Connecting steady states of a discrete diffusive energy balance climate model: branch of parametrized solutions and controllability results for the evolution model

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1 Introduction

In this communication we consider some simple Budyko-Sellers climate models of the type

$$(P)\begin{cases} y_t - (k(1-x^2)y_x)_x = R_a(x,y,v) - R_e(y,x,u) & x \in (-1,1), t>0, \\ y(x,0) = y_0(x) & x \in (-1,1), \end{cases}$$

where $k>0$, $R_a(x,y,v)$ is a bounded increasing function on y (the absorbed energy due to the co-albedo) of the and $R_e(y,x,u)$ is a strictly increasing function on y related to the Stefan-Boltzman radiation law with an emissivity u which, varying in some positive interval. Here u and v are taken as control variables (indicating the anthropogenerated actions on the rate of emissions on the greenhouse gases). This kind of methods were introduced, independently, in 1969 by M.I. Budyko and W.D. Sellers. The models have a diagnostic character and intended to understand the evolution of the global climate on a long time scale.

For some purposes it is useful to assume the presence of possible localized controls of the form $u(t)\chi_{(l_1,l_2)}$ and $v(t)\chi_{(l_1,l_2)}$ for some given latitude control interval $(l_1,l_2) \subset (-1,1)$. We shall assume here that $R_a(x,y,v)$ is closer to the model proposed by Sellers and so $R_a = u(t)\chi_{(l_1,l_2)}QS(x)\beta(y)$ with β a Lipschitz continuous, as for instance, $\beta(y) = m$ if $y < y_i, \beta(y) = m + (\frac{u-u_i}{u_w-u_i})(M-m)$ if $y_i \leq y \leq y_w$, $\beta(y) = M$ if $y > y_w$, where u_i and u_w are fixed temperatures closed to -10^0C and $m = \beta_i$ and $M = \beta_w$ represent the coalbedo in the ice-covered zone and the free-ice zone, respectively, $0 < \beta_i < \beta_w < 1$. Moreover, $S(x)$ is the *insolation function* and Q is the so-called *solar constant*. We assume $S : [-1,1] \to \mathbb{R}$, $S \in C^0([-1,1])$, $S_1 \geq S(x) \geq S_0 > 0$ for any $x \in [-1,1]$. We also assume that $R_e = u(t)\chi_{(l_1,l_2)}\mathcal{G}(y) - f(x)$ with $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ a continuous strictly increasing function such that $\mathcal{G}(0) = 0$, $\lim_{|s|\to\infty} |\mathcal{G}(s)| = +\infty$ and $f \in C^0([-1,1])$.

Our main goal is to consider the problem of transfering the system from a stationary state to another one. This type of problem was raised by J. von Neumann in a general context ([14]: see also [13] and [10]). Our study have two different parts: first we obtain a result on a connected branch of stationary solutions (for instance, as function of parameter Q and in the absence of any control: $(l_1,l_2) = (-1,1)$ and $u(t) = v(t) \equiv 1$). In a second part we shall use some techniques of the controllability theory of nonlinear systems of ODEs to analyze the transfering question by means of suitable controls.

As as mater of fact, we shall consider here only some simplified versions of problem (P). We shall concentrate our attention in the discrete version of (P) arising by a spatial difference scheme discretization (for a discretization by finite elements see [3]). There are several possible discrete simplified problems. For instance, to avoid technicalities concerning the degenerate diffusion, as in other precedent papers ([6]), we can replace the degenerate linear diffusion operator by the usual uniforme diffusion expression but then adding Neumann boundary conditions

$$(P_L)\begin{cases} y_t - ky_{xx} = R_a(x,y,v) - R_e(y,x,u) & x \in (-1,1), t > 0, \\ y_x(1,t) = y_x(-1,t) = 0 & t > 0, \\ y(x,0) = y_0(x) & x \in (-1,1). \end{cases}$$

Then, a spatial difference scheme discretization of problem (P_L) can be generated in the usual way: given $N \in \mathbb{N}$, we define $h = 2/(N-1)$ and we denote by $y_i(t)$ to the approximation of $y(-1+ih,t)$. Then, we consider the discrete algorithm

$$(\mathbf{P}_h)\begin{cases} \dot{\mathbf{y}}(t) - \mathbf{A}\mathbf{y}(t) + \mathbf{R}_e(\mathbf{y}(t),u(t)) - \mathbf{R}_a(\mathbf{y}(t),v(t)) = \mathbf{0}, \\ \mathbf{y}(0) = \mathbf{y}^0, \end{cases}$$

where $\mathbf{y}(t) := (y_1(t), y_2(t), ..., y_N(t))^T$, $u(t), v(t) \in \mathbb{R}$, with $u(t)$ and $v(t)$ appearing only in some coordinates associated to some $m \in \mathbb{N}$, $1 < m \leq N$ (the discretized control interval (l_1,l_2) is here represented by an interval of length $(m-1)h$). Problem (P_L) leads to the symmetric positive definite matrix $\mathbf{A}_L$ of $\mathbb{R}^{N\times N}$ given by

$$\mathbf{A}_L = \frac{k}{h^2}\begin{pmatrix} 1 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & 0 & ... \\ 0 & -1 & 2 & -1 & 0 \\ ... & 0 & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 1 \end{pmatrix},$$

$\mathbf{R}_a$:$\{-1,-1+h,...,+1\}\times\mathbb{R}^N\times\mathbb{R}^m \to \mathbb{R}^N$ is given by $\mathbf{R}_a(x_1,\ldots,x_N,y_1,\ldots,y_N,$ $v_1,\ldots,v_N) = (R_a(x_1,y_1,v_1(t)),\ldots,R_a(x_N,y_N,v_N(t)))^T$ and $\mathbf{R}_e$:$\{-1,-1+h,...,+1\}\times$ $\mathbb{R}^N\times\mathbb{R}^m \to \mathbb{R}^N$ $\mathbf{R}_e(x_1,\ldots,x_N,y_1,\ldots,y_N,u_1,\ldots,u_N) = (R_e(x_1,,y_1(t),u_1(t)),\ldots,R_e(x_N,y_N(t),u_N(t)))^T$, where we used the following notation: $u_j(t) \equiv 1$ if j is not one of the m coordinates where the control is located and $u_j(t) \equiv u(t)$ otherwise (and analogously for $v_j(t)$) and $x_i = -1 + (j-1)h$.

A different discrete approximation of problem (P), which maintains the peculiar deneracy of the diffusion leads also to the formulation $(\mathbf{P}_h)$ but with a different symmetric matrix $\mathbf{A}_D$ of $\mathbb{R}^{N\times N}$

$$\mathbf{A}_D = \frac{k}{h^2}\begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ -(1-x_2^2) & 2(1-x_2^2) & -(1-x_2^2) & 0 & ... \\ 0 & ... & ... & ... & 0 \\ ... & 0 & -(1-x_{N-1}^2) & 2(1-x_{N-1}^2) & -(1-x_{N-1}^2) \\ 0 & ... & 0 & 0 & 0 \end{pmatrix},$$

which results from the identity $(k(1-x^2)y_x)_x = k(1-x^2)y_{xx} - 2kxy_x$ when we neglect the transport term $2kxy_x$. Note that in that case the first and the last equations of $(\mathbf{P}_h)$ are uncoupled.

Although our results are true for a general value of $N \in \mathbb{N}$, for the sake of simplicity in the exposition, here we shall only consider the case of $N = 3$ and $m = 1$ leading to the vectorial formulation

$$(\mathbf{P}_Q) \begin{cases} \dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t), u(t), v(t), Q), \\ \mathbf{y}(0) = \mathbf{y}^0 \end{cases}$$

with $\mathbf{f}$:$\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ given by (when $\mathbf{A} = \mathbf{A}_N$)

$$\mathbf{f}(\mathbf{y}, u, v, Q) = \begin{pmatrix} \frac{k}{h^2}(y_2 - y_1) + QS(-1)\beta(y_1) - \mathcal{G}(y_1) + f(-1) \\ \frac{k}{h^2}(y_3 - 2y_2 + y_1) + vQS(0)\beta(y_2) - u\mathcal{G}(y_2) + f(0) \\ \frac{k}{h^2}(-y_3 + y_2) + QS(1)\beta(y_3) - \mathcal{G}(y_3) + f(1) \end{pmatrix}$$

and (when $\mathbf{A} = \mathbf{A}_D$)

$$\mathbf{f}(\mathbf{y}, u, v, Q) = \begin{pmatrix} QS(-1)\beta(y_1) - \mathcal{G}(y_1) + f(-1) \\ k(y_3 - 2y_2 + y_1) + vQS(0)\beta(y_2) - u\mathcal{G}(y_2) + f(0) \\ QS(1)\beta(y_3) - \mathcal{G}(y_3) + f(1) \end{pmatrix}.$$

2 A connected set of stationary solutions depending on Q

In this section we shall assume the absence of any control: $(l_1, l_2) = (-1, 1)$ and $u(t) = v(t) \equiv 1$. Our main goaI is to adapt the results of [7] and ([2])) to show that the set of stationary solutions $(\mathbf{y}^\infty, Q) \in \mathbb{R}^3 \times \mathbb{R}$, i.e. satisfying

$$(\mathbf{P}_Q^\infty) \qquad \mathbf{f}(\mathbf{y}^\infty, 1, 1, Q) = \mathbf{0},$$

is very large (depending on the parameter Q). We make the additional assumptions

(H_{f_∞}) there exist $C_f > 0$ such that $f(x_i) \le -C_f$

(H_β) β is Lipschitz increasing function and there exists $0 < m < M$ and $\epsilon > 0$ such that $\beta(r) = \{m\}$ for any $r \in (-\infty, -10 - \epsilon)$ and $\beta(r) = \{M\}$ for any $r \in (-10 + \epsilon, +\infty)$.

We note that since the matriz $\mathbf{A}$ is symmetric (and, at least, semdefinite positve) the strict monotonicity and the coercivedness assumed on $\mathcal{G}$ implies the existence of a unique $\mathbf{y}_m$ (respect.$\mathbf{y}_M$) solution of the problem $((\mathbf{P}_Q^\infty)_m)$ (respect.$(\mathbf{P}_Q^\infty)_M$) given by $(\mathbf{P}_Q^\infty)$ but replacing $\beta(y_i)$ by m (respect. by M). In the rest of the section we shall use several comparison arguments on $\mathbb{R}^3$. Here we shall use the following notation

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \le \overline{\mathbf{y}} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \end{pmatrix} \text{ if and only if } y_1 \le \overline{y}_1, y_2 \le \overline{y}_2 \text{ and } y_3 \le \overline{y}_3.$$

Analogously, the use of the strict inequality $<$ among vectors means that the strict inequality holds among all the components of the vectors. Finally, if $\alpha \in \mathbb{R}$ the notation $\alpha \leq \mathbf{y}$ means that $\alpha \leq (\mathbf{y})_i$ for $i = 1, 2, 3$.

We start by proving the existence of at least three solutions for suitable Q (in the line of [7]).

Theorem 1. *Let $\mathbf{y}_m$(respect.$\mathbf{y}_M$) be the (unique) solutions of the problem $(\mathbf{P}_Q^\infty)_m$ (respect. $(\mathbf{P}_Q^\infty)_M$).Then: i) for any $Q > 0$ there is a minimal solution $\underline{\mathbf{y}}$ (resp. a maximal solution $\overline{\mathbf{y}}$) of $(\mathbf{P}_Q^\infty)$. Moreover any other solution $\mathbf{y}$ must satisfy*

$$\mathbf{y}_m \ \leq \ \underline{\mathbf{y}} \ \leq \ \mathbf{y} \ \leq \ \overline{\mathbf{y}} \ \leq \ \mathbf{y}_M \tag{1}$$

$$\mathcal{G}^{-1}(QS_0 m + \min f) \leq (\mathbf{y}_m)_i \leq \mathcal{G}^{-1}(QS_1 m - C_f), \tag{2}$$

$$\mathcal{G}^{-1}(QS_0 M + \min f) \ \leq (\mathbf{y}_M)_i \leq \mathcal{G}^{-1}(QS_1 M - C_f) \text{ for } i = 1, 2, 3. \tag{3}$$

If we assume, in addition,

$$(H_{C_f}) \ \ \mathcal{G}(-10 - \epsilon) + C_f > 0 \ \textit{and} \ \frac{\mathcal{G}(-10 + \epsilon) - \min f}{\mathcal{G}(-10 - \epsilon) + C_f} \leq \frac{S_0 M}{S_1 m}$$

and define

$$Q_1 = \frac{\mathcal{G}(-10 - \epsilon) + C_f}{S_1 M} Q_2 = \frac{\mathcal{G}(-10 + \epsilon) - \min f}{S_0 M} \tag{4}$$

$$Q_3 = \frac{\mathcal{G}(-10 - \epsilon) + C_f}{S_1 m} Q_4 = \frac{\mathcal{G}(-10 + \epsilon) - \min f}{S_0 m}. \tag{5}$$

then: ii) if $0 < Q < Q_1$ (repect.$Q > Q_4$) then $(\mathbf{P}_Q^\infty)$ has a unique solution $\mathbf{y} = \mathbf{y}_m$, $(\mathbf{y}_m)_i < -10$, (repect.$\mathbf{y} = \mathbf{y}_M$, $(\mathbf{y}_M)_i > -10$) and

$$\mathcal{G}^{-1}(\min f) \leq \lim_{Q \searrow 0} \ \inf \|y\|_\infty \leq \lim_{Q \searrow 0} \ \sup \|y\|_\infty \leq \mathcal{G}^{-1}(-C_f),$$

iii) if $Q_2 < Q < Q_3$, then $(\mathbf{P}_Q^\infty)$ has at least three solutions, $\mathbf{y}_i$, $i = 1, 2, 3$ with $\mathbf{y}_1 = \mathbf{y}_M$, $\mathbf{y}_2 = \mathbf{y}_m$, and $\mathbf{y}_1 \geq \mathbf{y}_3 \geq \mathbf{y}_2$. **Idea of the Proof.** i) and ii) are consequence of the fact that the comparison principle holds for problems $(\mathbf{P}_Q^\infty)_m$, $(\mathbf{P}_Q^\infty)_M$ (since the systems are of cooperative type) and then the method of sub and supersolutions can be applied (see e.g. Pao [15]).The proof of iii) is divided into several steps. First, we construct two constant subsolutions $\mathbf{V}_i$ and two constant supersolutions $\mathbf{U}_i$ such that

$$\mathbf{V}_2 \ < \ \mathbf{U}_2 \ < \ -10 - \epsilon \ < \ -10 + \epsilon \ < \ \mathbf{V}_1 \ < \ \mathbf{U}_1, \tag{6}$$

proving the existence of, at least, two solutions of $(\mathbf{P}_Q^\infty)$. The existence of a third solution of $(\mathbf{P}_Q^\infty)$ is obtained by a topological fixed point argument. Let us show the convergence of the mentioned solution of $(\mathbf{P}_Q^\infty)$ to a third solution of $(P_{Q,f})$. For $\lambda < \lambda_0$ (a certain positive parameter) $\mathbf{U}_1$, $\mathbf{U}_2$ are supersolutions of $(\mathbf{P}_Q^\infty)$ and $\mathbf{V}_1$, $\mathbf{V}_2$ are subsolutions of $(\mathbf{P}_Q^\infty)$. So, arguing as in i) we obtain two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of $(\mathbf{P}_Q^\infty)$ such that

$$\begin{aligned} -10+\epsilon+\lambda_0 M &< \mathbf{V}_1 \leq \mathbf{y}_1 \leq \mathbf{U}_1 \\ V_2 &\leq \mathbf{y}_2 \leq \mathbf{U}_2 < -10-\epsilon. \end{aligned}$$

In order to prove that $(\mathbf{P}_Q^\infty)$ has a third solution u_3 different to u_1^λ and u_2^λ we apply a result due to Amann [1] (which is justified since the operator $\mathbf{F}(\mathbf{z}) := (\mathbf{A}+\mathbf{v}\mathcal{G})^{-1}(\mathbf{u}Q\mathbf{S}(\cdot)\beta(z)+\mathbf{f})$ is compact on the space $E=\mathbb{R}^3$).

Now we can show that it is possible to associate a bifurcation diagram for the special case of $f(x_i) = -C_f$, $\mathcal{G}(-10-\epsilon)+C>0$ and $\dfrac{\mathcal{G}(-10+\epsilon)+C}{\mathcal{G}(-10-\epsilon)+C} \leq \dfrac{S_2 M}{S_1 m}$.

Theorem 2 *If we denote by* Σ *the set of pairs* $(Q,\mathbf{y}) \in \mathbb{R}^+ \times \mathbb{R}^3$, *where* $\mathbf{y}$ *verifies* $(\mathbf{P}_Q^\infty)$ *then* Σ *contains an unbounded connected component containing the point* $(0,\mathcal{G}^{-1}(-\mathbf{C}))$.

Proof. We claim that the following result, due to Rabinowitz [16], can be applied to our case: "Let E a Banach space. If $F : I\!R \times E \to E$ is compact and $F(0,u) \equiv 0$, then Σ contains a pair of unbounded components C^+ and C^- in $I\!R^+ \times E$, $I\!R^- \times E$ respectively and $C^+ \cap C^- = \{(0,0)\}$". In order to do that we consider the translation of $\mathbf{y}$ given by $\mathbf{z} := \mathbf{y} - \mathcal{G}^{-1}(-\mathbf{C})$. Obviously, v is a solution of $(\mathbf{P}_Q^\infty)$ with $\hat{\mathcal{G}}(\sigma) = \mathcal{G}(\sigma + \mathcal{G}^{-1}(-C)) + C$ and $\hat{\beta}(\sigma) = \beta(\sigma + \mathcal{G}^{-1}(-C))$. We define $\hat{\Sigma}$ in an analogous way to Σ. Let $E = \mathbb{R}^3$ and define $\mathbf{F}(\mathbf{z}) := (\mathbf{A}+\mathbf{v}\mathcal{G})^{-1}(\mathbf{u}Q\mathbf{S}(\cdot)\beta(z)+\mathbf{f})$ is compact on the space $E=\mathbb{R}^3$. On the other hand, if $Q=0$ problem $(\mathbf{P}_Q^\infty)$ has a unique solution $v=0$, so $F(0,0)=0$. In conclusion $\hat{\Sigma}$ contains two unbounded components $\hat{C}^+$ and $\hat{C}^-$ on $I\!R^+ \times \mathbb{R}^3$ and $I\!R^- \times \mathbb{R}^3$ respectively and $\hat{C}^+ \cap \hat{C}^- = \{(0,0)\}$. Since Σ is a translation of $\hat{\Sigma}$ then Σ contains two unbounded components C^+ and C^- on $I\!R^+ \times \mathbb{R}^3$ and $I\!R^- \times \mathbb{R}^3$ respectively and that $C^+ \cap C^- = \{(0,\mathcal{G}^{-1}(-C))\}$. Since $Q \geq 0$ in the studied model, we are interested in C^+. In order to establish the behaviour of C^+, we also recall that for every $q>0$ there exists a constant $L=L(q)$ such that if $0 \leq Q \leq q$ then every solution $\mathbf{y}_Q$ of $(\mathbf{P}_Q^\infty)$ verifies $\|\mathbf{y}_Q\|_\infty \leq L(q)$. Since the principal component is unbounded its projection over the Q-axis is $[0,\infty)$. On the other hand, if Q is large enough $(\mathbf{P}_Q^\infty)$ has a unique solution $\mathbf{y}_Q$ and this solution is greater than $\mathcal{G}^{-1}(QS_0M - C)$. Since $\lim_{|s|\to\infty} |\mathcal{G}(s)| = +\infty$, then the unbounded branch C^+ containing $(0,\mathcal{G}^{-1}(-\mathbf{C}))$ should go to (∞,∞).

Remark 1. In the continuous problem it is well known that there are many other solutions which does not belongs to the branch C^+ of the above proof (see [8]). In some special cases (for instance, the zero-dimensional model: $k=0$ and constant coefficients) it is possible to characterize the different parts of the brach correspoponding to stable (and unstable) solutions.

Remark 2. Under symmetry conditions on $S(x)$ and $f(x)$ the branch C^+ is formed by symmetry stationary solutions $(\mathbf{y})_1 = (\mathbf{y})_3$.

Remark 3. It is not difficult to make a similar study about a branch of solutions when Q is fixed but which is taken as a variable parameter is the emmisivity u.

3 Connecting stationary solutions by means of controls

We consider the problem of transferring the system from a stationary state to another one (when $Q = Q_0$ is fixed) but now by means of suitable choices of the controls $u(t)$ and $v(t)$. In fact, we shall consider here only the case of a single control $v(t)$ and when both solutions are in the same connected component (the branch C^+). For the sake of simplicity, we shall consider the connection bettween an arbitrary (possibly unstable) symmetric state $(\mathbf{y}^0, v^0 Q_0)$ to a final stable symmetric one $(\mathbf{y}^f, v^f Q_0)$, both in the principal branch C^+. The case when $v(t)$ is fixed and the only control is $u(t)$ follows the same arguments. Finally, the case of two controls $u(t)$ and $v(t)$ is even easier. We start with the uniforme diffusion case $\mathbf{A} = \mathbf{A}_N$ with Neumann boundary conditions

Theorem 3. *i) Assume $\mathbf{A} = \mathbf{A}_N$, $u(t) \equiv 1$ and that the control $v(t)$ acts globaly in space ($(l_1, l_2) = (-1, 1)$). Let $(\mathbf{y}^f, Q_0 v^f)$ be a stable symmetric stationary solution in the branch C^+. Then, for any other symmetric state $(\mathbf{y}^0, v^0 Q_0)$ in C^+ there exists a time $T > 0$ and a piece-wise continuous control $v \in L^\infty(0, T)$ with $v(0) = v^0$ and $v(T) = v^f$ such that the solution $\mathbf{y}(t)$ of the problem $(\mathbf{P}_{Q_0})$ with initial datum $\mathbf{y}^0$ verifies that $\mathbf{y}(T) = \mathbf{y}^f$. ii) In the case of a localized control ($(l_1, l_2) \subsetneq (-1, 1)$) the same conclusion holds when, in addition, $(\mathbf{y}^0, v^0 Q_0)$ and $(\mathbf{y}^f, v^f Q_0)$ are closed enough.*

Proof. As in We divide the proof of i) in two different steps. In the first step, given an small $\epsilon > 0$ we connect $(\mathbf{y}^0, v^0 Q_0)$ with a point $(\mathbf{y}^f, Q_0 v^f)$ by means of the branch of stationary solutions C^+ and so, by means of a parametrization $(\mathbf{y}^*(\tau), Q(\tau))$ with $Q(\tau) = (1-\tau) v^0 Q_0 + \tau v^f Q_0$ for $\tau \in [0, 1]$. Obviously, this orbit does not need to be a solution of $(\mathbf{P}_{Q_0})$ but, given $\varepsilon > 0$, we can construct the function $[0, 1/\varepsilon] \to \mathbb{R}^3 \times \mathbb{R}$ given by $(\mathbf{y}^\varepsilon(t), v^\varepsilon(t)) = ((\mathbf{y}^*(\varepsilon t), Q(\varepsilon t))$ which is "almost" a solution since

$$\left\| \dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t), 1, v(t), Q) \right\| = O(\varepsilon).$$

Then, since $(\mathbf{y}^f, v^f Q_0)$ is stable we can assume that $\mathbf{y}^\varepsilon(T_\varepsilon)$ (with $T_\varepsilon = 1/\varepsilon$) is near $\mathbf{y}^f$. The second step consists in to connect $\mathbf{y}^\varepsilon(T_\varepsilon)$ with $\mathbf{y}^f$ by means of a control $\widehat{v}(t)$ for $t \in [T_\varepsilon, T]$, for some $T > T_\varepsilon$. This can be donne thanks to well-known results (see, e.g. [12], [17]) since the Kalman's condition for the linearized equation, near $(\mathbf{y}^f, v^f Q_0)$ holds. Note that due to the symmetry assumption we can reduce the system $(\mathbf{P}_{Q_0})$ to a system of only two equations leading to a linearization

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{y}(t) + \mathbf{B}u(t)$$

where $\mathbf{C} = \nabla_{\mathbf{y}} \mathbf{f}(\mathbf{y}^f, v^f Q_0)$ and $\mathbf{B} = \nabla_{\mathbf{u}} \mathbf{f}(\mathbf{y}^f, v^f Q_0)$, and so the Kalman's condition $\mathit{Range}(\mathbf{B}, \mathbf{CB}) = 2$ holds. ii) For a lozalized control $v(t)$ appearing only in the second equation of $(\mathbf{P}_{Q_0})$ the argument of connecting branch of stationary solutions C^+ may fail but at least we can apply the local controllabilidad results for nonliear equations since the Kalman's condition holds.

Remark 4. It is a courious fact that, in the case of the original 3-system $(\mathbf{P}_{Q_0})$, the necessary and sufficient condition in order to have the Kalman's condition for the linearized equation allows to see that there are other solutions (not necessarely symmetric) which does not satisfy it.

We end this section with the consideration of the degenerate case $\mathbf{A} = \mathbf{A}_D$. As indicated before, now the first and third equations of $(\mathbf{P}_{Q_0})$ are uncoupled and so the problem (neither its linearizations) can be locally controllable. Nevertheless, we can state some result on a relaxed notion of controllability given in terms of the reachability set:

Theorem 4. *i) Assume* $\mathbf{A} = \mathbf{A}_D$, $u(t) \equiv 1$ *and that the control* $v(t)$ *acts globaly in space* $((l_1, l_2) = (-1, 1))$. *Let* $(\mathbf{y}^f, Q_0 v^f)$ *be a stable symmetric stationary solution in the branch* C^+. *Then, for any other symmetric state* $(\mathbf{y}^0, v^0 Q_0)$ *in* C^+ *and for any* $\varepsilon > 0$ *there exists a time* $T^\epsilon > 0$ *and a piecewise continuous control* $v \in L^\infty(0, T^\epsilon)$ *with* $v(0) = v^0$ *and* $v(T^\epsilon) = v^f$ *such that the solution* $\mathbf{y}(t)$ *of the problem* $(\mathbf{P}_{Q_0})$ *with initial datum* $\mathbf{y}^0$ *verifies that* $\left\| \mathbf{y}(T^\epsilon) - \mathbf{y}^f \right\| \leq \varepsilon$. *ii) In the case of a localized control (*$(l_1, l_2) \subsetneq (-1, 1)$*) the same conclusion holds when, in addition,* $(\mathbf{y}^0, v^0 Q_0)$ *and* $(\mathbf{y}^f, v^f Q_0)$ *are closed enough.*

Proof. It is enough to apply the arguments of the proof of Theorem 3 replacing the local controllability condition for $(\mathbf{P}_{Q_0})$ by the fact that the reachability set is open since the Lie bracket condition is satisfied.

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