s,\tau,0) = U(t,\tau,0) = 1$. Now we evaluate $t = 2\tau$ and obtain

$$\partial_\xi v(2\tau,0,0,\lambda) = 1 - \lambda \int_\tau^{2\tau} b(s)ds = 1 - \lambda\beta \; (= 0, , \text{ if } \lambda = \lambda^*)$$

Moreover $\partial_\lambda \partial_\xi v(2\tau, 0, 0, \lambda) = -\beta \neq 0$. Hence $(\xi, \lambda) \longmapsto v(2\tau, 0, \xi, \lambda)$ satisfies the hypotheses of the Crandall-Rabinowitz theorem for $\lambda = \lambda^* = 1/\beta$ and the proof is finished.
Remark 1. Note that conditions (13) can be read in the form

$$u(2\tau,0,\xi) = \lambda \int_\tau^{2\tau} U(t,s,u(s,0,\xi))b(s)f(u(s-\tau,0,\xi))ds \tag{14}$$

and represents an implicit condition on $\xi(= u_0(0)), \lambda$ and $b(s)$. This condition makes sense once that Alekseev's nonlinear integral formula holds (for instance, some extensions without the condition $f'(0) = 1$ can be obtained). We point out that even in the abstract framework there are some suitables semilinear equations for which such a formula remain valid. We also can show that the formula holds for abstract semilinear equations of the form

$$(AP) \begin{cases} \frac{du}{dt} + Au + F(u) = 0 & t \in (0,T), \\ u(0) = u_0 \end{cases}$$

where $T > 0$, $A : D(A) \to \mathcal{P}(X)$ is a linear maximal monotone operator on a $X = L^2(\Omega)$ generating a compact semigroup and $F : \mathbb{R} \to \mathbb{R}$ is a convex increasing function. In that case, the equation satisfied by U is of the type $U' + AU = -(\partial F(u(t,s,\xi)))^0 U$ where $(\partial F(u(t,s,\xi)))^0$ is the element of the subdifferential $\partial F(u(t,s,\xi))$ of minimal norm. This would be detailed in a future paper and can be obtained by approximating F by its Yosida-approximation, by applying the results for the associated semilinear equation with Lipschitz perturbations and passing to the limit.

4. On the finite extinction time for some delayed PDEs.

Theorem 3. *Consider the problem*

$$(P_D)\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \lambda b(t) u(t-\tau,\mathbf{x}) = 0 & (0,+\infty)\times\Omega, \\ u(t,\mathbf{x}) = 0 & (0,+\infty)\times\partial\Omega, \\ u(s,\mathbf{x}) = u_0(s,\mathbf{x}) & (-\tau,0)\times\Omega, \end{cases}$$

with $\lambda > 0$. Let $u_0 \in C([-\tau,0] : L^\infty(\Omega))$ be such that

$$u_0(s,\mathbf{x}) = \mu(s)\varphi_n(\mathbf{x}),\ a.e.\ \mathbf{x}\in\Omega \tag{15}$$

with $\mu \in C([-\tau,0])$, where φ_n is n-th eigenfunction

$$\begin{cases} -\Delta\varphi_n = \lambda_n\varphi_n & in \quad \Omega, \\ \varphi_n = 0 & on \quad \partial\Omega. \end{cases}.$$

Assume b such that $b(t) \equiv 0$ for a.e. $t \in [0,\tau]\cup[2\tau,+\infty)$ and

$$\lambda e^{\lambda_n\tau}\int_\tau^{2\tau} b(s)ds = 1. \tag{16}$$

Then, there exists a function $W(t)$ with $W(t) \equiv 0 \quad \forall t \in [2\tau,+\infty)$ such that the solution u of (P_D) is given by $u(t,\mathbf{x}) = W(t)\varphi_n(\mathbf{x})$ a.e. $\mathbf{x}\in\Omega$ and for any $s \in [0,2\tau]$. Therefore u vanishes identically for $t \geq 2\tau$.

Proof. Consider the function $\underline{u}(t,\mathbf{x}) = \varphi_n(\mathbf{x})\,W(t)$. It is a routine matter to check that

$$\begin{cases} \underline{u}_t - \Delta\underline{u} + \lambda b(t)\underline{u}(t-\tau,\mathbf{x})) = \\ \quad \varphi_n(\mathbf{x})\,(W'(t) + \lambda_n W(t) + \lambda b(t) W(t-\tau)). \end{cases}$$

So, by taking $W(t)$ as solution of the ODE with delay

$$\begin{cases} W'(t) + \lambda_n W(t) + \lambda b(t) W(t-\tau) = 0, \\ W(s) = \mu(s), \quad for \quad s \in (-\tau,0), \end{cases}.$$

we find that $\underline{u}$ is a solution of (P_D) which must coincide with u by uniqueness of solutions. It remains to prove that $W(t) \equiv 0 \quad \forall t \in [2\tau,+\infty)$. But this is precisely the result of Lemma 2. Then, the function $u(t,\mathbf{x})$ vanishes globally in Ω.■

Remark 2. The result also gives the way in which the solution of the Dirichlet problem (P_D) reaches the identically zero state. We recall that in the case of semilinear equations with a strong absorption term, (1) $m \in (0,1)$, it is known that a "dead core" appears giving rise to a free (or moving) boundary defined as the boundary of the support of $u(t,.)$ ([2]). In fact, under symmetry assumptions on the initial data such a dead core ends being the complete domain Ω except a single point ([4]). Our result shows that, for some cases for which the finite-time extinction arises just by addition of a delay term, the decay to zero is spatially uniform on the whole domain Ω.. This proof also shows the way to a great variety of possible generalizations to other retarded equations associated to different linear problems as, e.g. the ones associated to higher order elliptic operators, the Stokes problem, etc.

Remark 3. If we consider, for instance, the zero controllability problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = v, & (0,+\infty)\times\Omega, \\ u(t,\mathbf{x}) = 0, & (0,+\infty)\times\partial\Omega, \\ u(0,\mathbf{x}) = U_0(\mathbf{x}), & \Omega, \end{cases}$$

where we want to find a control v such that $u(2\tau,x) = 0$ for a.e. $x\in\Omega$, we can use Theorem 3 to construct $v(t,\mathbf{x}) = -b(t)u(t-\tau,\mathbf{x})$. Since v becomes extinct after a finite time $t_b > 0$ we recover the typical characteristic of switched controls.

Remark 4. Whith slight changes, the above argument can be applied also to some homogeneous nonlinear problems such as

$$\frac{\partial u}{\partial t} - \Delta_p u + \lambda b(t)\left|u(t-\tau,\mathbf{x})\right|^{p-2} u(t-\tau,\mathbf{x}) = 0.$$

In that case the eigenfunctions are given by

$$\begin{cases} -\Delta_p\varphi_n = \lambda_n\left|\varphi_n\right|^{p-2}\varphi_n & in \quad \Omega, \\ \varphi_n = 0 & on \quad \partial\Omega. \end{cases} .$$

and the ordinary delayed differential equation becomes

$$W'(t) + \lambda_n\left|\mathbf{W}\right|^{p-2}\mathbf{W} + \lambda b(t)\left|W(t-\tau)\right|^{p-2} W(t-\tau) = 0.$$

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