Simplifying Nonlinear PDEs: Quasi-Linearization and Pseudo-Linearization Principles

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Dedicated to Professor Lakshmikantham in occasion of his 85th birthday

1 Introduction

Many evolution boundary value problems in PDEs can be reformulated as special cases of abstract Cauchy problems of the type

$$(ACP)\begin{cases} \frac{du}{dt}(t) + Au(t) \ni F(u) & \text{in } X, \\ u(0) = u_0. \end{cases} \tag{1}$$

where the operator A is, in fact, a nonlinear operator on a Banach space X and $F(u)$ represents the operator associated to a real continuous function f.

Different ideas have been introduced at the literature in order to get conclusions on the solution u but by using the information obtained trough some auxiliary simplified Cauchy problems. Here we shall pay attention to two different arguments of this nature.

The first of them is the so called "method of quasi-linearization" introduced by Bellman and Kalaba in [8] in order to prove the existence of solutions of nonlinear parabolic *semilinear* problems with A a second order linear elliptic operator and f written as $f = f_{cv} + f_{cn}$ with f_{cv} convex and f_{cn} concave by means of some iteration schemes. One of ours contributions to this respect is to avoid any additional assumption on the second derivative of the function f , in contrast with other results in the literature (see, for instance, Laksmikantham and Vatsala [27] and Carl and Laksmikantham [11], [12], [13]).To do that we shall combine an iterative scheme with some approximation arguments (we replace f by a regular approximation f^k) of the type

$$\begin{cases} \frac{d\overline{u}_{k+1}}{dt} + A\overline{u}_{k+1} \ni \\ \qquad f^k(\overline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) \qquad \text{in } X, \\ \frac{d\underline{u}_{k+1}}{dt} + A\underline{u}_{k+1} \ni \\ \qquad f^k(\underline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) \qquad \text{in } X, \\ \overline{u}_{k+1}(0) = u_0, \ \underline{u}_{k+1}(0) = u_0. \end{cases}$$

We also prove that this method can be extended beyond the linear assumption on A and, which is perhaps more useful, we formulate and prove this principle in the abstract framework of *T-accretive* operators in the Banach lattices $X = L^p(\Omega)$ for some $p \in [1, +\infty]$ or $X = C(\overline{\Omega})$, where Ω is a regular open bounded set of $\mathbb{R}^N$ allowing to get, as applications the case of quasilinear or fully nonlinear parabolic equations. It is also applicable to some multivalued equations, as the obstacle problem (something proposed in Laksmikantham [25]). Notice that, the above system of uncoupled equations replace the nonlinear term $F(u(t))$ by linear (zero order terms) as $[(f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)]\overline{u}_{k+1}$ and $[(f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)]\underline{u}_{k+1}$.As a matter of fact the quasilinearization method was introduced by R. Bellman and collaborators ([8]) in order to get the approximation of the solution of semilinear equations by means of a quadratic (sometimes called as *rapid*) convergence. We can prove something similar for concrete quasilinear equations as we explain at the end of Section 2.

The second method which keeps the main idea of simplify the initial nonlinear PDE is a principle, introduced by the authors in Casal -Díaz [16], and generalizes the *classical linearization principle.* We consider, this time, some abstract problems of the type

$$\begin{cases} \frac{du}{dt}(t) + Au(t) + Bu(t) \ni F(u_t(.)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0]. \end{cases} \tag{2}$$

on a Banach space X, where $u_t(\theta) = u(t+\theta), \theta \in [-\tau, 0]$ .We want to study the convergence, as $t \to \infty$, to the associated equilibria: $w \in D(A) \subset D(B) \subset X$ such that

$$Aw + Bw \ni F(\widehat{w}(.)),$$

where $\widehat{w} \in C := C([-\tau, 0] : X)$ is the function which takes constant values equal to w. Our main goal is to extend, to a broad class of nonlinear operators A, the usual linearized stability principle saying, roughly speaking, that for the special case of A linear (single valued) and B and F are differentiable, the asymptotic stability of the zero solution of the linearized equation,

$$\begin{cases} \frac{dv}{dt}(t) + Av(t) + \mathrm{D}B(w)v(t) = \mathrm{D}F(\widehat{w})v_t(.) & \text{in } X, \\ v(s) = u_0(s) & s \in [-\tau, 0]. \end{cases}$$

implies that $u(t : u_0) \to w$ as $t \to \infty$, at least if $u_0(.)$ is close enough to $\widehat{w}$. We point out that our results seem to be new even without the delayed and nonlocal term (i.e. for $F \equiv 0$). Our generalization is motivated by the fact that quite often the nonlinear operator A is not differentiable near some equilibrium points and so the *classical linearization principle* is not applicable.

The main motivation to keep A nonlinear after the process of linearization in the above papers was the study the stabilization of the uniform oscillations for the *complex Ginzburg-Landau equation* by means of some global delayed feedback. In fact, due to the important role of a controlling term, we consider in this section the case in which F depends of some delayed term $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau, 0]$ for some $\tau > 0$. It is a curious fact that even if the *complex Ginzburg-Landau equation* is formulated in terms of a linear (vectorial) diffusion operator A the usual representation for the unknown as $\mathbf{z}(x,t) = \rho(x,t)e^{i\phi(x,t)}$ leads the original system to a coupled nonlinear system of equations for ρ and ϕ which can be formulated again in the form $\frac{dz}{dt}(t) + \widetilde{A}z \ni \widetilde{F}(z)$ but with a nonlinear (and not everywhere differentiable) operator $\widetilde{A}$.

Many other examples can be appealed to justify the philosophy of keeping A non-linear after linearizing the rest of the terms of the equation. For instance, this is the case when A is multivalued, or nondifferentiable or a degenerate quasilinear operator. We point out that some relevant examples of nonlinear equations arise in the most different contexts (see, for instance, Díaz and Hetzer [20] for one example in Climatology).

The main conclusion of the pseudolinearization principle was formulated in terms of the condition that the operator $y \to Ay - \mathrm{D}F(w)y$ belongs to $\mathcal{A}(\omega^* : X)$, for some $\omega^* \in \mathbb{C}$ with $\mathrm{Re}\,\omega^* = \gamma^* < 0$ where the class of operators $\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathcal{P}(X)$,such that $A + \omega I$ is a *m-accretive* operator$\}$ (see Brézis [10] for the case of $X = H$ a Hilbert space and Bénilan, Crandal and Pazy [9], Vrabie [32] for the case of a general Banach space).

2 Abstract Quasilinearization Principle

Given Ω,a regular open bounded set of $\mathbb{R}^N$, we shall consider the abstract Cauchy problem (ACP) in the Banach lattice $X = L^p(\Omega)$ for some $p \in [1, +\infty]$ or $X = C(\overline{\Omega})$. Several definitions are in order first.

Definition 1 *An operator* $A : D(A) \subset X \longrightarrow 2^X$ *is called* m-accretive *if it is accretive and, in addition,* $R(I + \lambda A) = X$, *for each* $\lambda > 0$

The structural assumptions on the operators we shall assume in this section are the following

(H$_1$): $A \in \mathcal{A}_+(\omega : X)$, for some $\omega \in \mathbb{C}$, with $\mathcal{A}_+(\omega : X) = \{A : D(A) \subset X \to \mathcal{P}(X)$,such that $A + \omega I$ is a m-T-accretive operator$\}$. Moreover A satisfies the *property (M)* of Bénilan [9].

(H$_2$): The operator semigroup $T(t) : \overline{D(A)} \to X$, $t \geq 0$, generated by A, is compact. (see Vrabie [32]).

(H$_3$): $u_0 \in \overline{D(A)} \cap L^\infty(\Omega)$.

We shall assume that the nonlinear term $F(u)$ is generated through a continuous real function $f : \mathbb{R} \to \mathbb{R}$ satisfying that

(H$_4$): $f = f_{cv} + f_{cn}$ with f_{cv} convex and f_{cn} concave

Notice that, in contrast with previous works on the quasilinearization process, we do not require any assumption on the linearity of operator A neither on the differentiability of f (in the classical sense).

We define the notion of sub and supersolution of the original abstract Cauchy problem (ACP) : A couple of functions $\underline{u}, \overline{u} \in C([0,T] : X) \cap L^\infty((0,T) \times \Omega)$ are called sub (respect.) supersolutions of (ACP) if there exists $\underline{g}$ (respectively $\overline{g}$) in $L^1(0,T : L^\infty(\Omega))$ with $\underline{g} \leq 0$ (respectively$\overline{g} \geq 0$) such that $\underline{u}, \overline{u}$ are *mild solutions* of the problem

$$\begin{cases} \frac{du}{dt}(t) + A\underline{u}(t) \ni f(\underline{u}) + \underline{g} & \text{in } X, \\ \\ \underline{u}(0) = u_0, \end{cases}$$

(respectively

$$\begin{cases} \frac{du}{dt}(t) + A\overline{u}(t) \ni f(\overline{u}) + \overline{g} & \text{in } X, \\ \\ \overline{u}(0) = u_0 \end{cases}$$

in the case of $\overline{u}$). Notice that here we are identifying the operator $F(u)$ associated to f with the own function $f(u)$.We shall assume

(H$_5$): there exists $\underline{u}, \overline{u}$ sub and super solutions of (ACP).

Finally, as we shall combine some ordering and some approximation arguments we shall need

(H$_6$): the subdifferential operators ∂f_{cv} and $\partial(-f_{cv})$ are bounded on the set $I := [\inf ess_{t\in[0,T],x\in\Omega}\underline{u}(t,x),\ \sup ess_{t\in[0,T],x\in\Omega}\overline{u}(t,x)]$, i.e., $|b| \leq M$ for any $b \in \partial f_{cv}(r)$ or $b \in \partial(-f_{cn})(r)$, for any $r \in I$.

Remark. Since $\underline{u}, \overline{u}$ are bounded functions then I is a compact interval of $\mathbb{R}$. Moreover, by using some well known results (see, e.g. Brézis [10]) it shown that assumption (H$_6$) implies the existence of a sequence of auxiliary functions $f^k \in C^2(\mathbb{R})$ such that $f^k = f^k_{cv} + f^k_{cn}$ with $f^k_{cv}, f^k_{cn} \in C^2(\mathbb{R})$, f^k_{cv} convex and f^k_{cn} concave for any $k \in \mathbb{N}$, such that

$$\begin{cases} f^k_{cv} \nearrow f_{cv}, \text{ as } k \to \infty, & \text{uniformly on any compact interval of } I, \\ f^k_{cn} \searrow f_{cn}, \text{ as } k \to \infty, & \text{uniformly on any compact interval of } I, \end{cases} \tag{3}$$

Moreover $\left\|(f^k_{cv})_u(\eta), (f^k_{cn})_u(\eta)\right\|_{L^\infty(0,T:X')} \leq M_k \leq M$, for the same $M > 0$ given in (H$_6$), for any $\eta \in C([0,T] : X)$ such that $\underline{u} \leq \eta \leq \overline{u}$.

Finally, since the main goal is the approximation of the solution we can consider the uniqueness of solution question as an independent goal. So, we shall assume that

(H$_7$): Problem (ACP) has at most one mild solution.

Remark. This can be proved once we assume (H$_1$) and some extra condition on f such as, f is (globally) Lipschitz continuous.

In order to construct the iterative scheme we define the *pseudo linearized (approximated) abstract Cauchy system*

$$(PLACS)_k \begin{cases} \frac{d\overline{u}_{k+1}}{dt} + A\overline{u}_{k+1} \ni \\ \qquad f^k(\overline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) \\ \qquad +(f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1} - \overline{u}_k) & \text{in } X, \\ \frac{d\underline{u}_{k+1}}{dt} + A\underline{u}_{k+1} \ni \\ \qquad f^k(\underline{u}_k) + (f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) \\ \qquad +(f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1} - \underline{u}_k) & \text{in } X, \\ \overline{u}_{k+1}(0) = u_0, \ \underline{u}_{k+1}(0) = u_0. \end{cases}$$

Theorem 2 *Assume (H$_1$)-(H$_7$). Then, for any $k \in \mathbb{N}$ there exists $(\underline{u}_k, \overline{u}_k) \in L^\infty((0,T)\times\Omega)^2$ mild solutions of the system $(PLACS)$ and with $(\underline{u}_1, \overline{u}_1) = (\underline{u}, \overline{u})$. Moreover,the sequences $\{\underline{u}_k\}, \{\overline{u}_k\}$ converge in $C([0,T] : X)$ to $u \in L^\infty((0,T)\times\Omega)$ (unique) mild solution of (ACP) and we have that $\underline{u} \le u \le \overline{u}$.*

We shall prove the result in several steps.

i) *Existence of* $(\underline{u}_2, \overline{u}_2)$.The $(PLACS)_2$ is given by

$$(PLACS)_2 \begin{cases} \frac{d\overline{u}_2}{dt} + A\overline{u}_2 \ni f^1(\overline{u}) \\ \qquad +(f^1_{cv})_u(\underline{u})(\overline{u}_2 - \overline{u}) + (f^1_{cn})_u(\overline{u})(\overline{u}_2 - \overline{u}) & \text{in } X, \\ \frac{d\underline{u}_2}{dt} + A\underline{u}_2 \ni f^1(\underline{u}) \\ \qquad +(f^1_{cv})_u(\underline{u})(\underline{u}_2 - \underline{u}) + (f^1_{cn})_u(\overline{u})(\underline{u}_2 - \underline{u}) & \text{in } X, \\ \overline{u}_2(0) = u_0, \ \underline{u}_2(0) = u_0. \end{cases}$$

The existence (and uniqueness) of solution of this uncoupled system comes from the fact that $A \in \mathcal{A}_+(\omega : X)$, and that $f^1(\overline{u}) - (f^1_{cv})_u(\underline{u})\overline{u} - (f^1_{cn})_u(\overline{u})\overline{u}$, $f^1(\underline{u}) - (f^1_{cv})_u(\underline{u})\underline{u} - (f^1_{cn})_u(\overline{u})\underline{u} \in L^1(0,T : X)$ (recall that $\overline{u}, \underline{u}$ are bounded and that f^1 is continuous in $\mathbb{R}$).

ii) *Estimates on* $[\overline{u}_2 - \overline{u}]_+$ *and* $[\underline{u} - \underline{u}_2]_+$. By construction we get that

$$\frac{d(\overline{u}_2 - \overline{u})}{dt} + A\overline{u}_2 - A\overline{u} \ni a_1(t,x)(\overline{u}_2 - \overline{u}) - \overline{g} + f^1(\overline{u}) - f(\overline{u})$$

where $a_1(t,x) = (f^1_{cv})_u(\underline{u}(t,x)) + (f^1_{cn})_u(\overline{u}(t,x))$ and so, $\|a_1\|_{L^\infty((0,T)\times\Omega)} \leq M_1$. Since $\overline{g} \geq 0$ and $A + \omega I$ is a T-accretive operator we get the estimates

$$\begin{aligned}
&\max_{t\in[0,T]} \left\|[\overline{u}_2(t,.) - \overline{u}(t,.)]_+\right\|_{L^\infty(\Omega)} \leq e^{(\omega+M_1)T} T\,|\Omega| \left\|\left[f^1 - f\right]_+\right\|_{C(I)}, \\
&\max_{t\in[0,T]} \left\|[\overline{u}_2(t,.) - \overline{u}(t,.)]_-\right\|_{L^\infty(\Omega)} \leq \\
&\qquad e^{(\omega+M_1)T}(T\,|\Omega| \left\|\left[f^1 - f\right]_-\right\|_{C(I)} + \|\overline{g}\|_{L^1(0,T:L^\infty(\Omega))}).
\end{aligned}$$

The proof of the existence of $\underline{u}_2$ is analogous. In that case we get the estimates

$$\begin{aligned}
&\max_{t\in[0,T]} \left\|[\underline{u}(t,.) - \underline{u}_2(t,.)]_+\right\|_{L^\infty(\Omega)} \leq e^{(\omega+M_1)T} T\,|\Omega| \left\|\left[f - f^1\right]_+\right\|_{C(I)}, \\
&\max_{t\in[0,T]} \left\|[\underline{u}(t,.) - \underline{u}_2(t,.)]_-\right\|_{L^\infty(\Omega)} \leq \\
&\qquad e^{(\omega+M_1)T}(T\,|\Omega| \left\|\left[f^1 - f\right]_-\right\|_{C(I)} + \|\underline{g}\|_{L^1(0,T:L^\infty(\Omega))}).
\end{aligned}$$

Remark. If no regularization is needed and so $f^k = f$ then we get that $\underline{u} \leq \underline{u}_2$ and $\overline{u}_2 \leq \overline{u}$.
iii) *Proof of the inequality* $\underline{u}_2 \leq \overline{u}_2$. We have that

$$\frac{d(\overline{u}_2 - \underline{u}_2)}{dt} + A\overline{u}_2 - A\underline{u}_2 \ni [(f^1_{cv})_u(\underline{u}) + (f^1_{cn})_u(\overline{u})](\overline{u}_2 - \underline{u}_2) - F_1$$

with $F_1 = f^1(\overline{u}) - f^1(\underline{u}) + (f^1_{cv})_u(\underline{u})(\underline{u} - \overline{u}) + (f^1_{cn})_u(\overline{u})(\underline{u} - \overline{u})$. But, from the convexity of f^1_{cv} we get that for any $u, v \in I$, $f^1_{cv}(u) \geq f^1_{cv}(v) + (f^1_{cv})_u(v)(u - v)$. Analogously, the concavity of f^1_{cn} implies, for any $u, v \in I$, that $f^1_{cn}(u) \geq f^1_{cn}(v) + (f^1_{cn})_u(u)(u - v)$. Both properties imply that $F_1 \geq 0$ and so, by the T-accretiveness we get the conclusion.
iv) *Existence of* $(\underline{u}_k, \overline{u}_k)$ for $k \in \mathbb{N}$, $k > 1$. It is analogous to the above step. For instance, the forcing term (independent on $\overline{u}_{k+1}$) is now $f^k(\overline{u}_k) - (f^k_{cv})_u(\underline{u}_k)\overline{u}_k - (f^k_{cn})_u(\overline{u}_k)\overline{u}_k$ which, again, is in $L^\infty((0,T)\times\Omega)$.
v) *Estimates on* $[\overline{u}_{k+1} - \overline{u}_k]_+$ *and* $\left[\underline{u}_k - \underline{u}_{k+1}\right]_+$. By construction and ($H_6$) we get that

$$\frac{d(\overline{u}_{k+1} - \overline{u}_k)}{dt} + A\overline{u}_{k+1} - A\overline{u}_k \ni a_k(t,x)(\overline{u}_{k+1} - \overline{u}_k) + F_k$$

with $a_k(t,x) = (f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)$ (and so $\|a_k\|_{L^\infty((0,T)\times\Omega)} \leq M_k$) and $F_k = f^k(\overline{u}_k) - f^{k-1}(\overline{u}_{k-1}) + a_{k-1}(t,x)(\overline{u}_k - \overline{u}_{k-1})$. So, using the convexity of f^k_{cv} and the concavity of f^k_{cn} we get that $F_k \geq f^k(\overline{u}_k) - f^{k-1}(\overline{u}_k)$. Thus, by the T-accretiveness of A we get that

$$\begin{aligned}
&\max_{t\in[0,T]} \left\|[\overline{u}_{k+1}(t,.) - \overline{u}_k(t,.)]_+\right\|_{L^\infty(\Omega)} \leq \\
&\qquad e^{(\omega+M_1)T} T\,|\Omega| \left\|\left[f^k - f^{k-1}\right]_+\right\|_{C(I)}, \\
&\max_{t\in[0,T]} \left\|\left[\underline{u}_k(t,.) - \underline{u}_{k+1}(t,.)\right]_+\right\|_{L^\infty(\Omega)} \leq \\
&\qquad e^{(\omega+M_1)T} T\,|\Omega| \left\|\left[f^{k-1} - f^k\right]_+\right\|_{C(I)}.
\end{aligned}$$

Remark. If no regularization is needed and so $f^k = f^{k-1}$ then we get that $\underline{u}_k \leq \underline{u}_{k+1}$ and that $\overline{u}_k \leq \overline{u}_{k+1}$.
vi) *Proof of the inequality* $\underline{u}_{k+1} \leq \overline{u}_{k+1}$. As in step iii) we have

$$\tfrac{d(\overline{u}_{k+1}-\underline{u}_{k+1})}{dt} + A\overline{u}_{k+1} - A\underline{u}_{k+1} \ni$$
$$[(f^k_{cv})_u(\underline{u}_k) + (f^k_{cn})_u(\overline{u}_k)](\overline{u}_{k+1} - \underline{u}_{k+1}) - F_k,$$
$$F_k = f^k(\overline{u}_k) - f^k(\underline{u}_k) + [(f^k_{cv})_u(\underline{u}_{k-1}) + (f^1_{cn})_u(\overline{u}_{k-1})](\underline{u}_k - \overline{u}_{k-1}).$$

Using the convexity of f^k_{cv} and the concavity of f^k_{cn} we have that $F_k \geq 0$ and so, by the T-accretiveness we get the wanted comparison.
End of the proof of Theorem 1. The sequences $\{\underline{u}_k\}$ and $\{\overline{u}_k\}$ are uniformly bounded in $L^\infty((0,T)\times\Omega)$. In consequence, using assumption (H.6), the sequences $\{f^k(\overline{u}_k)+(f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1}-\overline{u}_k)+(f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1}-\overline{u}_k)\}$ and $\{f^k(\underline{u}_k)+(f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1}-\underline{u}_k)+(f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1}-\underline{u}_k)\}$ are also uniformly bounded in $L^\infty((0,T)\times\Omega)$. Thus, by the assumption (H2) there exists $\underline{U},\overline{U}$ such that $\{\underline{u}_k\}\to\underline{U}$ and $\{\overline{u}_k\}\to\overline{U}$ (strongly) in $C([0,T]:X)$ (and, at least, weakly in $L^\infty((0,T)\times\Omega)$). Finally, $\{f^k(\overline{u}_k)+(f^k_{cv})_u(\underline{u}_k)(\overline{u}_{k+1}-\overline{u}_k)+(f^k_{cn})_u(\overline{u}_k)(\overline{u}_{k+1}-\overline{u}_k)\}\to f(\overline{U})$ and $\{f^k(\underline{u}_k)+(f^k_{cv})_u(\underline{u}_k)(\underline{u}_{k+1}-\underline{u}_k)+(f^k_{cn})_u(\overline{u}_k)(\underline{u}_{k+1}-\underline{u}_k)\}\to f(\underline{U})$ and so $\underline{U},\overline{U}\in L^\infty((0,T)\times\Omega)^2$ are mild solutions of (ACP) which must coincide due the assumption (H$_7$). From steps i)-vi), passing to the limit, we get that $\underline{u}\leq u\leq\overline{u}$.■
Remark. If no regularization is needed and $f^k = f$ then it is easy to see that $\underline{u}\leq\underline{u}_2\leq...\leq\underline{u}_k\leq...\leq u\leq...\leq\overline{u}_k\leq\overline{u}_2\leq\overline{u}$.
Remark. Some abstract results of a difference nature (in which A is a regular function, and so, of difficult application to PDEs) can be found in Section 4.6 of the book [27] (see also the references cited there).
Remark. More general functions $f(t,x,u)$ can be also considered. For instance, it is possible to improve Theorem 1 by assuming the existence of ϕ convex and ψ concave such that $f = f_{cv} + f_{cn}$ with $f_{cv} + \phi$ convex and $f_{cn} + \psi$ concave. In that case, we assume merely that $A+\phi+\psi$ is a m-T-accretive operator on X. The method can be applied to systems of nonlinear pdes as well as to pdes with some delayed terms (see some iterative schemes in [29], [18] and [**?**]).

Many different examples are possible. For instance, we can consider

Example. $A : D(A)\to\mathcal{P}(L^1(\Omega))$ given by $Au = -div(|\nabla\phi(u)|^{p-2}\nabla\phi(u)) + \beta(u)$ with $D(A) = \{\phi(u)\in W^{1,1}(\Omega),\ u(x)\in D(\beta)$, a.e. $x\in\Omega,\ Au\in L^1(\Omega)$, $-\left|\frac{\partial\phi(u)}{\partial n}\right|^{p-2}\frac{\partial\phi(u)}{\partial n}\in\gamma(\phi(u))$ on $\partial\Omega\}$ where $p>1$, ϕ is continuous and increasing, and β,γ are maximal monotone graphs of $\mathbb{R}^2$ (not necessarily associated to differentiable functions).

One of the reasons argued by Bellman to introducing this method for semilinear equations is the quadratic (sometimes called as *rapid*) convergence. We can prove something similar for concrete quasilinear equations:

Theorem 3 *([**?**]). Let A as in the example with $p\geq 2$, $\phi(u) = u$, $\beta = 0$ and γ*

corresponding to Dirichlet boundary conditions and assume (H3)-(H7). Then

$$\begin{cases} \max_{t\in[0.T]} \|u(t)-\overline{u}_k(t)\|^2_{L^2(\Omega)} + \|u-\overline{u}_k\|^p_{L^p(0,T:W_0^{1,p}(\Omega)} \le \\ \qquad\qquad C(\|u-\overline{u}_{k-1}\|^2_{L^2((0,T)\times\Omega)} + \left\|u-\underline{u}_{k-1}\right\|^2_{L^2((0,T)\times\Omega)}), \\ \max_{t\in[0.T]} \|u(t)-\underline{u}_k(t)\|^2_{L^2(\Omega)} + \|u-\underline{u}_k\|^p_{L^p(0,T:W_0^{1,p}(\Omega)} \le \\ \qquad\qquad C(\|u-\overline{u}_{k-1}\|^2_{L^2((0,T)\times\Omega)} + \left\|u-\underline{u}_{k-1}\right\|^2_{L^2((0,T)\times\Omega)}). \end{cases}$$

3 Abstract Pseudolinearization Principle

As said at the Introduction, our main motivation to generalize the linearization principle comes from some previous works by the authors and collaborators ([**?**], [1]), [**?**]) dealing with the stabilization of the uniform oscillations for the *complex Ginzburg-Landau equation.* This stabilization takes place by means of some global delayed feedback. If, for instance,we consider the case in which the domain is $\Omega=(0,L_1)\times(0,L_2)$ with periodic boundary conditions, and define the faces of the boundary

$$\Gamma_j=\partial\Omega\cap\{x_j=0\}, \Gamma_{j+2}=\partial\Omega\cap\{x_j=L_j\}, j=1,2,$$

this problem can be stated as follows

$$(P_1)\begin{cases} \frac{\partial\mathbf{u}}{\partial t}-(1+i\epsilon)\Delta\mathbf{u}=(1-i\omega)\mathbf{u}-(1+i\beta)\left|\mathbf{u}\right|^{\mathbf{2}}\mathbf{u} \\ \qquad\qquad +\mu e^{i\chi_0}\mathbf{F}(\mathbf{u},t,\tau) & \Omega\times(0,+\infty), \\ \mathbf{u}|_{\Gamma_j}=\mathbf{u}|_{\Gamma_{j+2}}, \\ \qquad \left(-\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\Big|_{\Gamma_j}=\right)\frac{\partial\mathbf{u}}{\partial x_j}\Big|_{\Gamma_j}=\frac{\partial\mathbf{u}}{\partial x_j}\Big|_{\Gamma_{j+2}}\left(=\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\Big|_{\Gamma_{j+2}}\right) & \partial\Omega\times(0,+\infty), \\ \mathbf{u}(x,s)=\mathbf{u}_0(x,s) & \Omega\times[-\tau,0], \end{cases}$$

where $\mathbf{n}$ is the outpointing normal unit vector and

$$\mathbf{F}(\mathbf{u},t,\tau)=[m_1\mathbf{u}(t)+m_2\overline{\mathbf{u}}(t)+m_3\mathbf{u}(t-\tau,x)+m_4\overline{\mathbf{u}}(t-\tau)],$$

$$\text{with } \overline{\mathbf{u}}(s)=\frac{1}{|\Omega|}\int_\Omega\mathbf{u}(s,x)dx.$$

Here the parameters $\epsilon,\beta,\omega,\mu,\chi_0,m_i$ and τ are real numbers, in contrast with the solution $\mathbf{u}(x,t)=u_1(x,t)+iu_2(x,t)$.

This type of equations (called as of Stuart-Landau in absence of the diffusion term) arise in the study of the stability of reaction diffusion equations such as $\frac{\partial\mathbf{X}}{\partial t}-\mathbf{D}\Delta\mathbf{X}=\mathbf{f}(\mathbf{X}:\eta)$ where $\mathbf{X}:\Omega\times(0,+\infty)\to\mathbb{R}^n$ and η is a real scalar parameter when the deviation $\mathbf{v}$ from the uniform state solution $\mathbf{X}_\infty$ is developed asymptotically in terms of some multiple scales (see Kuramoto [23]).

Coefficient ε measures the degree to which the diffusion matrix $\mathbf{D}$ deviates from a scalar. With the basis of a sound experimental work, many recent studies of a more descriptive nature, but of a great originality and interest, have been written. In those studies the delay term $\mathbf{F}(\mathbf{u}, t, \tau)$ has been taken corresponding to $m_4 = 1, m_i = 0$ for $i = 1, 2, 3$ and introduced as a control mechanism (see Battogtokh and Mikhailov[7], Mertens *et al.* [28]).

If we focus our attention on the so called *slowly varying complex amplitudes* defined by $\mathbf{u}(x,t) = \mathbf{v}(x,t)e^{-i\omega t}$, thus, $\mathbf{v}$ satisfy

$$(P_2) \begin{cases} \frac{\partial \mathbf{v}}{\partial t} - (1+i\epsilon)\Delta \mathbf{v} = \mathbf{v} - (1+i\beta)|\mathbf{v}|^{\mathbf{2}}\, \mathbf{v} \\ \qquad +\mu e^{i\chi_0}\left[m_1\mathbf{v} + m_2\overline{\mathbf{v}}\right. \\ \qquad \left.+e^{i\omega\tau}(m_3\mathbf{v}(t-\tau,x) + m_4\overline{\mathbf{v}}(t-\tau))\right] & \Omega \times (0,+\infty), \\ \mathbf{v}|_{\Gamma_j} = \mathbf{v}|_{\Gamma_{j+2}}, \\ \qquad \left(-\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\big|_{\Gamma_j} = \right) \frac{\partial \mathbf{v}}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial \mathbf{v}}{\partial x_j}\Big|_{\Gamma_{j+2}} \left(= \frac{\partial \mathbf{v}}{\partial \mathbf{n}}\big|_{\Gamma_{j+2}}\right) & \partial\Omega \times (0,+\infty), \\ \mathbf{v}(x,s) = \mathbf{u}_0(x,s)e^{i\omega s} & \Omega \times [-\tau, 0]. \end{cases}$$

The existence and uniqueness of a solution of (P_1) can be proven once we assume, for instance, that $\mathbf{u}_0 \in \mathbf{C}([-\tau,0] : \mathbf{L}^2(\Omega))$. In the mentioned references we were interested in the stability analysis of the time-periodical function $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$. We can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t) = \mathbf{v}(x,t)e^{i\theta t}$ where $\mathbf{v}(x,t)$ is a solution of (P_2). Thus $\mathbf{z}(x,t)$ satisfies

$$(P_3) \begin{cases} \frac{\partial \mathbf{z}}{\partial t} - (1+i\epsilon)\Delta \mathbf{z} = (1+i\theta)\mathbf{z} - (1+i\beta)\,|\mathbf{z}|^2\, \mathbf{z} \\ \qquad +\mu e^{i\chi_0}\left[m_1\mathbf{z} + m_2\overline{\mathbf{z}}\right. \\ \qquad \left.+\, e^{i(\omega+\theta)\tau}(m_3\mathbf{z}(t-\tau,x) + m_4\overline{\mathbf{z}}(t-\tau))\right] & \Omega \times (0,+\infty), \\ \mathbf{z}|_{\Gamma_j} = \mathbf{z}|_{\Gamma_{j+2}}, \\ \qquad \left(-\frac{\partial \mathbf{z}}{\partial \overrightarrow{n}}\big|_{\Gamma_j} = \right) \frac{\partial \mathbf{z}}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial \mathbf{z}}{\partial x_j}\Big|_{\Gamma_{j+2}} \left(= \frac{\partial \mathbf{z}}{\partial \overrightarrow{n}}\big|_{\Gamma_{j+2}}\right) & \partial\Omega \times (0,+\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x,s)e^{i(\omega-\theta)s} & \Omega \times [-\tau, 0]. \end{cases}$$

Notice that now, $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_{uosc}(x,t)e^{i\theta t} = \mathbf{y} = \rho_0$ is an stationary solution of (P_3): i.e. $\mathbf{0} = (1+i\theta)\mathbf{y} - (1+i\beta)\,|\mathbf{y}|^2\,\mathbf{y} + \mu e^{i\chi_0}\left[m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3+m_4)\right]\mathbf{y}$.

The motivation to keep A nonlinear after the process of linearization (reason why we used the term of *pseudo-linearization principle*) comes from the fact that if we use the representation for the unknown of the delayed nonlinear equation (P_3) as $\mathbf{z}(x,t) = \rho(x,t)e^{i\phi(x,t)}$ then we arrive to a coupled nonlinear system of delayed equations for ρ and ϕ which can be described in terms of the representation operator given by $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$. Indeed, notice that $\mathbf{P}$ is nonlinear and that if $\mathbf{q} = (\rho,\phi)$ then $\mathbf{z}(x,t) = \mathbf{P}(\mathbf{q}(x,t))$ and the (P_3) can be formulated as $\frac{d\mathbf{P}(\mathbf{q}(\cdot,t))}{dt} + A\mathbf{P}(\mathbf{q}(\cdot,t)) + B\mathbf{P}(\mathbf{q}(\cdot,t)) = F(\mathbf{P}(\mathbf{q}(\cdot))_t)$. By using that the matrix $\mathbf{C}(\mathbf{q}(\cdot,t)) = grad\mathbf{P}(\mathbf{q}(\cdot,t))$ is not singular, we can arrive to the simpler formulation

$$\frac{d\mathbf{q}}{dt}(\cdot,t) + \mathbf{C}(\mathbf{q}(\cdot,t))^{-1}[A\mathbf{P}(\mathbf{q}(\cdot,t)) + B\mathbf{P}(q(\cdot,t))] = \mathbf{C}(\mathbf{q}(\cdot,t))^{-1}F(\mathbf{P}(\mathbf{q}(\cdot))_t). \quad (4)$$

Notice that, although this delayed system can be also (formally) linearized (this is the procedure followed in Battogtokh and Mikhailov [7] and Mertens *et al.* [28]) the above diffusion operator $\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot,t))$ becomes now quasilinear on $\mathbf{q}$ and thus the mathematical justification is much more delicate.

Other examples, given in Section 3, justify also the philosophy of keeping A non-linear after linearizing the rest of the terms of the equation. For instance, this is the case when A is multivalued, or nondifferentiable or a degenerate quasilinear operator. We point out that some relevant examples of nonlinear functional equations arise in the most different contexts (see, for instance, Díaz and Hetzer [20] for one example in Climatology, Chukwu [19] for a family of examples dealing with the wealth of nations and the general exposition made in Hale [22]).

Coming back to the abstract formulation, the structural assumptions we shall assume in this paper are the following

(H1): $A \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, with

$$\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathcal{P}(X), \text{ such that } A + \omega I \text{ is a } m - accretive \text{ operator}\},$$

(see Brezis [10] for the case of $X = H$ a Hilbert space and the works by Benilan, Crandall, Pazy and others for the case of a general Banach space: see the monographs [9] and [32]),

(H2)≡(H$_2$): the operators semigroup $T(t) : \overline{D_X(A)}^X \to X$, $t \geq 0$, generated by A, is compact

(H3): $B \in \mathcal{A}(0 : X)$, B is single valued, Fréchet differentiable, and B is dominated by A; i.e.

$$D_X(A) \subset D_X(B) \text{ and } |Bu| \leq k\left|A^0u\right| + \sigma(|u|) \quad (5)$$

for any $u \in D_X(A)$ and for some $k < 1$ and some continuos function $\sigma : \mathbb{R} \to \mathbb{R}$, where, here and in what follows, $|.|$ denotes the norm in the space X (in contrast with the norm in space C which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $\left|A^0u\right| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$,

(H4): $F : C \to X$ satisfies a local Lipschitz condition, i.e., for any $R > 0$ there exists $L(R) > 0$ such that

$$|F(\phi) - F(\psi)| \leq L(R) \|\phi - \psi\| \text{ for any } \phi, \psi \in C \text{ and } \|\phi\|, \|\psi\| \leq R. \tag{6}$$

(H5): there exists $\delta^F > 0$ such that $F : B^X_{\delta^F}(\widehat{w}) \to X$ is Fréchet differentiable with the Fréchet derivative $\mathrm{D}F(\widehat{w})$ given by $\mathrm{D}(F(\widehat{w}))\phi = \int_{-\tau}^{0} d\eta(\theta)\phi(\theta), \phi \in C$, for $\eta : [-\tau, 0] \to B(X, X)$ of bounded variation and the Fréchet derivative is locally Lipschitz continuous, where

$$B^X_{\delta^F}(\widehat{w}) = \left\{ \phi \in C; \|\phi - \widehat{x}\| < \delta^F \right\},$$

We further assume the main condition of our arguments:

(H6): the operator $y \to Ay + By - \mathrm{D}F(\widehat{w})(e^{\omega \cdot} y)$ belongs to $\mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ with $\mathrm{Re}\,\omega = \gamma < 0$ where $e^{\omega \cdot} v \in C$ is defined by

$$(e^{\omega \cdot} v)(s) = e^{\omega s} \widehat{v}(s), \text{ with } \widehat{v}(s) = v, \text{ for any } s \in [-\tau, 0] \text{ for } v \in X. \tag{7}$$

In order to treat the case in which B is differentiable we introduce the conditions

(H7): there exists a Banach space Y and there exists $\delta^B > 0$ such that B is Fréchet differentiable as function from $B_{\delta^B}(w)$ into Y, with the Fréchet derivative $\mathrm{D}B(w)$ locally Lipschitz continuous, where

$$B_{\delta^B}(w) = \left\{ z \in D(B); \; |w - z| < \delta^B \right\},$$

(H8) the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w}) \left(e^{\omega^* \cdot} y \right)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\mathrm{Re}\,\omega^* = \gamma^* < 0$.

Theorem 4 *Assume (H1)-(H6). Then there exists $\alpha > 0$, $\epsilon > 0$ and $M \geq 1$ such that if $u_0 \in B^X_\epsilon(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ then the solution $u(\cdot : u_0)$ of (1) exists on $[-\tau, +\infty)$ and*

$$|u(t : u_0) - w| \leq M e^{-\alpha t} \|u_0 - \widehat{w}\|, \text{ for any } t > 0. \tag{8}$$

Moreover, if we also assume (H7), that (H1)-(H5) holds on the space Y and (H8) then there exists $\alpha^* > 0$, $\epsilon^* \in (0, \epsilon]$ and $M^* \geq 1$ such that if $u_0 \in B^{X \cap Y}_{\epsilon^*}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ and for any $t > 0$,then

$$|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \leq M^* e^{-\alpha^* t} (\|u_0 - \widehat{w}\|_X + \|u_0 - \widehat{w}\|_Y). \tag{9}$$

Proof. From assumptions (H4) and (H5)

$$F(\phi) = F(\widehat{w}) + \mathrm{D}F(\widehat{w})(\phi - \widehat{w}) + G^F(\widehat{w}, \phi), \text{ for any } \phi \in B^X_{\delta^F}(\widehat{w}).$$

Moreover since $\mathrm{D}F(\widehat{w})$ is locally Lipschitz continuous, there exists a continuous increasing functions b_X^F such that

$$\left|G^F(\widehat{w},\phi)\right| \leq b_X^F(\|\phi-\widehat{w}\|)\,\|\phi-\widehat{w}\|\,,\ \text{for any } \phi\in B^X_{\delta^F}(\widehat{w}). \tag{10}$$

Then

$$\frac{du}{dt}(t)-\frac{dw}{dt}+Au(t)-Aw+Bu(t)-Bw-\mathrm{D}F(\widehat{w})(u_t-\widehat{w})\ni -G^F(\widehat{w},u_t). \tag{11}$$

We now use assumption (H6). We claim that we can find a constant constant $K\geq 1$ and such that

$$\|u_t-\widehat{w}\|\leq Ke^{\gamma t}\|u_0-\widehat{w}\|+\int_0^t Ke^{\gamma(t-s)}\left|G^F(\widehat{w},u_s)\right|ds. \tag{12}$$

Indeed, as $u(t)$ and w are "integral solutions" in the sense of Benilan (see. e.g. [9]), then, by (H6), if we multiply (11) by $u(t)-w$ (by using the usual semi inner-braket $[,]$: see, for instance Benilan, Crandall and Pazy [9] or Vrabie [32] (Section 1.4)) we get that

$$|u(t)-w|\leq Ke^{\gamma(t-t_0)}\,|u(t_0)-w|+\int_{t_0}^t Ke^{\gamma(t-s)}\left|G^F(\widehat{w},u_s)\right|ds \tag{13}$$

for any $t\geq t_0\geq 0$ (see, for instance, Benilan, Crandall and Pazy [9] or Vrabie [32] Theorem 1.7.5). Then,

$$|u(t)-w|\leq Ke^{\gamma t}\|u_0-\widehat{w}\|+\int_0^t Ke^{\gamma(t-s)}\left|G^F(\widehat{w},u_s)\right|ds \tag{14}$$

for any $t\geq 0$. Finally, since (14) holds trivially for $t\in[-\tau,0]$ we get (12) by taking the maximum, in (13), on intervals of the form $[t-\tau,t]$ for any $t\geq 0$. Now, let $R\in(0,\delta^F)$ be chosen so that

$$b_X^F(R)<(-\gamma)/(4K). \tag{15}$$

Define $\epsilon=\min\left\{R/(2K),\delta_X^F\right\}$. Let us show that if $u_0\in B_\epsilon^X(\widehat{w})$ then the associated solution u of (1) exists and $\|u_t-\widehat{w}\|<R$ for all $t\geq 0$. Thanks to assumption (H2) we can apply some maximal continuation results (see, for instance, Chapter 3 of Vrabie [32], or Chapter 2 of Wu [33] when A is linear), it suffices to show that there exists no $t_1>0$ so that $\|u_{t_1}\|=R$ and $\|u_t\|<R$ for $t\in[0,t_1)$. By contradiction, if there exists such a t_1, then on $[0,t_1]$ we have

$$\begin{aligned}\|u_t-\widehat{w}\| &\leq Ke^{\gamma t}\|u_0-\widehat{w}\|+\int_0^t Ke^{\gamma(t-s)}\left|G^F(\widehat{w},u_s)\right|ds\\ &\leq Ke^{\gamma t}\|u_0-\widehat{w}\|+2Kb_X^F(R)\int_0^t e^{\gamma(t-s)}\|u_s-\widehat{w}\|\,ds.\end{aligned}$$

In particular, at $t = t_1$ we have

$$\|u_{t_1} - \widehat{w}\| \leq K\epsilon + \frac{2Kb_X^F(R)}{(-\gamma)} R \leq R,$$

a contradiction to the choice of t_1.
Finally, to end the proof, let $u_0 \in B_\epsilon^X(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ and let u the associated solution of (1). Since we have shown that $\|u_t - \widehat{w}\| \leq R$ for all $t \geq 0$ we get that

$$\|u_t - \widehat{w}\| \leq Ke^{\gamma t} \|u_0 - \widehat{w}\| + Kb_X^F(R) \int_0^t e^{\gamma(t-s)} \|u_s - \widehat{w}\| \, ds \tag{16}$$

holds for all $t \geq 0$. Thus, by using the Gronwall's inequality, we get

$$\|u_t - \widehat{w}\| \leq Ke^{[\gamma - Kb(R)]t} \|u_0 - \widehat{w}\| \leq Ke^{(\gamma/2)t} \|u_0 - \widehat{w}\|, u_0 \in B_\epsilon^X(\widehat{w})$$

which shows (8).
In order to show the decay estimate (9), we repeat the same arguments as before but now on the space Y. Then, from assumptions (H3) on Y and (H7), there exist δ_Y^F and δ_X^B such that

$$B(z) = B(w) + \mathrm{D}B(w)\,(z - w) + G^B(w, z), \text{ for any } z \in B_{\delta_X^B}(w),$$
$$F(\phi) = F(\widehat{w}) + \mathrm{D}F(\widehat{w})\,(\phi - \widehat{w}) + G^F(\widehat{w}, \phi), \text{ for any } \phi \in B^Y_{\delta_Y^F}(\widehat{w}).$$

where now

$$B_{\delta_X^B}(w) = \left\{ z \in D_X(B) \cap D_Y(B);\ \ |w - z| < \delta_X^B \right\},$$

$$B_{\delta_Y^F}(\widehat{w}) = \left\{ \phi \in C; \|\phi - \widehat{x}\|_Y < \delta_Y^F \right\},$$

and, as before, $\|.\|_Y$ denotes the norm on the space $C_Y := C([-\tau, 0] : Y)$. Moreover, there exists two continuous increasing functions b_X^B and b_Y^F such that

$$\left|G^B(w, z)\right|_Y \leq b_X^B(|w - z|)\,|w - z|\,, \text{ for any } z \in B_{\delta_X^B}(w), \tag{17}$$
$$\left|G^F(\widehat{w}, \phi)\right|_Y \leq b_Y^F(\|\phi - \widehat{w}\|_Y)\,\|\phi - \widehat{w}\|_Y\,, \text{ for any } \phi \in B_{\delta_Y^F}(\widehat{w}). \tag{18}$$

Now

$$\frac{du}{dt}(t) - \frac{dw}{dt} + Au(t) - Aw + \mathrm{D}B(w)(u(t) - w) - \mathrm{D}F(\widehat{w})(u_t - \widehat{w}) \ni G^B(w, u(t)) - G^F(\widehat{w}, u_t). \tag{19}$$

Thus, by using (H8) and arguing as in the first part we get that there exists a constant constant $K^* \geq 1$ such that

$$\|u_t - \widehat{w}\|_Y \leq K^* e^{\gamma^* t} \|u_0 - \widehat{w}\|_Y + \int_0^t K^* e^{\gamma^*(t-s)} (\left|G^B(w, u(s))\right|_Y + \left|G^F(\widehat{w}, u_s)\right|_Y) ds \tag{20}$$

and then, by taking $\delta = \min(\delta_X^B, \delta_Y^F)$ and $R^* \in (0, \delta)$ such that

$$\max(b_X^B(R^*), b_Y^F(R^*)) < (-\gamma)/(4K), \tag{21}$$

we obtain that

$$\|u_t - \widehat{w}\|_Y \leq K^* e^{\gamma^* t} \|u_0 - \widehat{w}\|_Y$$
$$+K^* \int_0^t e^{\gamma^*(t-s)} (b_X^B(R^*) \|u_s - \widehat{w}\|_X + b_Y^F(R^*) \|u_s - \widehat{w}\|_Y) ds. \tag{22}$$

We define $\widetilde{R} = \min(R, R^*)$, $\widetilde{K} = \max(K, K^*)$, $\widetilde{\gamma} = \max(\gamma, \gamma^*) < 0$ and $\epsilon^* = \min\left\{\widetilde{R}/(2\widetilde{K}), \delta\right\}$. Then, if $u_0 \in B_{\epsilon^*}^{X \cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ and we assume, for instance, that $\widetilde{\gamma} = \gamma$, by adding (16) and (22) we deduce that

$$\|u_t - \widehat{w}\|_X + \|u_t - \widehat{w}\|_Y \leq \widetilde{K} e^{\widetilde{\gamma} t} (\|u_0 - \widehat{w}\|_X + e^{(\gamma^* - \gamma)t} \|u_0 - \widehat{w}\|_Y)$$
$$+\widetilde{K} \int_0^t e^{\widetilde{\gamma}(t-s)} [(b_X^F(\widetilde{R}) + b_X^B(\widetilde{R}) e^{(\gamma^* - \gamma)t}) \|u_s - \widehat{w}\|_X$$
$$+ b_Y^F(R^*) e^{(\gamma^* - \gamma)t} \|u_s - \widehat{w}\|_Y] ds. \tag{23}$$

and the estimate (9) follows, again, by Gronwall's inequality.■

Remark 5 *It is not difficult to show that the assumption (H8) is implied (when A is linear) by the condition: "if* $\lambda \in \mathbb{C}$ *is given so that there exists* $y \in D(B) \setminus \{0\}$ *such that* $Ay + \mathrm{D}B(w)y - \lambda y \ni \mathrm{D}F(\widehat{w})\left(e^{\lambda \cdot} y\right)$ *then* $\mathrm{Re}\,\lambda > 0$*". This allow to see Theorem 4.1 of Wu [33] (see also Parrot [30] and its references) as an special case of our abstract result with* $B = 0$*. In that case the "variation of the constants formula" can be used to get a different proof of the theorem since A is linear. Notice that if* $B \neq 0$ *and* $D(B) \subsetneq X$ *then the arguments of the proof of Wu [33] do not work (in spite of the claimed in the Example 4.8 given there).*

Remark 6 *When A is linear, as in the case without delay, assumption (H7) implies that the zero solution of the linearized problem* $\frac{dU}{dt}(t) + AU(t) + \mathrm{D}B(w)U(t)$-$\mathrm{D}F(\widehat{w})U_t(.) = 0$ *in X, is locally asymptotically stable (Wu [33]).*

Remark 7 *It is possible to prove the existence of global solutions for a general class of initial data (not necessarily near* $\widehat{w}$*) by using that* $A + B \in \mathcal{A}(\omega : X)$*, for some* $\omega \in \mathbb{C}$*, some truncation of the nonlocal term* $F(u_t)$ *and passing to the limit by the compactness of the semigroup generated by A (see Vrabie [32] for some related results).*

An easy adaptation of the above proof leads to the following linearization result (now on a possibly smaller neighborhood of w) when A is differentiable

Theorem 8 *The conclusion of the above result remains true if we assume, additionally, that condition (H7) also holds for A and we replace condition (H8) by*

(H9): *the operator* $y \to \mathrm{D}A(w)y + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\,(e^{\omega\cdot}y)$ *belongs to* $\mathcal{A}(\omega)$, *for some* $\omega \in \mathbb{C}$ *with* $\mathrm{Re}\,\omega = \gamma < 0$ ■

3.1 Example 1. The complex Ginzburg-Landau equation with a global delayed mechanism

Motivated by the special form of the nonlinear term of the equation in (P_3) we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$ (notice that, in contrast with the case of scalar equations (see Parrot [30]) the space $\mathbf{L}^\infty(\Omega)$ is not suitable space to check assumption (H1): see [6]). A detailed analysis of the associated diffusion operator is consequence of some previous results in the literature: see, for instance, Amann ([4])). Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} \Delta & -\epsilon\Delta \\ \epsilon\Delta & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, if $\epsilon \neq 0$ the diffusion matrix has a non zero antisymmetric part. In particular, A is the generator of a semigroup of contractions $\{T(t)\}_{t\geq 0}$ on X and the compactness of the semigroup is consequence of the compactness of the inclusion $D(A) \subset X$ (notice that, since $N = 2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems.

Concerning the rest of the terms of the equation in (P_3), we define $B\mathbf{u} = (1+i\beta)\,|\mathbf{u}|^2\,\mathbf{u}$ with $D(B) = \mathbf{L}^{12}(\Omega)$. By using the characterization of the semi inner-braket $[,]$ for the spaces $L^p(\Omega)$ (see, for instance Benilan, Crandall and Pazy [9]) it is easy to see that $\mathbf{B}$ verifies (H3). Moreover, by the results on the Frechet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi [5]) we get that (H7) holds, with $\mathrm{D}B(\mathbf{y})\mathbf{v} = 3(1+i\beta)\,|\mathbf{y}|^2\,\mathbf{v}$, if we take $Y = \mathbf{L}^{4/3}(\Omega)$. It can be found in the above mentioned reference that assumption (H7) does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

The nonlocal term is defined, by

$$F(\mathbf{u}_t) = (1+i\theta)\mathbf{u}(t) + \mu e^{i\chi_0}\left[m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + e^{i(\omega+\theta)\tau}(m_3\mathbf{u}(t-\tau) + m_4\overline{\mathbf{u}}(t-\tau))\right],$$

is locally Lipschitz continuous and its Frechet derivative is given by

$$\begin{aligned}\mathrm{D}F(\widehat{\mathbf{y}})\mathbf{v}(t) = &-(1+i\theta)\mathbf{v}(t) - \mu e^{i\chi_0}\left[m_1\mathbf{v}(t) + m_2\overline{\mathbf{v}}(t)\right.\\ &\left. - e^{i(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau) - m_4\overline{\mathbf{v}}(t-\tau))\right]\end{aligned} \quad (24)$$

since for any $\phi \in C$, the non-local operator $\phi \to \frac{1}{|\Omega|}\int_\Omega \phi(s)dx$ is linear and we can write $\mathrm{D}F(\widehat{\mathbf{y}})\phi = \int_{-\tau}^0 d\eta(s)\phi(s)$, with

$$\begin{aligned} d\eta(s)\mathbf{v}(s) \;=\; &\delta_0(s)(1+i\theta)\mathbf{v}(s) \\ &+\mu e^{i\chi_0}\left[\delta_0(s)(m_1\mathbf{v}(s) + m_2\overline{\mathbf{v}}(s)) + e^{i(\omega+\theta)\tau}\delta_{-\tau}(s)(m_3\mathbf{v}(s) + m_4\overline{\mathbf{v}}(s))\right]\end{aligned} \quad (25)$$

for any $\mathbf{v} \in C([-\tau,\infty)\colon \mathbf{L^4}(\Omega))$ and any $s \in [-\tau,\infty)$, where $\delta_0(s), \delta_{-\tau}(s)$ denote the Dirac delta at the points $s = 0$ and $s = -\tau$ respectively. By well-known results, we have that $\eta : [-\tau, 0] \to B(X, X)$ has a bounded variation and so, conditions (H4) and (H5) hold (and analogously replacing X by Y).

Finally, assumption (H6) can be read as a condition on the stationary state $\mathbf{y}$ (a study of the eigenvalue of operator A can be found, for instance, in Temam [31]).

Remark 9 *By introducing the representation operator* $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$ *it is clear that the quasilinear operator* $A\mathbf{P}(\mathbf{q})$ *obtained from the operator* $A\mathbf{u}$=-$(1 + i\epsilon)\Delta\mathbf{u}$ *satisfies also condition* $A \in \mathcal{A}(\omega)$ *(since* $\mathbf{P}$ *is merely a change of variables). We point out that,*

$$A\mathbf{P}(\mathbf{q}) = -(1 + i\epsilon)[\Delta\rho - \rho\left|\nabla\phi\right|^2 + i(2\nabla\rho\cdot\nabla\phi + \rho\Delta\phi)]e^{i\phi}.$$

Then, the "formal linearization" of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ *at* $\mathbf{q}^*(x, y) := \mathbf{y} \equiv \rho_0$ *becomes*

$$D\mathbf{E}(\mathbf{q}^*)(\rho e^{i\phi}) = -(1 + i\epsilon)[\Delta\rho + i\rho_0\Delta\phi]e^{i\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ *needs a slight modification of the above linear expression.*■

3.2 Example 2. Case in which A is nonlinear and nondifferentiable

It is not difficult to adapt the results of the first example to the case in which the vectorial operator is given by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} A_1 & -\epsilon\Delta \\ \epsilon\Delta & A_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{26}$$

with $A_i : D(A_i) \to \mathcal{P}(L^4(\Omega))$ two (possibly different) m-accretive operators in $L^4(\Omega)$, as for instance,

$$\left\{ \begin{array}{c} A_i u = -div(\left|\nabla u\right|^{p_i-2} \nabla u) + \beta_i(u) \\ D(A_i) = \{u \in W^{1,1}(\Omega) \cap L^4(\Omega),\ u(x) \in D(\beta) \text{ a.e. } x \in \Omega,\ A_i u \in L^4(\Omega) \\ \text{and } -\left|\frac{\partial u}{\partial n}\right|^{p_i-2} \frac{\partial u}{\partial n} \in \gamma_i(u) \text{ on } \partial\Omega\} \end{array} \right.$$

where $p_i \in (1, +\infty)$ and β_i, γ_i are maximal monotone graphs of $\mathbb{R}^2$ (not necessarily associated to differentiable functions). We send the reader to Vrabie [32] (and its references) for the study of the assumptions (H1) and (H2) for each of the nonlinear operators A_i. We point out that the structure of the nonlinear diffusion operator (26) allows to guarantee that the diffusion operator is m-accretive in $\mathbf{L}^4(\Omega)$. The same holds also on $\mathbf{L}^{4/3}(\Omega)$.■

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