5. Dead cores

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The paper [i] below deals with several qualitative properties of the solutions of the quasilinear elliptic partial differential equations

$$\operatorname{div}\{A(|Du|)Du\} = f(u) \quad \text{in } \Omega, \tag{1}$$

where A and f satisfy some structure conditions which do not imply their differentiability. Most of the paper is devoted to the "monotone case" in which $f \in C(\mathbb{R})$, $f(0) = 0$, f is nondecreasing on $\mathbb{R}$, and $f(u) > 0$ for $u > 0$. This "monotone case" is also considered (for a more special diffusion term) in the presence of some weight coefficients, including for example the equations of Matukuma or Batt-Faltenbacher-Horst type. For the sake of the presentation, only nonnegative solutions are considered. A final short section considers also the "nonmonotone case", in which f is assumed to be nondecreasing only near 0.

Here Ω is an open and connected set in $\mathbb{R}^n$, $n \geq 1$, not necessarily bounded. We follow the usual terminology of the chemical engineering literature by saying that a solution u which is strictly positive on $\partial\Omega$ has a dead core if u vanishes on a set of positive measure. The main difficulty in studying this problem lies in obtaining the existence and behavior of the free boundary defined as the boundary of the dead core. This behavior had been studied previously in connection with other problems for the case of unbounded domains. It is difficult to be absolutely precise at this point, but it seems that this type of free boundary was first considered, in the class of second order elliptic equations, by H. Brezis and G. Stampacchia in a series of papers, starting in 1973 (see references in [4]) on the "classical" problem of the flow past a given profile with prescribed velocity at infinity. In this special case, the location of the free boundary associated to an auxiliary semilinear problem leads to some sharp estimates on the maximum velocity. The study of solutions with compact support on unbounded domains was extended later to more general equations. The compactness of the support of the solution of some variational inequalities (of obstacle type) formulated in terms of the multivalued semilinear equation $Lu + \beta(u) \ni \psi(x)$ in Ω, where L is a linear second order elliptic operator and β is a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \beta(0)$ was proved in [3]. Other nonlinear variational problems having a solution with compact support were studied earlier by other authors (J. Auchmuty and R. Beals, L. Berkowitz and H. Pollard, and R. Redheffer). The auxiliary problems in those case involved higher order semilinear equations.

It seems important to point out that this type of free boundary was studied almost twenty years before in an entirely different framework: the solutions of some degenerate quasilinear parabolic equations such as the porous medium equation. The compactness of the support of the solution $u(t,\cdot)$, for any fixed $t>0$ when $u(0,\cdot)$ has compact support was proved for the first time in the literature in 1958 by O. A. Oleinik, A. S. Kalashnikov and Chzou Yui Lin. Some special solutions were considered even earlier by G. I. Barenblatt, Ya. B. Zeldovich and later by many other distinguished specialists. This relevant class of parabolic problems, attracted the attention of other authors too numerous to mention here.

In 1975, Ph. Benilan, H. Brezis and M.G. Crandall [2] studied the support of the solution of the semilinear equation $-\Delta u+\beta(u)\ni\psi$ in $\mathbb{R}^N$ considering the case in which the monotone term $\beta(u)$ is not necessarily multivalued at the origin as assumed in the above-mentioned papers. They proved that the necessary and sufficient condition on β in order to get a solution with compact support is given by the integrability at the origin of a suitable improper integral. Starting in 1978, this criterion was extended to the case of quasilinear problems of the type $-\text{div}\ (|\nabla u|^{p-2}\nabla u)+\beta(u)\ni\psi,\ p>1$ in several papers by J. I. Díaz and M. A. Herrero (see, *e.g.*, [7]), to a new criterion which now applies to Lipschitz functions $\beta(u)$ if $p>2$.

The above results were extended later in many directions in the literature. For instance, the study of the semilinear elliptic equation but now on a bounded domain Ω (and with $\psi=0$ on Ω and $u=1$ on the boundary) was studied by C. Bandle, R. Sperb and I. Stakgold. They showed that the same criterion of [2] is, in some sense, the necessary and sufficient condition on β for the formation of an internal free boundary (the boundary of the *dead core*). The general idea of the construction of super and subsolutions vanishing at suitable points of the domain (and without requiring the condition Ω unbounded) was first carried out in 1984 by J. I. Díaz and J. Hernández for the case of the dead core problem. It was later extended systematically to the class of general quasilinear equations of the type (1) in the 1985 monograph [5]. Some further results were published by J. I. Díaz, J. E. Saa and U. Thiel in 1990. These types of results attained their maximum elegance and deepness with the 2006 paper by Pucci and Serrin [i].

They improve all the previous papers in the literature by using only arguments which can be considered today as "elementary" in the theory of elliptic partial differential equations: in the style of the famous 1927 paper by E. Hopf. Their presentation is highly pedagogical. They study the optimality of their main assumptions. For instance, the above-mentioned criteria of [2] and [7] now become $\displaystyle\int_{0+}\frac{ds}{H^{-1}(F(s))}<+\infty$, where $H(\rho)=\displaystyle\int_0^{\Phi(\rho)}\Phi^{-1}(s)ds$, with $\Phi(\rho)=\rho A(\rho)$ and $F(u)=\displaystyle\int_0^u f(s)ds$. The criterion on the balance between the data and the domain (see Section 1.2b of [5]) was here made sharper for the special case in which Ω is a ball and the boundary datum is a constant. Several steps of their study (perhaps the finest and newest parts of the paper) have some interest by themselves. For instance Theorem 4.1 proves the strict separation among solutions outside their dead cores, even if the strong maximum principle can not be applied directly.

The short section on the "nonmonotone case" presents some special solutions (labeled here, for the first time, as *burst within the core solutions*) which can be built, like a puzzle, due to the lack of uniqueness of solutions. The literature on nonmonotone equations leading to this type of free boundary is also quite long (see, *e.g.*, [8], several previous papers by Serrin and coauthors and the exposition of [6]), but never before has the problem been treated in the generality considered in this paper. The deepness and clarity of the techniques seem to make it possible to obtain a bifurcation diagram with respect to a chosen parameter.

Finally, we briefly mention that some appropriate energy methods can be applied in studying this type of free boundary, even for higher order equations or systems of equations (see, *e.g.*, the monograph [1]).

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