Hopf bifurcation and bifurcation from constant oscillations to a torus path for delayed complex Ginzburg-Landau equations *

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Summary. We consider the complex Ginzburg-Landau equation with feedback control given by some delayed linear terms (possibly dependent of the past spatial average of the solution). We prove several bifurcation results by using the delay as parameter. We start proving a Hopf bifurcation result for the equation without diffusion (the so-called Stuart-Landau equation) when the amplitude of the delayed term is suitably chosen. The diffusion case is considered first in the case of the whole space and later on a bounded domain with periodicity conditions. In the first case a linear stability analysis is made with the help of computational arguments (showing evidence of the fulfillment of the delicate transversality condition). In the last section the bifurcation takes place starting from an uniform oscillation and originates a path over a torus. This is obtained by the application of an abstract result over suitable functional spaces.

Key words: delayed complex Ginzburg-Landau equations, Hopf bifurcation, torus bifurcation, linearization, uniform oscillations

1 Introduction

1.1 Reaction-diffusion equations and the complex Ginzburg-Landau equation

The evolution of a chemical system consisting of n species which are reacting with each other and allowed to diffuse in a spatially extended medium, is generally

*To the memory of Maria Luisa Menéndez: excellent mathematician, admirable colleague and great person.

described by a n-component reaction-diffusion equation for the $n-$concentrations $\mathbf{c}(x,t)$

$$\partial_t \mathbf{c} = \mathbf{F}(\mathbf{c};p) + \mathbf{D}\Delta\mathbf{c}, \tag{1}$$

where $\mathbf{F}$ denotes the typically nonlinear reaction term representing chemical kinetics, $\mathbf{D}\Delta\mathbf{c}$ the diffusion term (being $\mathbf{D}$ the diffusion matrix) and p a scalar control parameter. We assume that this system has a homogeneous, stationary solution $\mathbf{c_s}$ which undergoes a *Hopf bifurcation* at $p = p_0$: i.e., for $p \in (p_0, p_0 + \varepsilon)$ the stationary solution $\mathbf{c_s}$ becomes a time periodic solution, at least for $\varepsilon > 0$ small enough.

It has been shown by Kuramoto and others that the dynamics of any reaction-diffusion system (1) in the vicinity of a Hopf bifurcation is described, by means of suitable parametrizations, by a nonlinear parabolic equation with complex coefficients, the so-called *complex Ginzburg-Landau equation* (CGLE), see, e.g., [12, 8]. The relation between reaction-diffusion systems and the CGLE has been treated in many texts, here we will follow the presentation of [10].

After a convenient choice of variables $\mathbf{X} = \mathbf{c} - \mathbf{c_s}$ (the *concentration deviations*) and $\epsilon = p - p_0$, the system can be reformulated as

$$\partial_t \mathbf{X} = \mathbf{J}\mathbf{X} + \mathbf{f}(\mathbf{x},\epsilon) + \mathbf{D}\Delta\mathbf{X},$$

where $\mathbf{J}$ is the Jacobian matrix for the homogeneous system evaluated at $\mathbf{X_s} = \mathbf{0}$, i.e. $\mathbf{F}(\mathbf{c};p) - \mathbf{F}(\mathbf{c}_s;p_0) = \mathbf{J}\mathbf{X} + \mathbf{f}(\mathbf{x},\epsilon)$. At the *bifurcation point*, $\mathbf{J}$ has two imaginary eigenvalues $\pm \mathrm{i}\omega_0$, being ω_0 the so-called *Hopf frequency*. The corresponding *right eigenvectors* $\mathbf{e}_1$ and $\mathbf{e}_2 = \bar{\mathbf{e}}_\mathbf{1}$ (normalized with left eigenvectors $\mathbf{e}_i^+$ according to $\mathbf{e}_i^+\mathbf{e}_j = \delta_{ij}$) span the *center subspace* E^c of the homogeneous solution. The *center manifold* W^c is tangent to E^c at $\mathbf{X} = \mathbf{0}$, $\epsilon = 0$. The other $n-2$ eigenvalues are all assumed to be large and negative. This assures that a homogeneous solution converges fast toward W^c provided that $\mathbf{X}$ and ϵ are sufficiently small (for details and further references see [10]).

This allows us to express the concentration deviations $\mathbf{X}$ in terms of *amplitude coordinates* $\mathbf{Y} \in E^c$ by

$$\mathbf{X} = \mathbf{Y} + \mathbf{h}(\mathbf{Y},\epsilon).$$

This equation describes a mapping from coordinates in the center subspace E^c onto the center manifold W^c. The function $\mathbf{h}(\mathbf{Y},\epsilon)$ is selected in such a way to successively eliminate as many nonlinear terms as possible from the kinetic equations starting from the lowest order [10]. Each kind of bifurcation is characterized by the specific terms which cannot be eliminated (the so-called *resonant terms*). In this way we obtain a general equation valid for all reaction-diffusion equations undergoing a given bifurcation. In the case of the Hopf bifurcation, neglecting the diffusion term, to third order we obtain the so-called *Stuart-Landau equation*

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = (\mathrm{i}\omega_0 + \sigma_1\epsilon)Y - g|Y|^2 Y,$$

where Y is a complex amplitude given by $\mathbf{Y} = Y\mathbf{e}_1 + \overline{Y}\mathbf{e}_2$. The parameters σ_1 and g are complex and given by solutions of lengthy equations given in [10]. The Stuart-Landau equation represents the *normal form* of a homogeneous system close to a Hopf bifurcation. Performing a similar derivation, but including diffusion, we arrive at

$$\partial_t Y = (\mathrm{i}\omega_0 + \sigma_1\epsilon)Y - g|Y|^2 Y + d\Delta Y,$$

with $d = \mathbf{e}_1^+ \cdot \mathbf{D}\mathbf{e}_1$. After rescaling of space, time, and introducing A for Y, we finally arrive at the rescaled complex Ginzburg-Landau equation

$$\partial_t A = (1 - \mathrm{i}\omega)A - (1 + \mathrm{i}\alpha)|A|^2 A + (1 + \mathrm{i}\beta)\Delta A, \tag{2}$$

where A is the *complex oscillation amplitude*, ω the *linear frequency parameter*, α the *nonlinear frequency parameter*, and β the *linear dispersion coefficient*. All reaction-diffusion systems sufficiently close to a Hopf bifurcation are described by the complex Ginzburg-Landau equation. The specific details of the original system are incorporated in the parameter values. If one wishes to express the solution of the CGLE in the original variables, to first order the concentrations of the chemical species are expressed by

$$\mathbf{c} = \mathbf{c_s} + \sqrt{\epsilon}(Y(x,t)\mathbf{e}_1 + \overline{Y}(x,t)\mathbf{e}_2).$$

Different scalings of the CGLE are considered in the literature [3]. Here, we assume that the Hopf frequency is not scaled out, and hence contributes to ω in Eq. (2). We also send the reader to Appendix B of [12] for the detailed derivation of the CGLE associated to the Brusselator model.

1.2 On feedback control using delayed terms

Over the decades, the complex Ginzburg-Landau equation has been studied intensively because of its frequent appearance in different contexts of science, and its rich repertoire of different spatio-temporal wave patterns like plane waves, spiral waves, or localized hole solutions [3]. Remarkable, even if the Hopf bifurcation is supercritical, and hence the limit cycle a stable solution of the Stuart-Landau equation, the oscillations in the spatially-extended system may be unstable. The resulting states of spatiotemporal chaos appear if the Benjamin-Feir-Newell criterion $1 + \alpha\beta < 0$ is fulfilled, a phenomenon that is induced by the diffusive coupling and that is therefore genuine to a system with spatial degrees of freedom.

Considerable efforts have been made to understand this type of chaotic behavior and to apply methods to suppress this kind of turbulence and replace it by regular dynamics. In the context of the reaction-diffusion systems, the introduction of forcing terms or global feedback terms have been shown to be efficient ways to control turbulence [13, 11]. Still, control of chaotic states in nonlinear systems is a wide field of research that we cannot review here [15].

Global feedback methods, where a spatially independent quantity (or, e.g., a spatial average of a space-dependent quantity) is coupled back to the system dynamics, have attracted much attention since in many cases the models are simpler and easier to be carried out experimentally. Nevertheless, local methods have gained interest in recent years since they allow to access other solutions of the systems and may also be implemented, such as in the light-sensitive BZ reaction or in neurophysiological experiments [13].

Feedback methods with an explicit time delay amplify the range of possibilities of control that can be applied to the system and provide the researcher with an additional adjustable parameter. On the level of the mathematical description, the model equations become delay differential equations [9, 4]. Obviously, time delay feedback can be applied to any solution of the dynamics, not necessarily to a chaotic one.

1.3 Main results

In this paper we analyze several bifurcation effects produced by the delay time in the behavior of solutions of the complex Ginzburg-Landau equation with this type of feedback.

In Section 2 we prove a Hopf bifurcation result for the equation without diffusion (the Stuart-Landau equation) when the amplitude of the delayed term is suitably chosen. This simplified formulation has the advantage that closed analytical solutions are possible and the necessary eigenvalue computations can be carried out in full. The diffusion case is considered firstly in the case of the whole space (Section 3) and later on a bounded domain with periodicity conditions (Section 4).

In the case in which the space is the whole $\mathbb{R}$ (we consider here the one-dimensional case) we performed a linear stability analysis of uniform oscillations with respect to spatiotemporal perturbations following the treatment made in [16]: we express the complex oscillation amplitude A as the superposition of a homogeneous mode H (corresponding to uniform oscillations) with spatially inhomogeneous perturbations,

$$A(x,t) = H(t) + A_+(t)\mathrm{e}^{\mathrm{i}\kappa x} + A_-(t)\mathrm{e}^{-\mathrm{i}\kappa x}\,.$$

With the help of computational arguments we get several bifurcation diagrams where, besides the delay time it is possible to use the feedback magnitude term. Among many other detailed informations, we obtain numerical evidence of the fulfillment of the delicate transversality condition.

The paper ends by analyzing the case in which the bifurcation takes place starting from an uniform oscillation and originating a path over a torus. This time the study is carried out in two spatial dimensions over a rectangle in which we impose periodic boundary conditions. We show the applicability of an abstract result ([22]) to our formulation thanks to a suitable choice of the involved functional spaces. In this way, the spatial perturbations can be considered in their greatest generality.

The presentation of this chapter is very condensed due to limit space. A more detailed study will be published elsewhere.

2 Hopf bifurcation for the Stuart-Landau equation with a time delay feedback

For the purposes of clarity and ease of understanding, we start by considering in this section a very simplified version of the general model to be given later which has the advantage that closed analytical solutions are possible and the necessary eigenvalue computations can be carried out in full. Unfortunately, such precise calculations are not available for the general model and a fairly complete graphical-numerical study will be given in exchange.

Equation (2) reads

$$\partial_t A = (1 - \mathrm{i}\omega)A - (1 + \mathrm{i}\alpha)\,|A|^2\,A + (1 + \mathrm{i}\beta)\Delta A.$$

In the Stuart-Landau equation, the diffusion term is absent, which amounts to restricting our study to the *spatially homogeneous* solutions (which always satisfy periodic boundary conditions as it will be formulated in Section 4). On the other hand, we assume that a *delayed linear feedback term* is added, so the equation under study in this section will be

$$\partial_t A = (1 - \mathrm{i}\omega)A - (1 + \mathrm{i}\alpha)\,|A|^2\,A + m_1 A + m_3 A(t-\tau). \tag{3}$$

More general control terms will be considered in the remaining sections of the paper. The change of variables $\mathbf{w}(t) = \mathrm{e}^{-\mathrm{i}\phi t}A(t)$ gives

$$\partial_t \mathbf{w} = (1 - \mathrm{i}\omega - \mathrm{i}\phi)\mathbf{w} - (1 + \mathrm{i}\alpha)\,|\mathbf{w}|^2\,\mathbf{w} + m_1\mathbf{w} + m_3\mathrm{e}^{-\mathrm{i}\phi\tau}\mathbf{w}(t-\tau). \tag{4}$$

We now choose $\phi = -\alpha - \omega$ and $m_3 = -\mathrm{e}^{\mathrm{i}\phi\tau}m_1$ and denote the stationary solution of

$$\partial_t \mathbf{w} = (1 + \mathrm{i}\alpha)(\mathbf{w} - |\mathbf{w}|^2\,\mathbf{w}) + m_1\left[\mathbf{w} - \mathbf{w}(t-\tau)\right]. \tag{5}$$

by $\mathbf{w}_0$.

In order to check if at some critical value of the delay $\tau = \tau^*$ a Hopf bifurcation takes place, we linearize the equation around $\mathbf{w}_0 = 1$ and check whether a pair of complex eigenvalues $\lambda(\tau) = a(\tau) \pm \mathrm{i}b(\tau)$ of the linearization cross transversally the imaginary axis away from the origin, i.e., they satisfy $a(\tau^*) = 0$, $b(\tau^*) \neq 0$ and $a'(\tau^*) \neq 0$ (see, e.g., [22]).

Observe now that the complex term $|v|^2\,v$, although perfectly differentiable from the real point of view (in fact, the complex map $z \longmapsto |z|^2\,z = z^2\bar{z}$ is *real-analytic*), is *not an analytic* (or holomorphic) function from the complex viewpoint. Therefore it becomes convenient at this point to abandon the complex notation and write the system in real form ($\mathbf{w} = u + \mathrm{i}v$) as follows

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \left(1 - (u^2 + v^2)\right) \begin{pmatrix} u \\ v \end{pmatrix} + m_1 \begin{pmatrix} u - u(t-\tau) \\ v - v(t-\tau) \end{pmatrix}.$$

Let us fix our attention to the stationary solution $\mathbf{w}_0 = (u_0, v_0) = (1, 0)$. The linearization around $\mathbf{w}_0$ is given by

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + m_1 \begin{pmatrix} U - U(t-\tau) \\ V - V(t-\tau) \end{pmatrix} \tag{6}$$

and the eigenvalue-eigenvector pairs associated to this vector equation are the solutions of (6) of the special form $U(t) = \mathrm{e}^{\lambda t}U_0,\ V(t) = \mathrm{e}^{\lambda t}V_0$ where $\lambda \in \mathbb{C}$ and U_0, V_0 are (possibly complex) constant (nonzero) 2-vectors. One thus easily finds

$$\lambda \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} -2 + m_1 & 0 \\ -2\alpha & m_1 \end{pmatrix} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} - m_1\mathrm{e}^{-\lambda\tau} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix},$$

thus arriving to the *characteristic equation*

$$\begin{vmatrix} \lambda + 2 - m_1 + m_1\mathrm{e}^{-\lambda\tau} & 0 \\ 2\alpha & \lambda - m_1 + m_1\mathrm{e}^{-\lambda\tau} \end{vmatrix} = 0.$$

This means that we have a double collection of eigenvalues: those satisfying $\lambda - m_1 + m_1\mathrm{e}^{-\lambda\tau} = 0$ and those satisfying $\lambda + 2 - m_1 + m_1\mathrm{e}^{-\lambda\tau}$. Denoting $\lambda = a + \mathrm{i}b$, we identify two classes of eigenvalues:

$$\lambda - m_1 + m_1 \mathrm{e}^{-\lambda\tau} = 0 \Longleftrightarrow \begin{cases} a - m_1 + m_1 \mathrm{e}^{-a\tau}\cos b\tau = 0 \\ b - m_1 \mathrm{e}^{-a\tau}\sin b\tau \end{cases} \text{(Class 1)}$$

$$\lambda + 2 - m_1 + m_1 \mathrm{e}^{-\lambda\tau} = 0 \Longleftrightarrow \begin{cases} a + 2 - m_1 + m_1 \mathrm{e}^{-a\tau}\cos b\tau = 0 \\ b - m_1 \mathrm{e}^{-a\tau}\sin b\tau \end{cases} \text{(Class 2)}$$

We now look for values $\tau = \tau^*$ for which $a = 0$ and $b \neq 0$. We find no eigenvalues of this kind for Class 1, since $-1 + \cos b\tau = 0$ implies $\sin b\tau = 0$, and hence $b = 0$ from the second equation.

However, Class 2 does give us some useful values:

$$2 - m_1 + m_1 \cos b\tau = 0 \Longrightarrow \cos b\tau = \frac{m_1 - 2}{m_1},$$
$$b - m_1 \sin b\tau = 0 \Longrightarrow \sin b\tau = \frac{b}{m_1}.$$

Thus,

$$1 = \cos^2 b\tau + \sin^2 b\tau = \left(\frac{m_1 - 2}{m_1}\right)^2 + \frac{b^2}{m_1^2} \Longrightarrow b^2 = m_1^2 - (m_1 - 2)^2 = 4(m_1 - 1).$$

Hence, if $m_1 > 1$, we have

$$\cos b\tau = \frac{m_1 - 2}{m_1} \Longrightarrow b\tau = \arccos\left(\frac{m_1 - 2}{m_1}\right)$$

which is well defined for every $m_1 > 1$.

Summarizing, the set of values

$$b^* = 2\sqrt{m_1 - 1}, \ \tau^* = \frac{1}{b^*}\left[\arccos\left(\frac{m_1 - 2}{m_1}\right) + 2k\pi\right]$$

corresponds to a (possible) bifurcation point of Hopf type. For instance, for $m_1 = 2$ we have $b^* = 2$ and $\tau^* = k\pi + \pi/4$.

We now need to compute the derivative $a'(\tau^*)$. It is easier now to go back to the complex formulation of Class 2 eigenvalues

$$\lambda + 2 - m_1 + m_1 \mathrm{e}^{-\lambda\tau} = 0,$$

and find $\mathrm{d}\lambda/\mathrm{d}\tau$ by implicit differentiation:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + m_1 \mathrm{e}^{-\lambda\tau}\left(-\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\tau - \lambda\right) = 0 \Longrightarrow \frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \frac{\lambda \mathrm{e}^{-\lambda\tau}}{1 - m_1 \mathrm{e}^{-\lambda\tau}\tau} = \frac{\lambda}{1 - m_1 \mathrm{e}^{\lambda\tau}\tau}.$$

Concentrating on the specific values $b^* = 2$ and $\tau^* = \pi/4$ we find, at the bifurcation values $\tau^*, \ \lambda^* = \mathrm{i}b^*$, that

$$\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right|_{(\tau^*, \lambda^*)} = \frac{\mathrm{i}b^*}{1 - m_1 \mathrm{e}^{\mathrm{i}b^*\tau^*}\tau^*} = -\frac{4\pi}{\pi^2 + 4} + \frac{8}{\pi^2 + 4}\mathrm{i}.$$

Hence

$$\frac{\mathrm{d}a}{\mathrm{d}\tau}(\tau^*) = -\frac{4\pi}{\pi^2 + 4} \neq 0$$

and the transversality condition is satisfied. Therefore, a Hopf bifurcation occurs, and a periodic orbit of approximate period

$$T \simeq \frac{2\pi}{b(\tau^*)} = \pi$$

exists for delay values τ near τ^*.

Remark 1. To decide the sub- or supercritical character of the bifurcation a much longer analysis is necessary. On the other hand, for $\tau > 1/2$ there are always positive *real* eigenvalues coming from the first class, which means that the stationary point has become already unstable *before* the delay reaches $\tau^* = \pi/4$ value. Hence the periodic orbit cannot capture the stability lost by the stationary point, since that stability was already lost.

3 Hopf bifurcation for the complex Ginzburg-Landau equation on the whole space and with delayed time feedback

We come back to the consideration of the complex Ginzburg-Landau equation subjected to a time-delay feedback with local and global terms but now for the case of a spatial domain given by the whole space:

$$\begin{aligned} \partial_t A &= (1 - \mathrm{i}\omega)A - (1 + \mathrm{i}\alpha)|A|^2 A + (1 + \mathrm{i}\beta)\partial_{xx} A + F, \\ F &= \mu \mathrm{e}^{\mathrm{i}\xi} \left[m_1 A + m_2 \langle A \rangle + m_3 A(t-\tau) + m_4 \langle A(t-\tau) \rangle \right], \end{aligned} \tag{7}$$

where

$$\langle A \rangle = \frac{1}{L} \int_0^L A(x,t)\, \mathrm{d}x$$

denotes the spatial average of A over a one-dimensional medium of length L. There are many previous works in the literature dealing with such type of formulations: [6, 7, 17, 16].

Extensive simulations [17] and an analytical stability analysis [16] for a special case representing a Pyragas-type feedback [14] ($m_3 = -m_1 = m_l$, $m_4 = -m_2 = m_g$) showed the range of patterns that can be stabilized as function of the local and global feedback terms. If the feedback is global, uniform oscillations can be stabilized for a large range of feedback parameters, while as the contribution of the local feedback term becomes larger, the parameter regions increase where the homogeneous fixed point solution, standing waves and traveling waves are found.

Uniform oscillations $A(t) = \rho_0 \exp(-\mathrm{i}\theta t)$ are a solution of Eqs. (7) with amplitude and frequency given by

$$\begin{aligned} \rho_0 &= \sqrt{1 + \mu(m_g + m_l)(\cos(\xi + \theta\tau) - \cos\xi)}, \\ \theta &= \omega + \alpha + \mu(m_g + m_l)\left[\alpha(\cos(\xi + \theta\tau) - \cos\xi) - (\sin(\xi + \theta\tau) - \sin\xi)\right]. \end{aligned}$$

In [16], we performed a linear stability analysis of uniform oscillations with respect to spatiotemporal perturbations. There, we expressed the complex oscillation amplitude A as the superposition of a homogeneous mode H (corresponding to uniform oscillations) with spatially inhomogeneous perturbations,

$$A(x,t) = H(t) + A_+(t)e^{\mathrm{i}\kappa x} + A_-(t)e^{-\mathrm{i}\kappa x}. \tag{8}$$

Notice that here we are using the fact that the equation takes place on the whole space, which allows the justification of the spatially inhomogeneous perturbations of the form $A_+(t)e^{\mathrm{i}\kappa x} + A_-(t)e^{-\mathrm{i}\kappa x}$. Inserting Eq. (8) into Eq. (7), and assuming that the amplitudes $A_\pm$ are small, we obtain a set of equations for H, A_+, and

A_-^* (see [16] for details of this derivation). To investigate linear stability of uniform oscillations with respect to spatiotemporal perturbations, we make the ansatz

$$\begin{aligned} A_+ &= A_+^0 \exp(-\mathrm{i}\theta t)\exp(\lambda t), \\ A_-^* &= A_-^{*0} \exp(\mathrm{i}\theta t)\exp(\lambda t), \end{aligned} \tag{9}$$

where $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ is a complex eigenvalue. Using ansatz (9), we arrive at the following eigenvalue equation:

$$F = (A + \mathrm{i}B - \mathrm{i}\lambda_2 + D_1 + \mathrm{i}D_2)(A - \mathrm{i}B - \mathrm{i}\lambda_2 + C_1 + \mathrm{i}C_2), \tag{10}$$

where we have defined

$$\begin{aligned} F &= (1+\alpha^2)\rho_0^4, \\ A &= 1 - \lambda_1 - 2\rho_0^2 - \kappa^2, \\ B &= \theta - \omega - 2\alpha\rho_0^2 - \beta\kappa^2, \\ C_1 &= \mu m_l e^{-\lambda_1\tau}\cos(\xi + \theta\tau + \lambda_2\tau) - \mu m_l \cos\xi, \\ C_2 &= -\mu m_l e^{-\lambda_1\tau}\sin(\xi + \theta\tau + \lambda_2\tau) + \mu m_l \sin\xi, \\ D_1 &= \mu m_l e^{-\lambda_1\tau}\cos(\xi + \theta\tau - \lambda_2\tau) - \mu m_l \cos\xi, \\ D_2 &= \mu m_l e^{-\lambda_1\tau}\sin(\xi + \theta\tau - \lambda_2\tau) - \mu m_l \sin\xi. \end{aligned}$$

We point out that the above eigenvalue equation can be obtained also by a formal linearization argument involving the Fréchet derivatives as in the next section. There is no general analytic solution to Eq. (10) for $\lambda_{1,2}$. Thus, Eq. (10) must be solved numerically for a given set of parameters. We keep the CGLE parameters α, β, ω and the feedback parameters m_l, m_g, and ξ constant and solve Eq. (10) with the FindRoot routine of the Mathematica package [21]. We then find, for each point in the (τ, μ)-space, the functional dependence of λ_1 and λ_2 on κ. Notice that if we assume $\kappa = 0$ the study can be applied to the case of the Stuart-Landau equation, as in Section 2.

In general, Eq. (10) has multiple solutions, reflected by multiple branches in the dispersion relation. Stability is determined by the sign of λ_1. The curves $\lambda_1(\kappa)$ either lie below $\lambda_1 = 0$, so that uniform oscillations are stable, or they display an interval of κ-values, where $\lambda_1 > 0$, so that uniform oscillations are unstable. At criticality, we have $\lambda_1 = 0$, $\partial_\epsilon \lambda_1 \neq 0$, where ϵ stands for either μ or τ. For the critical wavenumber κ_c, there are two possibilities: $\kappa_c = 0$ or $\kappa_c \neq 0$ ($\pm\kappa_c$ are solutions, although below, we consider only $\kappa_c > 0$ without loss of generality).

Two instabilities are particularly important in our system: the first one is associated with $\kappa_c > 0$ and $\lambda_2(\kappa_c) = 0$, and the second one with $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$. In Figure 1, we show as an example the control diagram in (μ, τ)-space for $m_l = 0.4$, $m_g = 0.6$. Stable uniform oscillations are observed above the solid curve and to the right of the dotted curve. At the solid curve, uniform oscillations become unstable with respect to perturbations with $\kappa_c > 0$ and $\lambda_2(\kappa_c) = 0$, at the dotted curve, with $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$. In Figure 2(a,b), the dispersion relations $\lambda_{1,2} = \lambda_{1,2}(\kappa)$ are shown for three τ values close to criticality, demonstrating clearly the nature of the underlying instability. In Figure 2(c), we show that λ_1 crosses $\lambda_1 = 0$ as τ is varied, hence demonstrating transversality. As the uniform oscillations become unstable with respect to a mode with complex conjugated eigenvalues and since ρ_0 remains finite, we infer the presence of a secondary Hopf bifurcation.

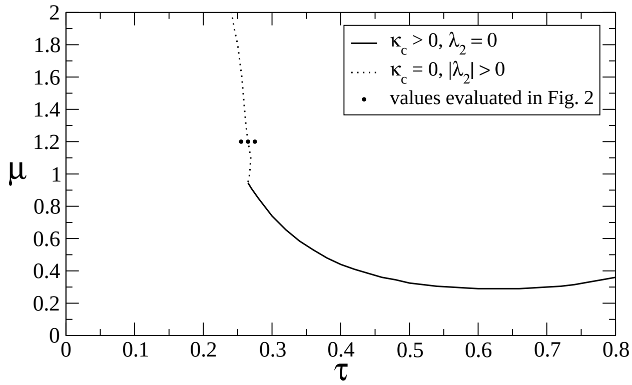

Fig. 1. Control diagram in (μ, τ)-space for $m_l = 0.4$, $m_g = 0.6$. The other parameters are $\alpha = -1.4$, $\beta = 2$, $\omega = 2\pi - \alpha$, $\xi = \pi/2$. At the solid curve, uniform oscillations become unstable with respect to perturbations with $\kappa_c > 0$ and $\lambda_2(\kappa_c) = 0$, at the dotted curve, with $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$. The dots indicate parameter values further studied in Figure 2.

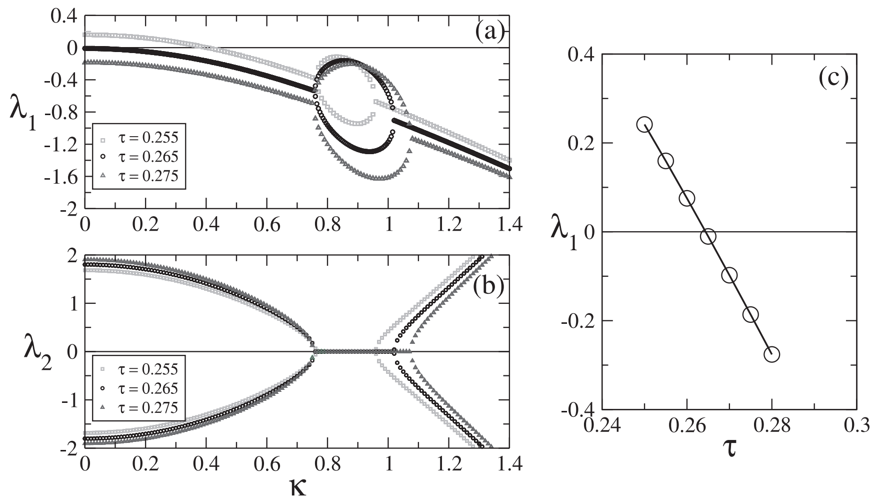

Fig. 2. Dispersion relations for three parameter sets close to criticality: $\tau = 0.255$ (light grey squares), $\tau = 0.265$ (black circles), $\tau = 0.275$ (dark grey triangles). (a) Real part of the eigenvalue as function of the wavenumber κ. (b) Imaginary part of the eigenvalue. The instability is characterized by $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$ and occurs for $\mu = 1.2$ at $\tau = 0.264399$. (c) Real part of the eigenvalue as function of τ, demonstrating transversality.

4 Hopf bifurcation for the delayed CGLE in a bounded domain

In this section we consider the case of two spatial dimensions varying on the domain $\Omega = (0, L_1) \times (0, L_2)$ (note a slight change of notation with respect to Sect. 3). Our goal is to show a bifurcation phenomenon near uniform oscillations for the CGLE in terms of the delay term as parameter. We define the faces of the boundary

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}\,, \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}\,,\ j = 1, 2,$$

on which we assume periodic boundary conditions and, hence, the problem under study can be formulated as

$$(P_1)\begin{cases} \partial_t \mathbf{u} - (1+\mathrm{i}\beta)\Delta\mathbf{u} = (1-\mathrm{i}\omega)\mathbf{u} - (1+\mathrm{i}\alpha)|\mathbf{u}|^2\mathbf{u} & \\ \qquad\qquad +\mu \mathrm{e}^{\mathrm{i}\xi}\mathbf{F}(\mathbf{u},t,\tau) & \Omega\times(0,\infty), \\ \left.\mathbf{u}\right|_{\Gamma_j} = \left.\mathbf{u}\right|_{\Gamma_{j+2}}, & \\ \left(-\left.\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|_{\Gamma_j} =\right) \left.\frac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_{j+2}} \left(= \left.\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|_{\Gamma_{j+2}}\right) & \partial\Omega\times(0,\infty), \\ \mathbf{u}(x,s) = \mathbf{u}_0(x,s) & \Omega\times[-\tau,0], \end{cases}$$

where $\mathbf{n}$ is the outpointing normal unit vector, and

$$\mathbf{F}(\mathbf{u},t,\tau) = [m_1\mathbf{u}(x,t) + m_2\langle\mathbf{u}(t)\rangle + m_3\mathbf{u}(x,t-\tau) + m_4\langle\mathbf{u}(t-\tau)\rangle]$$

with

$$\langle\mathbf{u}(s)\rangle = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}(x,s)\mathrm{d}x.$$

Again, the parameters $\alpha, \beta, \omega, \mu, \xi, m_i$ and τ are real, while $\mathbf{u}(x,t) = \mathbf{u}_1(x,t) + \mathrm{i}\mathbf{u}_2(x,t)$ is complex.

We study the stability of *uniform oscillations*, i.e., solutions of (P_1) of the form $\mathbf{v}_{\mathrm{uo}}(t) = \rho_0\mathrm{e}^{-\mathrm{i}\theta t}$ which determines completely ρ_0 and θ. We are interested in the Hopf bifurcation close to $\mathbf{v}_{\mathrm{uo}}(t)$ which gives rise to some paths on a suitable torus (for a different study dealing with invariant tori see [18]).

In order to avoid the application of very sophisticated techniques (dealing with periodic solutions), we can reduce the study to the Hopf bifurcation near a stationary solution of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t) = \mathbf{v}(x,t)\mathrm{e}^{\mathrm{i}\theta t}$ where $\mathbf{v}(x,t)$ is a solution of (P_1). Thus, $\mathbf{z}(x,t)$ satisfies

$$(P_2)\begin{cases} \partial_t \mathbf{z} - (1+\mathrm{i}\beta)\Delta\mathbf{z} = (1+\mathrm{i}\theta)\mathbf{z} - (1+\mathrm{i}\alpha)|\mathbf{z}|^2\mathbf{z} + \mu\mathrm{e}^{\mathrm{i}\xi}\times & \\ \times[m_1\mathbf{z} + m_2\langle\mathbf{z}\rangle + \mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3\mathbf{z}(t-\tau) + m_4\langle\mathbf{z}(t-\tau)\rangle)] & \Omega\times(0,\infty), \\ \left.\mathbf{z}\right|_{\Gamma_j} = \left.\mathbf{z}\right|_{\Gamma_{j+2}}, & \\ \left(-\left.\frac{\partial \mathbf{z}}{\partial \mathbf{n}}\right|_{\Gamma_j} =\right) \left.\frac{\partial \mathbf{z}}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial \mathbf{z}}{\partial x_j}\right|_{\Gamma_{j+2}} \left(= \left.\frac{\partial \mathbf{z}}{\partial \mathbf{n}}\right|_{\Gamma_{j+2}}\right) & \partial\Omega\times(0,\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x,s)\mathrm{e}^{\mathrm{i}(\omega-\theta)s} & \Omega\times[-\tau,0]. \end{cases}$$

Now, $\mathbf{v}_{\mathrm{uo}}(t) = \rho_0\mathrm{e}^{-\mathrm{i}\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_{\mathrm{uo}}(t)\mathrm{e}^{\mathrm{i}\theta t} = \mathbf{z}_\infty = \rho_0$ is an stationary solution of (P_2), i.e.,

$$\mathbf{0} = (1+\mathrm{i}\theta)\mathbf{z}_\infty - (1+\mathrm{i}\alpha)\left|\mathbf{z}_\infty\right|^2\mathbf{z}_\infty + \mu\mathrm{e}^{\mathrm{i}\xi}\left[m_1 + m_2 + \mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3+m_4)\right]\mathbf{z}_\infty.$$

4.1 The abstract Hopf bifurcation theorem for semilinear functional equations

We shall apply to our setting an abstract result due to J. Wu (see [22], Theorem 2.1) stated for problems of the type

$$\begin{cases} \frac{du}{dt}(t) + Au(t) = L(\mu, u_t(.)) + g(u_t(.)) & in\ X, \\ u(s) = u_0(s) & s \in [-\tau, 0]. \end{cases}$$

on a Banach space X, where $u_t : [-\tau, 0] \to X$, under the following list of conditions:

(H_1) A generates an analytic compact semigroup $\{T(t)\}_{t\geq 0}$;

(H_2) The point spectrum of A consists of a sequence of real number $\{\mu_k\}_{k\geq 1}$ with the corresponding eigenspace M_k and the projection $P_k : X \to M_k$. Moreover, if $\sum_{k=1}^{\infty} x_k = 0$ for $x_k \in M_k$ then each x_k must be zero;

(H_3) Every $x \in D(A)$ has a unique expression $x = \sum_{k=1}^{\infty} P_k x$ and $Ax = \sum_{k=1}^{\infty} \mu_k P_k x$;

(H_4) The mapping $L : \mathbb{R} \times C \to X$ (with $C := C([-\tau, 0] : X))$ is C^k-smooth $(k \geq 4)$ and is given by

$$L(\mu, \phi) = \int_{-\tau}^{0} \phi(\theta) \mathrm{d}\eta(\mu, \theta)$$

for any $(\mu, \phi) \in \mathbb{R} \times C$, for a function $\eta(\mu, .)$: $[-\tau, 0] \to B(X, X)$ of bounded variation. Moreover, $L(\mu, P_k\phi) \in M_k$, $k \geq 1$, $\phi \in C$ and $L(\mu, \sum_{k=1}^{\infty} P_k\phi) = \sum_{k=1}^{\infty} L(\mu, P_k\phi)$ for any $\phi \in C$ such that $\sum_{k=1}^{\infty} P_k\phi \in C$, where $P_k\phi$ is defined by $(P_k\phi)(\theta) = P_k\phi(\theta)$ for $\theta \in [-\tau, 0]$;

(H_5) $g : \mathbb{R} \times C \to X$ has k-th-continuous Fréchet derivatives with $g(\mu, 0) = 0$ and $Dg(\mu, 0) = 0$ for $\mu \in \mathbb{R}$;

(H_6) There exists $\mu_0 \in \mathbb{R}$ and $\omega_0 > 0$ such that $\pm \mathrm{i}\omega_0$ are simple characteristic values of the linear equation

$$\dot{u}(t) + Au(t) = L(\mu_0, u_t(.)) \tag{12}$$

and all other characteristic values have negative real parts;

(H_7) *Transversality condition.* If μ is near μ_0 the eigenvalues of the corresponding problem (12) are given by $\lambda(\mu) = \alpha(\mu) + \mathrm{i}\omega(\mu)$, $\lambda(\mu_0) = \mathrm{i}\omega_0$, $\lambda(\mu)$ is C^k-smooth in μ and

$$\alpha'(\mu_0) \neq 0.$$

Remark 2. A careful reading of the proof of Theorem 2.1 of [22] allows to see that the use of the same notation u_t in the terms $L(\mu, u_t(.))$ and $g(u_t(.))$ does not needs that the kernels envolved in each of the possible nonlocal terms be exactly the same. So, in particular, the conclusion remains valid in the special case in which $g(u_t(.)) = g(u(.))$, i.e., without delay or neutral term.

4.2 Applications of the abstract result to the delayed CGLE on a bounded domain

Motivated by the special form of the nonlinear term of the equation in (P_2) we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$. A detailed analysis of the associated diffusion operator is consequence of some previous results in the literature: see, e.g., Amann [1]. Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} \Delta & -\beta\Delta \\ \beta\Delta & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, if $\beta \neq 0$ the diffusion matrix has a nonzero antisymmetric part. In particular, A is the generator of a semigroup of contractions $\{T(t)\}_{t\geq 0}$ on X and the compactness of the semigroup is consequence of the compactness of the inclusion $D(A) \subset X$ (notice that, since $N = 2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems. A study of the eigenvalues of A can be found, e.g., in Temam [19].

Concerning the rest of the terms of the equation in (P_2), we define $g(\mathbf{u}) = -(1 + \mathrm{i}\alpha)\,|\mathbf{u}|^2\,\mathbf{u}$ with $D(g) = \mathbf{L}^{12}(\Omega)$. By using the characterization of the semi inner-braket $[,]$ for the spaces $L^p(\Omega)$ (see, e.g., Benilan, Crandall and Pazy [5]) it is easy to see that $\mathbf{B} = -\mathbf{g}$ is an accretive operator on X, which is dominated by A; i.e.,

$$D_X(A) \subset D_X(B) \text{ and } |Bu| \leq k\left|A^0u\right| + \sigma(|u|)$$

for any $u \in D_X(A)$, some $k < 1$ and some continuous function $\sigma : \mathbb{R} \to \mathbb{R}$.

Here and in what follows, $|.|$ denotes the norm in the space X (in contrast to the norm in space C which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $\left|A^0u\right| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$. In particular, the operator $A + B$ is also an accretive operator on X.

In order to calculate the Fréchet differential of Nemitsky operator $g(\mathbf{u})$, it is useful to start analyzing the Gateaux derivative of the complex function $\mathbf{h}(\mathbf{z}) := \|\mathbf{z}\|^2\,\mathbf{z}$ in the direction of an arbitrary vector $\mathbf{v}$ of $\mathbb{C}$

$$\lim_{\substack{\beta\in\mathbb{R}\\ |\beta|\to 0}} \frac{\mathbf{h}(\mathbf{z}_0 + \beta\mathbf{v}) - \mathbf{h}(\mathbf{z}_0)}{|\beta|} = \mathbf{z}_0^2\overline{\mathbf{v}} + 2\,\|\mathbf{z}_0\|^2\,\mathbf{v}.$$

Then, we identify the Fréchet differential of operator $g(\mathbf{u})$ as

$$D\mathbf{B}(\mathbf{y})\mathbf{v} = (1 + \mathrm{i}\alpha)[\mathbf{y}^2\overline{\mathbf{v}} + 2\,\|\mathbf{y}\|^2\,\mathbf{v}]. \tag{13}$$

Since we have $\|D\mathbf{B}(\mathbf{y})\| \leq c\,\|\mathbf{y}\|^2$, by the results on the Fréchet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi [2]) we get that, if we take $Y = \mathbf{L}^{4/3}(\Omega)$, then exists $\delta^B > 0$ such that $\mathbf{B}$ is Fréchet differentiable as function from $B_{\delta^B}(w) = \left\{z \in D(\mathbf{B});\ |w - z| < \delta^B\right\}$ into Y, and that the Fréchet derivative is locally Lipschitz continuous.

The nonlocal term is defined by

$$\begin{aligned} F(\mathbf{u}_t) &= (1 + \mathrm{i}\theta)\mathbf{u}(t) \\ &\quad + \mu\mathrm{e}^{\mathrm{i}\xi}\left[m_1\mathbf{u}(t) + m_2\langle\mathbf{u}(t)\rangle + \mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3\mathbf{u}(t-\tau) + m_4\langle\mathbf{u}(t-\tau)\rangle)\right], \end{aligned}$$

is locally Lipschitz continuous and its Fréchet derivative is given by

$$\begin{aligned} \mathrm{D}F(\widehat{\mathbf{y}})\ \mathbf{v}(t) &= -(1 + \mathrm{i}\theta)\mathbf{v}(t) \\ &\quad - \ \mu\mathrm{e}^{\mathrm{i}\xi}\left[m_1\mathbf{v}(t) + m_2\langle\mathbf{v}(t)\rangle - \mathrm{e}^{\mathrm{i}(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau) - m_4\langle\mathbf{v}(t-\tau)\rangle)\right]. \end{aligned}$$

In consequence, the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega^*\cdot}y\right)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\mathrm{R}e\omega^* = \gamma^* < 0$. This means that the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega^*\cdot}y\right) + \omega^*y$ is accretive in $Y = \mathbf{L}^{4/3}(\Omega)$. We recall (see Ambrosetti and Prodi [2]) that this differentiability of B does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

We also recall that in [6] the existence (and uniqueness) of a mild solution of problem (P_2) was obtained through a pseudolinearization argument near a stationary solution $\widehat{w}$:

Theorem 1 ([6]). *Assume* $(H_1)-(H_7)$. *Then there exists* $\alpha > 0$, $\beta > 0$ *and* $M \geq 1$ *such that if* $u_0 \in B_\beta^X(\widehat{w})$, $u_0(s) \in D_X(B)$ *for any* $s \in [-\tau, 0]$ *then the solution* $u(\cdot : u_0)$ *of (12) exists on* $[-\tau, +\infty)$ *and*

$$|u(t:u_0) - w| \leq M\mathrm{e}^{-\alpha t} \|u_0 - \widehat{w}\|, \text{ for any } t > 0.$$

Moreover, there exists $\alpha^* > 0$, $\beta^* \in (0, \beta]$ *and* $M^* \geq 1$ *such that if* $u_0 \in B_{\beta^*}^{X\cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ *for any* $s \in [-\tau, 0]$ *then, for any* $t > 0$,

$$|u(t:u_0) - w|_X + |u(t:u_0) - w|_Y \leq M^* \mathrm{e}^{-\alpha^* t} (\|u_0 - \widehat{w}\|_X + \|u_0 - \widehat{w}\|_Y).$$

We can get better a priori estimates on the sup norm of the solution $\mathbf{u}$ if we assume more regular initial data in such a way that $u_0 \in B_{\beta^*}^{X\cap Y}(\widehat{w})$, $u_0(s) \in D(A) \cap D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$. Indeed, the solution can be found (after technical arguments) as a fixed point for the application $f \to Q_1(Q_2(f))$, with $w = Q_2 f$ (for $f \in W^{1,1}(0, T : X)$, for any arbitrary $T > 0$) being the solution of the problem

$$\begin{cases} \frac{dw}{dt}(t) + Aw(t) + B(w(t)) = f(t) \text{ in } X, \\ w(0) = w_0, \end{cases}$$

and Q_1 a suitable operator (see [20], Theorem 5.3.1). Since X is a reflexive Banach space, we know (see, e.g., [5], Lemma 7.8) that $w_0 \in D(A) \cap D_X(B)$ implies that $w(t) \in D(A) \cap D_X(B)$ for a.e. $t \in (0, T)$ and that

$$\|Aw(t)\|_X \leq C(\|Aw_0\|_X + \|B(w_0)\|_X, \|f\|_{W^{1,1}(0,T:X)}).$$

Thus, by the Sobolev imbedding theorems we know that

$$\|w(t)\|_{\mathbf{C}(\overline{\Omega})} \leq M$$

for a.e. $t \in (0, T)$ with $M = M(\|Aw_0\|_X + \|B(w_0)\|_X, \|f\|_{W^{1,1}(0,T:X)})$. In particular, this property remains true for the fixed point of $Q_1(Q_2(f))$ (see [20], Theorem 5.3.1) and thus

$$\|\mathbf{u}(t)\|_{\mathbf{C}(\overline{\Omega})} \leq M^*$$

for a suitable $M^* = M*(\|Au_0\|_{C([-\tau,0];X)} + \|B(w_0)\|_{C([-\tau,0];X)}, F)$. In consequence, without any loss of generality we can replace function $\mathbf{g}$ by the truncated one $\mathbf{g}_{M^*}(\mathbf{u})$:

$$\mathbf{g}_{M^*}(\mathbf{u}) = \begin{cases} -(1+\mathrm{i}\alpha)\,|\mathbf{u}|^2\,\mathbf{u} \quad \text{if } |\mathbf{u}| \leq M^*, \\ \\ -2(1+\mathrm{i}\alpha)\,(2M^*)^2\,\mathbf{u} \text{ if } |\mathbf{u}| \geq M^*, \end{cases}$$

and with $\mathbf{g}_{M^*}(\mathbf{u})$ a C^k-smooth function generating an accretive operator $\mathbf{B}_{M^*} = -\mathbf{g}_{M^*}$ on X dominated by A as before. This proves that, at least for regular initial data, $\mathbf{u}$ coincides with the solution of

$$\begin{cases} \frac{du}{dt}(t) + Au(t) = L(\mu, u_t(.)) + g_{M^*}(u_t(.)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0]. \end{cases}$$

Thanks to this argument we can verify now the assumption (H_5) since by the results of Ambrosetti and Prodi (see [2], Sect. 3, Chap. 1) we know that the Nemitsky operator associated to g_{M^*} has k-th-continuous Fréchet derivatives on any $\mathbf{L}^p(\Omega)$, $p > 1$.

Remark 3. By introducing the representation operator $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{\mathrm{i}\phi}$ it is clear that the quasilinear operator $A\mathbf{P}(\mathbf{q})$ obtained from the operator $A\mathbf{u}$=-(1+iβ)$\Delta\mathbf{u}$ satisfies also condition $A \in \mathcal{A}(\omega)$ (since $\mathbf{P}$ is merely a change of variables). We point out that

$$A\mathbf{P}(\mathbf{q}) = -(1+\mathrm{i}\beta)[\Delta\rho - \rho\,|\nabla\phi|^2 + \mathrm{i}(2\nabla\rho\cdot\nabla\phi + \rho\Delta\phi)]\mathrm{e}^{\mathrm{i}\phi}.$$

Then, the formal linearization of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ at $\mathbf{q}^*(x,y) := \mathbf{y} \equiv \rho_0$ becomes

$$D\mathbf{E}(\mathbf{q}^*)(\rho\mathrm{e}^{\mathrm{i}\phi}) = -(1+\mathrm{i}\beta)[\Delta\rho + \mathrm{i}\rho_0\Delta\phi]\mathrm{e}^{\mathrm{i}\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ needs a slight modification of the above linear expression. Nevertheless by applying the representation operator $\mathbf{P}$, after the linearization used in the abstract theorem, we get a curious result relating two nonlinear problems which are closed (in some sense) in the same spirit as the *pseudo-linearization principle* obtained in [6].

4.3 Some comments on the associated transversality assumption

Concerning problem (P_2), we give an outline of the study of eigenvalues and its implications on the associated transversality condition. The eigenvalue equation can be obtained by a linearization argument involving the Fréchet derivative of the nonlinear part, as in the preceding section.

As usual, the linear structure of the equation leads to the search of nontrivial solutions $\mathbf{z}(x)$ of the form $\mathbf{A_k}w^j_{\mathbf{k}}(x)$, with $j = 1,2$, where $w^j_{\mathbf{k}}(x)$ are the eigenfunctions for the usual Laplacian operator Δ with periodic boundary conditions on $\Omega = (0,L_1)\times(0,L_2)$. The eigenvalues of this problem are given by

$$\lambda^0_0 = 0,\ \lambda^0_{\mathbf{k}} = 4\pi\left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2}\right);\quad k_1,k_2 \in \mathbb{N}$$

with the associate eigenfunctions

$$w_0 = \frac{1}{\sqrt{|\Omega|}}, w^1_{\mathbf{k}} = \sqrt{\frac{2}{|\Omega|}}\cos 2\pi\mathbf{k}\mathbf{x}, w^2_{\mathbf{k}} = \sqrt{\frac{2}{|\Omega|}}\sin 2\pi\mathbf{k}\mathbf{x},\ \text{with}\ \ |\Omega| = L_1L_2,$$

where we have written $\mathbf{kx} := \left(\frac{k_1}{L_1}x_1 + \frac{k_2}{L_2}x_2\right)$. This study can be found in Temam [19]. We introduce the notation $\lambda_{\mathbf{k}} = a_{\mathbf{k}} + \mathrm{i}b_{\mathbf{k}}$ for the real and imaginary parts of the eigenvalues of the problem, and taking into account Fréchet derivative of the nonlinear part (13), the eigenvalue equations for the problem (P_2) are

$$\begin{cases} (a_{\mathbf{k}} + \mathrm{i}b_{\mathbf{k}})[v_r + \mathrm{i}v_i] - (1+\mathrm{i}\beta)(-\lambda_{\mathbf{k}})[v_r + \mathrm{i}v_i] = \\ (1+\mathrm{i}\theta)[v_r + \mathrm{i}v_i] - (1+\mathrm{i}\alpha)[3\rho_0^2 v_r + \mathrm{i}\rho_0^2 v_i] + \\ \mu\mathrm{e}^{\mathrm{i}\xi}\left[m_1 + m_2\delta_{0\mathbf{k}} + \mathrm{e}^{-a\tau+\mathrm{i}(\omega+\theta-b)\tau}(m_3 + m_4\delta_{0\mathbf{k}})\right][v_r + \mathrm{i}v_i], \end{cases}$$

where v_r and v_i are the real and imaginary parts of the linearization $\mathbf{v}$, and $\delta_{0\mathbf{k}}$ denotes the Kronecker delta function. We arrive at

$$\begin{cases} a_{\mathbf{k}}v_r - b_{\mathbf{k}}v_i = & -\lambda^0_{\mathbf{k}}v_r + \beta\lambda^0_{\mathbf{k}}v_i + \left(\left[1-3\rho_0^2\right]v_r + \left[\alpha\rho_0^2-\theta\right]v_i\right) + \\ & \mu(m_1+m_2\delta_{0k})\left[v_r\cos\xi - v_i\sin\xi\right] + \left\{\mu e^{-a_{\mathbf{k}}\tau}(m_3+m_4\delta_{0k})\right. \\ & \left.\left[\cos(\xi+(\omega+\theta-b_{\mathbf{k}})\tau)v_r - \sin(\xi+(\omega+\theta-b_{\mathbf{k}})\tau)v_i\right]\right\}, \\ b_{\mathbf{k}}v_r + a_{\mathbf{k}}v_i = & -\beta\lambda^0_{\mathbf{k}}v_r + \lambda^0_{\mathbf{k}}v_i + (v_i+\theta v_r) - \left[\rho_0^2v_i - 3\alpha\rho_0^2v_r\right] + \\ & \mu(m_1+m_2\delta_{0k})\left[v_r\sin\xi + v_i\cos\xi\right] + \left\{\mu e^{-a_{\mathbf{k}}\tau}(m_3+m_4\delta_{0k})\right. \\ & \left.\left[\sin(\xi+(\omega+\theta-b_{\mathbf{k}})\tau)v_r + \cos(\xi+(\omega+\theta-b_{\mathbf{k}})\tau)v_i\right]\right\} \end{cases}$$

To show the procedure, without loss of generality, we consider the case

$$m_3 + m_4\delta_{0\mathbf{k}} = 0. \tag{14}$$

This represents a special, and important, choice of the combination of instantaneous and delayed terms in the global feedback, none of them necessarily zero. The equations for the eigenvalues become

$$\begin{cases} a_{\mathbf{k}}v_r - b_{\mathbf{k}}v_i = -\lambda^0_{\mathbf{k}}v_r + \beta\lambda^0_{\mathbf{k}}v_i + \left(\left[1-3\rho_0^2\right]v_r + \left[\alpha\rho_0^2-\theta\right]v_i\right) + \\ \qquad \mu(m_1+m_2\delta_{0k})\cos\xi v_r - \mu(m_1+m_2\delta_{0k})\sin\xi v_i \\ b_{\mathbf{k}}v_r + a_{\mathbf{k}}v_i = -\beta\lambda^0_{\mathbf{k}}v_r + \lambda^0_{\mathbf{k}}v_i + (v_i+\theta v_r) - \left[\rho_0^2v_i - 3\alpha\rho_0^2v_r\right] + \\ \qquad \mu(m_1+m_2\delta_{0k})\sin\xi v_r + \mu(m_1+m_2\delta_{0k})\cos\xi v_i \end{cases}$$

If we call

$$\begin{aligned} C_1\left(\mu,m_1,m_2,\xi,\lambda^0_{\mathbf{k}}\right) &= 1-\lambda^0_{\mathbf{k}} - \mu(m_1+m_2\delta_{0k})\cos\xi, \\ C_2\left(\mu,m_1,m_2,\xi,\lambda^0_{\mathbf{k}}\right) &= 1+\lambda^0_{\mathbf{k}} + \mu(m_1+m_2\delta_{0k})\cos\xi, \\ D\left(\beta,\mu,m_1,m_2,\xi,\lambda^0_{\mathbf{k}}\right) &= -\beta\lambda^0_{\mathbf{k}} + \mu(m_1+m_2\delta_{0k})\sin\xi, \end{aligned}$$

we obtain

$$\begin{cases} \left(a_{\mathbf{k}} - \left[C_1-3\rho_0^2\right]\right)v_r - \left(b_{\mathbf{k}} + \left[\alpha\rho_0^2-\theta-D\right]\right)v_i = 0 \\ \left(b_{\mathbf{k}} - \left[-3\alpha\rho_0^2+\theta+D\right]\right)v_r + \left(a_{\mathbf{k}} - \left[C_2-\rho_0^2\right]\right)v_i = 0 \end{cases}$$

The compatibility of this system implies

$$\det\begin{pmatrix} a_{\mathbf{k}} - \left[C_1-3\rho_0^2\right] & -b_{\mathbf{k}} - \left[\alpha\rho_0^2-\theta-D\right] \\ b_{\mathbf{k}} - \left[-3\alpha\rho_0^2+\theta+D\right] & a_{\mathbf{k}} - \left[C_2-\rho_0^2\right] \end{pmatrix} = 0,$$

that is

$$\begin{cases} \left(a_{\mathbf{k}} - \left[C_1-3\rho_0^2\right]\right)\left(a_{\mathbf{k}} - \left[C_2-\rho_0^2\right]\right) = \\ \left(b_{\mathbf{k}} - \left[-3\alpha\rho_0^2+\theta+D\right]\right)\left(b_{\mathbf{k}} + \left[\alpha\rho_0^2-\theta-D\right]\right). \end{cases} \tag{15}$$

This expression is of the same type as (10) and, similarly, there is no general analytic solution for $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$. Thus, Eq. (15) must also be solved numerically for a given set of parameters, to find the numerical values of the eigenvalues as in the equation (10). One of the relevant parameter spaces of the representation is the one of (τ,μ) because they are the parameters of the perturbation.

Although the explicit analytical representation of the functions $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ is not possible, we can still say something analytic in the study of the transversality, already proved by the numerical computation of Sect. 3. From the equation (15), it is possible to find the implicit derivative

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}a_{\mathbf{k}}\right]_{a_{\mathbf{k}}=0}.$$

The analytic computation are rather involved. We show how to proceed in a simpler, and still very important example

$$m_1 + m_2\delta_{0\mathbf{k}} = 0, \tag{16}$$

where a remark similar as the one made for the expression (14) remains valid, in this case for the local part of the perturbation. For the case (16), we have

$$C_1\left(\mu, m_1, m_2, \xi, \lambda_{\mathbf{k}}^0\right) = 1 - \lambda_{\mathbf{k}}^0,$$

$$C_2\left(\mu, m_1, m_2, \xi, \lambda_{\mathbf{k}}^0\right) = 1 + \lambda_{\mathbf{k}}^0,$$

$$D\left(\beta, \mu, m_1, m_2, \xi, \lambda_{\mathbf{k}}^0\right) = -\beta\lambda_{\mathbf{k}}^0.$$

If we expand Eq. (15) for this case,

$$\begin{cases} a_{\mathbf{k}}^2 - 2\left[1 - 2\rho_0^2\right]a_{\mathbf{k}} + \left(\left[1 - \lambda_{\mathbf{k}}^0 - 3\rho_0^2\right]\left[1 + \lambda_{\mathbf{k}}^0 - \rho_0^2\right]\right) = \\ -b_{\mathbf{k}}^2 + 2\left[-\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 + \theta\right]b_{\mathbf{k}} + \left(\left[-\beta\lambda_{\mathbf{k}}^0 + 3\alpha\rho_0^2 + \theta\right]\left[+\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 - \theta\right]\right), \end{cases}$$

and differentiate implicitly

$$\begin{cases} 2a_{\mathbf{k}}\frac{\mathrm{d}}{\mathrm{d}\tau}a_{\mathbf{k}} - 2\left[1 - 2\rho_0^2\right]\frac{\mathrm{d}}{\mathrm{d}\tau}a_{\mathbf{k}} - a_{\mathbf{k}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2\left[1 - 2\rho_0^2\right]\right) + \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\left(1 - \left(\lambda_{\mathbf{k}}^0\right)^2 - 2\left[2 + \lambda_{\mathbf{k}}^0\right]\rho_0^2 + 3\rho_0^4\right) = \\ -2b_{\mathbf{k}}\frac{\mathrm{d}}{\mathrm{d}\tau}b_{\mathbf{k}} + 2\left[-\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 + \theta\right]\frac{\mathrm{d}}{\mathrm{d}\tau}b_{\mathbf{k}} - b_{\mathbf{k}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2\left[-\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 + \theta\right]\right) + \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left[-\beta\lambda_{\mathbf{k}}^0 + 3\alpha\rho_0^2 + \theta\right]\left[+\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 - \theta\right]\right). \end{cases}$$

The derivative of the real part $a_{\mathbf{k}}$ in the value $a_{\mathbf{k}} = 0$ can be written as

$$\begin{cases} \left[-2(1 - 2\rho_0^2)\frac{\mathrm{d}}{\mathrm{d}\tau}a_{\mathbf{k}}\right]_{a_{\mathbf{k}}=0} = \\ \left[-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(1 - \left(\lambda_{\mathbf{k}}^0\right)^2 - 2\left[2 + \lambda_{\mathbf{k}}^0\right]\rho_0^2 + 3\rho_0^4\right)\right]_{a_{\mathbf{k}}=0} \\ +2\left[-b_{\mathbf{k}}\frac{\mathrm{d}}{\mathrm{d}\tau}b_{\mathbf{k}} + \left[-\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 + \theta\right]\frac{\mathrm{d}}{\mathrm{d}\tau}b_{\mathbf{k}} - b_{\mathbf{k}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left[-\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 + \theta\right]\right)\right]_{a_{\mathbf{k}}=0} \\ + \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left[-\beta\lambda_{\mathbf{k}}^0 + 3\alpha\rho_0^2 + \theta\right]\left[+\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2 - \theta\right]\right)\right]_{a_{\mathbf{k}}=0}. \end{cases}$$

The coefficient of the derivative of $a_{\mathbf{k}}$,

$$-2(1 - 2\rho_0^2) = -2\left[1 - 2(1 + \mu\cos\xi)\right] = 2(1 + 2\mu\cos\xi)$$

does not vanish either for stability reasons as can be seen, e.g., in [6] and references therein.

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