ON THE FINITE TIME EXTINCTION PHENOMENON FOR SOME NONLINEAR FRACTIONAL EVOLUTION EQUATIONS

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ABSTRACT. The finite extinction time phenomenon (the solution reaches an equilibrium after a finite time) is peculiar to certain nonlinear problems which solutions exhibit an asymptotic behavior entirely different from the typical behavior of solutions of linear problems. The main goal of this work is to extend many of the results in the literature to the case in which the ordinaty time derivative is replaced by a fractional differentiation. Several concrete examples of quasilinear fractuiional partial differential equations are presented. The results can be also applied in the framework of suitable nonlinear Volterra integrodifferential equations

1. EXTENDED ABSTRACT

The main goal of this work is to extend many of the results in the literature on the finite time extinction phenomenon to the case in which the ordinary time derivative is replaced by a fractional differentiation. In order to fix ideas, let $\Omega \subset \mathbf{R}^N$, $N \geq 1$, be a general open set, let $Q_\infty = \Omega \times (0, +\infty)$, $\Sigma_\infty = \partial\Omega \times (0, +\infty)$, and consider a fractional evolution boundary value problem formulated as

$$\left\{ \begin{array}{rcll} \frac{\partial^\alpha u}{\partial t^\alpha} + Au & = & f(x,t) & \text{in } Q_\infty, \\ Bu & = & g(x,t) & \text{on } \Sigma_\infty, \\ u(x,0) & = & u_0(x) & \text{on } \Omega. \end{array} \right. \tag{1}$$

Here, $\alpha \in (0, 1]$ (although some remarks will be devoted to some $\alpha \in (1, 2)$), Au denotes a nonlinear operator (usually in terms of u and the partial differentials of u), Bu denotes a boundary operator and the data $f(x,t)$, $g(x,t)$ and $u_0(x)$ are given functions. For simplicity, we are assuming that A and B are *autonomous operators,* i.e. without time depending coefficients, nevertheless our treatment will allow the case of systems of equations (where $u(x,t) = \mathbf{u}(x,t) \in \mathbf{R}^m$ with $m > 1$).

In the study of the stabilization of solutions, as $t \to +\infty$, it is usually assumed that

$$f(x,t) \to f_\infty(x) \quad \text{and} \quad g(x,t) \to g_\infty(x) \quad \text{as } t \to +\infty,$$

in some functional spaces and the main task is to prove that

$$u(x,t) \to u_\infty(x), \text{as } t \to +\infty,$$

in some topology of a suitable functional space, with $u_\infty(x)$ solution of

$$\left\{ \begin{array}{rcll} Au_\infty & = & f_\infty(x) & \text{in } \Omega, \\ Bu_\infty & = & g_\infty(x) & \text{on } \partial\Omega. \end{array} \right. \tag{2}$$

Here we are interested in a stronger property. We start by assuming that

$$\begin{array}{ll} f(x,t) = 0 & \text{for any } t \geq T_f \ , \\ g(x,t) = 0 & \text{for any } t \geq T_g \ . \end{array} \tag{3}$$

for some We arrive to the following natural phenomenon of the *extinction in finite time:*

Definition 1. *Let u be a solution of the evolution boundary value problem (1). We will say that $u(x,t)$ possesses the property of extinction in a finite time if there exists $t^* < \infty$ such that*

$$u(x,t) \equiv 0, \ \textit{on } \Omega, \textit{for any } t \geq t^*.$$

Although some of our results can be formulated in an abstract framework involving suitable (possibly) multivalued operators

$$\left\{ \begin{array}{l} \frac{d^\alpha u}{dt^\alpha}(t) + Au(t) \ni f(t) \quad \text{in } X, \\ \qquad u(0) = u_0, \end{array} \right.$$

with X a Banach space, here we shall present here only a family of general problems, arising in many contexts (as e.g. the study of the nonlinear heat equation with absorption)

$$(\mathcal{P}) \left\{ \begin{array}{rl} \frac{\partial^\alpha}{\partial t^\alpha}\left(u\,|u|^{\gamma-1}\right) - \text{div}\,\left(|\nabla u|^{p-2}\,\nabla u\right) + |u|^{\sigma-1}\,u = f & \text{in } Q := \Omega \times (0,+\infty), \\ u = 0 & \text{on } \sum_T = \Gamma \times (0,+\infty). \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{array} \right.$$

where $\Omega \subset \mathbb{R}^N$ is a bounded of boundary Γ, $\gamma > 0$, $\sigma > 0$, $1 < p < \infty$, $\lambda \geq 0$. In the rest of the paper we shall assume that

$$\text{there exists} T_0 \geq 0 \text{ such that } f(t,.) = 0, \text{if } t > T_0.$$

The existence of solutions is a particular case of the abstract results on Volterra intregro-differential equations with accretive operators (see [2] and its long list of references which seems to be started in 1963 with the paper by A. Friedman [10]).

Remark. It seems that the application of the very fine techniques of nonlinear operators on Banach spaces to the case of nonlinear fractional partial differential equations was not advertised enough in an important amount of books (excellent from other point of views) on fractional differential equations (see e.g. the book by Kilbas, Srivastava and Trujillo [11] containing 928 references).

We prove that under suitable conditions, the “solution” (notion to be made precise) satisfy an integral energy inequality leading to the extinction in a finite time of the function

Theorem. *Let* $u \in L^1_{loc}(T_0, +\infty : W_0^{1,p}(\Omega))$ *for some* $p > 1$ *such that* $\exists$ $\gamma, k, c > 0, \lambda \geq 0, \sigma > k-1$ *for which* $|u|^{\gamma+k}, |u|^{\sigma+k}, |Du|^p\,|u|^{k-1}$ 2208$L^1_{loc}(T_0, +$221e : $L^1(3a9))$ *and for* t$\in$ $(T_0, +\infty)$, *where* $y(t)$=$\int_\Omega |u|^{\gamma+k} dx$. *Assumethat or andlet Thenthereexistsa* $\mathrm{T}_e \in (T_0, +\infty)$ *such that* $u(t,.) \equiv 0$ *in* Ω $\forall t \geq T_e$.

Some relevant choices of the parameters γ, p, σ which provide the fulfillment of the above conditions are: $p = 2$, $\gamma = 1$ and $\sigma < 1$; $\sigma = 1$, $p = 2$ and $\gamma > 1$; $\sigma = 1$, $\gamma = 1$and $p < 2$.

One ingredient of the proof uses several technical results dealing with the spatial partial differential operator which were previously obtained by some energy methods in [**?**]

The second ingredient concerns the treatment of the fractional time derivative. We prove that under suitable conditions we have that

$$\left\langle \frac{\partial^\alpha}{\partial t^\alpha}(|u|^{\gamma-1} u), |u|^{k-1} u \right\rangle \geq C \int_0^T \left[\frac{d}{dt} \int_\Omega |u(t,.)|^{\gamma+k} dx \right] dt \tag{4}$$

for some $C > 0$.

Remark. This formula was, initially justified in a pioneering paper by J.L. Lions [12], for the special case of the ordinary time differentiation and $\gamma = k = 1$ and later extended by many other authors to other cases of γ and k but always for the ordinary time differentiation). Which is more extraordinary is that this formula is true in the fractional case for some $\alpha \in (1, 2)$. We also point out that (4) allows to conclude the monotonicity (or accretiveness) of the fractional differentiation operator in a more direct way that other proofs in the literature (which seems are due to Clement-Nohel, Gripenberg and Clement-Pruss by means of more sophisticated arguments).
PONER LOS DETALLES DE ESAS REFERENCIAS

Here we shall prove (4) only for the case $\gamma = k = 1$.The more general situation will be given in [8].

Poner ahora las cuentas de Teresa (pidiendo decreciente y convexo) y lo de las hojas de mi variante del dia 12.

Poner tambien las experiencias numéricas del manuscrito enviado por Teresa en agosto

An idea of the rest of the proof of Theorem 1.

PONER AQUI LAS NOTAS DE TERESA DEL 2003 QUE SE APLICAN A LA ODEFRACCIONARIA SATISFECHA POR $y(t)$=$\int_\Omega |u|^2 dx$.

Acknowledgments. The research of JID was partially supported by the projects MTM2005-03463 of the DGISGPI (Spain) and CCG06-UCM/ESP-1110 of the DGUIC of the CAM and the UCM..........

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