Partially flat surfaces solving k–Hessian perturbed equations

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To José María Montesinos in occasion of his 70th birthday: a deep teacher and a good colleague

ABSTRACT

In this paper we study a free boundary arising when a kind of diffusion involving Hessian functions is placed in balance with an absorption term (zero order nonlinear term of the own solution u). The diffusion operator is the k^{th} elementary symmetric function of the eigenvalues of the Hessian matrix $\mathrm{D}^2 u$ and the absorption is a real increasing function vanishing at the origin such that the $(k+1)$–root of its primitive is integrable near the origin.

The surface associated to this solution has a strictly convex part and some flat sides. The junction between both regions of the surface behaves like a free boundary due to the degeneracy of the elliptic leading part of the equation on this interface.

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1. Introduction

Given $u \in \mathcal{C}^2(\Omega)$ we denote by $\lambda(\mathrm{D}^2u) = \big(\lambda_1, \ldots, \lambda_{\mathrm{N}}\big) \in \mathbb{R}^{\mathrm{N}}$ the eigenvalues of D^2u. Then we consider the k^{th} elementary symmetric function

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2u\big)\big] = \sum_{1\leq i_1<i_2<\cdots<i_k\leq \mathrm{N}} \lambda_{i_1}\cdots\lambda_{i_k}, \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N}>1$. Obviously, k is an integer number taking value in $[1,\mathrm{N}]$. Therefore, the case $k=1$ corresponds to the Laplacian operator $\mathcal{S}_1\big[\lambda\big(\mathrm{D}^2u\big)\big] = \Delta u$ while it is a fully nonlinear operator in the other choices of k. For example, the choice $k=2$ leads to $\mathcal{S}_2\big[\lambda\big(\mathrm{D}^2u\big)\big] = \dfrac{1}{2}\Big(\big(\Delta u\big)^2 - |\mathrm{D}^2u|^2\Big)$ and $k=\mathrm{N}$ leads to the Monge–Ampere operator $\mathcal{S}_{\mathrm{N}}\big[\lambda\big(\mathrm{D}^2u\big)\big] = \det \mathrm{D}^2u$. Such kind of eigenvalues products are of relevance in the study of calibrate geometry (see [15]).

We note that a kind of Strong Maximum Principle holds for the *admissible* solutions of

$$\mathcal{S}_k[\lambda(\mathrm{D}^2u)] \geq 0 \quad \text{in } \Omega.$$

Therefore they can not assume an interior maximum or minimum value unless are constant solutions. The main goal of this paper is to prove that this kind of positivity information can be violated generating a dead core in Ω whenever the Hessian function is balanced against suitable absorptions. This paper will extend our previous work [8] dealing with the perturbed Monge-Ampere equations to the case of arbitrary k–Hessian equations for any available k.

More precisely, we focus our the attention on the equation

$$\mathcal{F}(\mathrm{D}^2u, \mathrm{D}u, u) = 0 \quad \text{in } \Omega, \tag{1.2}$$

for the fully nonlinear operator

$$\mathcal{F}(\mathrm{D}^2u, \mathrm{D}u, u) \doteq -\mathcal{S}_k[\lambda(\mathrm{D}^2u)] + \eta g\big(|\mathrm{D}u|\big) f\big(u-h\big), \quad u \in \mathcal{C}^2 \tag{1.3}$$

with $\eta>0$, $g\in\mathcal{C}^1(\mathbb{R}_+;\mathbb{R}_+)$, $f\in\mathcal{C}(\mathbb{R})$ and $h\in\mathcal{C}(\Omega)$. The operator $\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2u\big)\big]$ is only elliptic when $u\in\mathcal{C}^2(\Omega)$ is k–admissible [3], namely the eigenvalues $\lambda(\mathrm{D}^2u)$ lie in the open symmetric convex cone, $\overline{\Gamma}_k$, in $\mathbb{R}^{\mathrm{N}}$ with vertex at the origin (see Section 2), then some compatibility is required on the structure of the equation (1.2) when is restricted to k-admissible solutions. In fact, the operator is degenerate elliptic on the symmetric definite non–negative matrices (see the comments in Section 2). As it will be proved in Theorem 3.2 (see also Remark 3.3), the compatibility is based on

$$h \text{ is locally } k\text{-admissible on } \overline{\Omega} \text{ and } h \leq u \text{ on } \partial\Omega. \tag{1.4}$$

We emphasize that if f is too flat near the origin (see (1.7) below) and $u(x_0) > h(x_0)$ at some $x_0\in\partial\Omega$ or $\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2h(x_0)\big)\big] > 0$ at some point $x_0\in\Omega$ then $\mathcal{F}(\mathrm{D}^2u, \mathrm{D}u, u)$ is non-degenerate in path-connected open sets Ω (see Corollary 2).

The paper is organized as follows. In Section 2 we collect in a short way several comments on the notions of solutions. In Section 3 we obtain some weak maximum principles for the associated boundary value problem to (1.2) and get an existence of solutions result. Section 4 deals with the study of flat regions: we give some sufficient conditions for its occurrence as well as some estimates on its location. The consideration of unflat solutions is carried out in Section 5. The results can be considered, in some sense, as necessary conditions for the existence of flat solutions in terms of the zero order term of the equation.

One of the main consequence of the Weak Maximum Principle is the comparison result for which one deduces $h \leq u$ on $\overline{\Omega}$, provided (1.4) holds, i.e., h behaves as a kind of lower "obstacle" for the solution u (see Remark 3.3 below). Therefore, under (1.4) for any $\varphi \in \mathcal{C}(\partial\Omega)$, the boundary value proble considered in this papers is

$$\begin{cases} \mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] = \eta g\big(|\mathrm{D}u|\big) f\big(u-h\big) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded $(k-1)$–covex open set of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} > 1$ (see below). We note that the usual restriction on the non negativity of the right hand side is here supplied by (1.4). We emphasize that since the right hand side of the equation needs not to be strictly positive in some region of Ω, the ellipticity of the Hessian function and the regularity $\mathcal{C}^2$ of solutions cannot be "a priori" guaranteed. The so-called "viscosity k–admissible solutions" or the "generalized k–admissible solutions" are adequate notions in order to weaken the non-degeneracy hypothesis on the operator. By using the Weak Maximum Principle and well known methods we prove, in Theorem 3.2, the existence of a unique generalized solution of (1.5). By a simple reasoning we obtain estimates on the gradient $\mathrm{D}u$. Bounds for the second derivatives $\mathrm{D}^2 u$ can be deduced from second order estimates (see Remark 3.3 below).

Since $h \leq u$ holds on $\overline{\Omega}$, the junction between the regions where $\{u = h\}$ and $\{h < u\}$ is a *free boundary*, thus it is not known a priori. This free boundary can be defined also as the boundary of the set of points $x \in \Omega$ for which $\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u(x)\big)\big] > 0$. Obviously, since the interior of the regions $\{u = h\}$ and $\{\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] = 0\}$ coincide, we must have $\mathcal{S}_k[\lambda(\mathrm{D}^2 h)] = 0$ in these interior region. The occurrence and localization of this free boundary is studied in Section 4 whenever $h(x)$ has *flat regions*

$$\mathrm{Flat}(h) = \bigcup_{\alpha} \{x \in \Omega : h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha,\ \mathbf{p}_\alpha \in \mathbb{R}^{\mathrm{N}},\ a_\alpha \in \mathbb{R}\} \neq \emptyset,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^{\mathrm{N}}$. As it will be proved, the free boundary appears under two different types of conditions on the data: a precise behavior of the zeroth order term

$$\int_{0^+} \mathrm{F}(t)^{-\frac{1}{k+1}} dt < \infty \tag{1.6}$$

(or $0 < \mathrm{q} < k$ for $f(t) = t^{\mathrm{q}}$), where $\mathrm{F}(t) = \displaystyle\int_0^t f(s)ds$, and a suitable balance between the "size" of the regions of Ω where h is flat and the "size" of the data φ and h.

We shall show here how the theory on free boundaries (essentially the boundary of the support of $u - h$), developped for a class of quasilinear operators in divergence form, can be extended to the case of the solution of (1.2) inside of flat regions of h, where $u_h = u - h$ solves

$$\mathcal{S}_k[\lambda(\mathrm{D}^2 u_h] = \lambda g\big(|\mathrm{D}u|\big) f(u_h).$$

This kind of question has been extensively studied in the monography [9], mainly for the quasilinear p–Laplacian operator (see also [7] for fully nonlinear operators). In fact, the results of this paper were suggested in [9] and performed in [8] for the Monge–Ampere operator. The main existence criterion for the free boundary is strongly based on the condition (1.6). Clearly, it coincides with the corresponding main assumption used in [9] for the Laplacian operator. Since the strict k-admissibilty must be removed, a critical size of the data is required: the parameter η governs these kind of condition (see (4.30) below). For instance, the second mentioned balance for given φ and Ω is satisfied if η is large enough.

In Theorems 3 and 5 below we prove the occurrence of the free boundary and give some estimates on its localization. We also prove that if $h(x)$ growths moderately (in a suitable way) near the region where it ceases to be flat then the free boundary region associated to the flattens of u (*i.e.* the region where $u_h = u - h$ vanishes) may coincide with the boundary of the set where h is flat (see Theorem 6 for $f(t) = t^{\mathrm{q}}$, $\mathrm{q} < \mathrm{N}$). The estimates on the localization of the free boundary are optimal, in the class of nonlinearities $f(s)$ satisfying (1.6).

In Section 5, by means of a Strong Maximum Principle for u_h, we prove that the condition

$$\int_{0^+} \mathrm{F}(t)^{-\frac{1}{k+1}} = \infty \tag{1.7}$$

(or $k \leq \mathrm{q}$ for $f(t) = t^{\mathrm{q}}$) is a necessary condition for the non-existence of such free boundary (see Theorem 8 and Corollary 2). More precisely, we shall prove that under this condition the solution can not have any flat region. This can be regarded as an extension of [20] to the non divergence case (see also [7], [9] and [16]). As it was pointed out, the condition $k \leq \mathrm{q}$ implies non-degenerate ellipticity of problem (3.6) under very simple assumptions, such as $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 h(x_0)\big)\big] > 0$ at some point $x_0 \in \Omega$ for path-connected open set Ω (see Corollary 2).

Finally, we note that all contributions coincide with the relative ones of [8] whenever $k = \mathrm{N}$.

2. Notations and other preliminaries comments

For any matrix $\mathbf{A} \in \mathcal{M}(\mathbb{R}, \mathrm{N} \times \mathrm{N})$, we consider the sum of the $k \times k$, $1 \leq k \leq \mathrm{N}$ principal minors, here denoted by

$$\mathcal{S}_k\big[\lambda(\mathbf{A})\big] = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq \mathrm{N}} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{2.1}$$

where $\lambda(\mathbf{A}) = \big(\lambda_1, \ldots, \lambda_{\mathrm{N}}\big)$ are the eigenvalues of $\mathbf{A}$. Obviously, k is an integer number taking value in $[1, \mathrm{N}]$. The more popular examples are $\mathcal{S}_1[\lambda(\mathbf{A})] =$ trace of $\mathbf{A}$ and $\mathcal{S}_{\mathrm{N}}[\lambda(\mathbf{A})] = \det(\mathbf{A})$. The main important case appears when $\mathbf{A} = \mathrm{D}^2 u$ for some function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, where Ω is an open set of $\mathbb{R}^{\mathrm{N}}$, for which $\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big]$ is called the k^{th} elementary symmetric function. If we define the cone

$$\Gamma_k = \left\{ \big(\lambda_1, \ldots, \lambda_{\mathrm{N}}\big) \in \mathbb{R}^{\mathrm{N}} : \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq \mathrm{N}} \lambda_{i_1} \cdots \lambda_{i_j} > 0, \ \forall j = 1, \ldots, k \right\}$$

the k–convex functions are introduced by the condition $\lambda\big(\mathrm{D}^2 u\big) \in \overline{\Gamma}_k$. Then, a function is 1–admissible if and only if it is sub–harmonic and a function N–admissible must be convex, because $\det \mathrm{D}^2 u > 0$ implies convexity by the Sylvester criterion. The expression $\mathcal{S}_k\big[\lambda(\mathbf{A})\big]$ are denoted alternatively as "principal invariants" of the tensor $\mathbf{A}$ (see [13, p.15]) as they are used in Continuum Mechanics. For instance the 2^{nd} elementary symmetric function plays a fundamental role in the study of Non–Newtonian fluids and Mooney–Rivlin materials (see [13, p. 174 and p. 192]).

Also we deduce that a k–admissible smooth function, for any $1 \leq k \leq \mathrm{N}$, is sub–harmonic because

$$\Gamma_{\mathrm{N}} \subset \cdots \subset \Gamma_k \subset \cdots \subset \Gamma_1.$$

Moreover the set of the k–admissible functions is a convex cone in $\mathcal{C}^2(\Omega)$. Since one proves that the matrix

$$\left\{ \frac{\partial \mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big]}{\partial \mathrm{D}_{ij} u} : \ 1 \leq i, j \leq \mathrm{N} \right\} \tag{2.2}$$

is positive semi–definite if u is k–admissible, the k^{th} elementary symmetric operator $\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big]$, in short the Hessian operator, is non–negative and degenerate elliptic on the convex cone of the k–admissible smooth functions (see [21]). Another main property useful to our reasoning: as function of $\mathrm{D}^2 u$

$$\Lambda_k\big(\mathrm{D}^2 u\big) = \big(\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big]\big)^{\frac{1}{k}}$$

is concave on the convex cone of the k–admissible functions. Finally we note that a compatibility geometric assumption must be required when one prescribes boundary values. In order to simplify it we only consider, for a while, smooth at the boundary k-admissible functions u vanishing on $\partial\Omega$. For any fixed point $x_0 \in \partial\Omega$, by suitable translations and rotations of coordinates if necessary, we may assume that x_0 is the origin and that locally $\partial\Omega$ is given by $x_{\mathrm{N}} = \Psi(x')$ such that $\mathbf{n} = (0, \ldots, 0, 1)$ is the inner normal of $\partial\Omega$ at x_0, where $x' = (x_1, \ldots, x_{\mathrm{N}-1})$. Then differentiating the boundary condition $u(x', \Psi(x')) = 0$ we get

$$\mathrm{D}_{ij}u(0) + \mathrm{D}_{\mathrm{N}}u(0)\mathrm{D}_{ij}\Psi(0) = 0.$$

Since u is sub–harmonic one has $\mathrm{D}_{\mathrm{N}}u(0) < 0$ whence

$$\frac{\partial \mathcal{S}_k\big[\lambda(\mathrm{D}^2 u)\big]}{\partial \mathrm{D}_{\mathrm{NN}}u} = |\mathrm{D}_{\mathrm{N}}u(0)|^{k-1} \sum_{1\leq i_1<i_2<\cdots<i_k\leq \mathrm{N}-1} \kappa_{i_1}\cdots\kappa_{i_k}$$

because the principal curvatures, $\kappa = (\kappa_1, \ldots, \kappa_{\mathrm{N}-1})$, of $\partial\Omega$ at x_0 are the eigenvalues of $\mathrm{D}_{ij}\Psi(0)$. So that, as the matrix given by (2.2) is positive semi–definite we know one the following condition holds:

$$\sum_{1\leq i_1<i_2<\cdots<i_k\leq \mathrm{N}-1} \kappa_{i_1}\cdots\kappa_{i_k} \geq c_0 > 0 \quad \text{on } \partial\Omega \text{ for some constant } c_0.$$

This defines the $(k-1)$–convex domains that we will consider in this paper. Clearly, when $k = \mathrm{N}$ this geometric condition is equivalent to the usual convexity.

Many previous expositions on the nature of the k–admissible functions can be found in the literature (see for instance the survey [21] or [4]).

As it was proved by several methods [3, 4, 18, 21], there exists a k–admissible $\mathcal{C}^2$ solution of the general boundary value problems as

$$\begin{cases} \mathcal{S}_k\big[\lambda(\mathrm{D}^2 u)\big] = \mathrm{H}(\mathrm{D}u, u, x), & \text{on } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

under suitable assumptions on Ω, $\mathrm{H} > 0$ and φ. A main question arises: what happens if $\mathrm{H} \geq 0$. Now the degenerate ellipticity may occur and in general the regularity $\mathcal{C}^2$ of solutions can not be guaranteed. As it was pointed out in the Introduction, the so called "viscosity solutions" or the "generalized solutions" the adequate notions of solutions in our study. In fact, by means of reasoning as in [14], it can be proved that for $(k-1)$–convex domains Ω both notions coincide.

A short description of all that is as follows. First of all, the smooth k–admissibilty notion must be weaken. So, from now, by a k–admissible function u in Ω we mean an upper semi–continuous function in Ω such that $\{u = \infty\}$ has measure zero and

$$\int_\Omega u a_{ij}\mathrm{D}^2_{ij}\phi \leq 0 \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega),\ \phi \geq 0,$$

for any matriz $\mathbf{A} = \{a_{ij}\}$ with eigenvalues in

$$\Gamma_k^* = \big\{\lambda^* \in \mathbb{R}^{\mathrm{N}} : \ \langle \lambda^*, \lambda \rangle \leq 0, \ \lambda \in \Gamma_k \big\}.$$

This implies that an upper semi–continuous function u is k–admissible is it is sub–harmonic with respect to the operator $\mathcal{L} = \sum a_{ij} \mathrm{D}^2_{ij}$ for any matriz $\mathbf{A} = \{a_{ij}\}$ with eigenvalues in Γ_k^*. Certainly, this non–smooth notion is consistent with the smooth k–admissible notion.

On the other hand, if $u \in \mathcal{C}^2(\Omega)$ is a non–negative k–admissible function the measure $\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big]dx$ has the important property that if $\{u_j\}_j \subset \mathcal{C}^2(\Omega)$ are smooth k–admissible functions which converge to a k–admissible function in Ω everywhere then $\big\{\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u_j\big)\big]dx\big\}_j$ converge weakly to a measure μ. With this property one proves

Theorem 1 ([21]) *For any k–admissible function u, there exists a Radon measure $\mu_k[u]$ such that:*

1. $\mu_k[u] = \mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big]dx$ *if* $u \in \mathcal{C}^2(\Omega)$,
2. *if* $\{u_j\}_j$ *are k–admissible functions which converge to a k–admissible function u a.e. then* $\big\{\mu_k[u_j]\}_j \to \mu[u]$ *weakly as measure.* □

Then we arrive to

Definition 2.1 *A k–admissible function u on Ω is a "generalized solution" of*

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] = \mathrm{H}(\mathrm{D}u, u, x), \quad \textit{on } \Omega$$

if

$$\mu[u](\mathrm{E}) = \int_{\mathrm{E}} \mathrm{H}(\mathrm{D}u, u, x)dx$$

for any Borel set $\mathrm{E} \subset \Omega$.

The continuity property of u on $\overline{\Omega}$ is compatible with the usual realization of the Dirichlet boundary condition $u = \varphi$. Here we may considered the weaker assumption $\mathrm{H} \geq 0$ which can not be removed. Certainly, the definition can be extended to locally k–admissible functions u on Ω, for which u can be constant on some subset of Ω. This notion of generalized solution is specific of the Hessian operator, but other notion of solutions are available as it happens with other type of fully nonlinear equations in non divergence form. The most usual is the so called "viscosity solution" introduced by M.G. Crandall and P.L. Lions (see [6]):

Definition 2.2 *A k–admissible function u on Ω is a viscosity solution of the inequality*

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] \geq \mathrm{H}(\mathrm{D}u, u, x) \quad \textit{in } \Omega \qquad \textit{(viscosity sub–solution)}$$

if for every smooth k–admissible function Φ on Ω for which

$$(u-\Phi)(x_0) \geq (u-\Phi)(x) \quad \textit{locally at } x_0 \in \Omega$$

one has

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2\Phi(x_0)\big)\big] \geq \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), x_0\big).$$

Analogously, one defines the viscosity solution of the reverse inequality

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] \leq \mathrm{H}(\mathrm{D}u, u, x) \quad \textit{in } \Omega \qquad \textit{(viscosity super–solution)}$$

as a k–admissible function u on Ω such that for every smooth k–admissible function Φ on Ω for which

$$(u-\Phi)(x_0) \leq (u-\Phi)(x) \quad \textit{locally at } x_0 \in \Omega$$

one has

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2\Phi(x_0)\big)\big] \leq \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), (x_0)\big).$$

Finally, when both properties hold we arrive to the notion of viscosity solution of

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] = \mathrm{H}(\mathrm{D}u, u, x) \quad \textit{in } \Omega.$$

Note that the k–admissible condition on u and Φ are extra assumptions with respect to the usual notion of viscosity solution (see [6]). This is needed here because the Hessian operator is only degenerate elliptic on this class of functions. In fact, the k–admissible condition on the test function Φ is only required for the correct definition of super–solutions in the viscosity sense, because if $u-\Phi$ attains a local maximum at $x_0 \in \Omega$ for a k–admissible function u on Ω and $\Phi \in \mathcal{C}^2(\Omega)$, reasoning as in [14], one can deduce

$$\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2\Phi(x_0)\big)\big] \geq 0.$$

Reasoning again as in [14], it is possible to prove the equivalence

u is a generalized solution of (2.3) *if and only if u is a viscosity solution of* (2.3),

provided that Ω is a $(k-1)$–convex domain and $\mathrm{H} \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times \mathbb{R} \times \Omega)$.

As an illustrative result on the complementary regularity, one proves that $\mathrm{H}(\mathrm{D}u, u, x) \in \mathrm{L}^p(\Omega)$, $p > \dfrac{\mathrm{N}}{2k}$, implies that u is Hölder continuous, provided $k \leq \dfrac{\mathrm{N}}{2}$ (see [21, Corollary 9.1]).

3. Weak Maximum Principle

In this section we obtain some comparison and existence results for the equation (1.2). They will show that the nature of the viscosity solution is an intrinsic property associated with the Maximum Principle.

Theorem 2 (Weak Maximum Principle I) *Let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ where u is k–admissible in Ω. Suppose*

$$\mathcal{F}(\mathrm{D}^2 u, \mathrm{D}u, u) \leq 0 \leq \mathcal{F}(\mathrm{D}^2 v, \mathrm{D}v, v) \quad \textit{in } \Omega. \tag{3.1}$$

Then

$$(u - v)(x) \leq \sup_{\partial\Omega} [u - v]_+, \quad x \in \Omega.$$

In particular,

$$|u - v|(x) \leq \sup_{\partial\Omega} |u - v|, \quad x \in \Omega.$$

whenever the equalities hold in (3.1).

Proof. By continuity there exists $x_0 \in \overline{\Omega}$ where $[u-v]_+$ attains the maximum value on $\overline{\Omega}$. We claim that $[u-v]_+(x_0) = 0$, whence the result follows. Indeed, if $x_0 \in \Omega$ and $[u-v]_+(x_0) > 0$, the matrix $\mathrm{D}^2(v-u)(x_0)$ is positive semidefinite. In particular, the function $v-u$ is k–admissible at x_0. Consider the function $\Lambda_k(\mathbf{A}) \doteq \left(\mathcal{S}_k\left[\lambda(\mathbf{A})\right]\right)^{\frac{1}{k}}$ which is homogeneous of degree 1 and concave on the convex cone of the set of matrices having eigenvalues in Γ_k. Since the convexity of this set of matrices implies that the sum of two k–admissible functions is also k-admissible, the function $v = (v-u) + u$ is k–admissible at x_0. Then

$$\Lambda_k\left[\mathrm{D}^2 v(x_0)\right] = 2^k \Lambda_k\left[\frac{1}{2}\mathrm{D}^2 v(x_0)\right] \geq \Lambda_k\left[\mathrm{D}^2(v-u)(x_0)\right] + \Lambda_k\left[\mathrm{D}^2 u(x_0)\right] \geq \Lambda_k\left[\mathrm{D}^2 u(x_0)\right]$$

leads to the contradiction

$$\begin{aligned} 0 &\leq \Lambda_k\left[\mathrm{D}^2 v(x_0)\right] - \Lambda_k\left[\mathrm{D}^2 u(x_0)\right] \\ &\leq \big(g\big(|\mathrm{D}v(x_0)|\big) f\big(v(x_0)\big) - h(x_0)\big)^{\frac{1}{k}} - \big(g\big(|\mathrm{D}u(x_0)|\big) f\big(u(x_0)\big) - h(x_0)\big)^{\frac{1}{k}} < 0. \end{aligned}$$

□

Remark 3.1 We note that the monotonicity on u on the zeroth order terms, $f(u-h)$, is the only assumption required on the structure of the equation and that our argument is strongly based on the notion of viscosity solution. An analogous estimate holds by changing the roles of u_1 and u_2 (but then we do not require the $\mathcal{C}^2$ function u_1 to be k–admissible). Note also that we did not assume any convexity condition on the domain Ω. When Ω is $(k-1)$–convex these results can be extended to the class of the generalized solutions through the mentioned equivalence between such solutions and the viscosity solutions. □

A very simple (and important fact) was used in our precedent arguments: if $u_1 \in \mathcal{C}^2$ and $u_2 - u_1 \in \mathcal{C}^2$ are k–admissible functions on a ball $\mathbf{B}$ then

$$\mathcal{S}_k\left[\lambda\big(\mathrm{D}^2 u_2\big)\right] \geq \mathcal{S}_k\left[\lambda\big(\mathrm{D}^2 u_1\big)\right] \quad \text{in } \mathbf{B}.$$

This simple inequality can be extended to the case where u_1 and $u_2 - u_1$ are k–admissible functions on a ball $\mathbf{B}$, with $u_1 = u_2$ on $\partial\mathbf{B}$, by the "monotonicity formula"

$$\mu[u_2](\mathbf{B}) \geq \mu[u_1](\mathbf{B}) \tag{3.2}$$

(see [21]). In this way, the Weak Maximum Principle can be extended to the class of generalized solutions.

Theorem 3.1 (Weak Maximum Principle II) *Let $h_1, h_2 \in \mathcal{C}(\overline{\Omega})$. Let $u_1, u_2 \in \mathcal{C}(\overline{\Omega})$ where u_1 is locally k–admissible in Ω. Suppose*

$$-\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u_1\big)\big] + g\big(|\mathrm{D}u_1|\big)f\big(u_1 - h_1\big) \leq -\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u_2\big)\big] + g\big(|\mathrm{D}u_2|\big)f\big(u_2 - h_2\big) \quad \text{in } \Omega \tag{3.3}$$

in the generalized solutions sense. Then

$$(u_1 - u_2)(x) \leq \sup_{\partial\Omega}[u_1 - u_2]_+ + \sup_{\Omega}[h_1 - h_2]_+, \quad x \in \overline{\Omega}. \tag{3.4}$$

In particular,

$$|u_1 - u_2|(x) \leq \sup_{\partial\Omega}|u_1 - u_2| + \sup_{\Omega}|h_1 - h_2|, \quad x \in \overline{\Omega}, \tag{3.5}$$

whenever the equality holds in (3.3).

Proof. As above, we only consider the case where the maximum of $[u_1 - u_2]_+$ on $\overline{\Omega}$ is achieved at some $x_0 \in \Omega$ with $[u_1 - u_2]_+(x_0) > 0$. Therefore, $\big(u_1 - u_2\big)(x) > 0$ and convex in a ball $\mathbf{B}_{\mathrm{R}}(x_0)$, for R small. Let $\Omega^+ = \{u_1 > u_2\} \supseteq \mathbf{B}_{\mathrm{R}}(x_0)$. We construct $\widehat{u}_1(x) = u_1(x) + \gamma\big(|x - x_0|^2 - \mathrm{M}^2\big) - \delta$, where $\mathrm{M} > 0$ is large and $\gamma, \delta > 0$ such that $\widehat{u}_1 < u_1$ on $\partial\Omega^+$ and the set $\Omega^+_{\gamma,\delta} = \{\widehat{u}_1 > u_2\}$ is compactly contained in Ω and contains $\mathbf{B}_\varepsilon(x_0)$ for some ε small. By choosing γ, δ properly, we can assume that the diameter of $\Omega^+_{\gamma,\delta}$ is small so that u_1 and therefore $u_2 = (u_2 - u_1) + u_1$ are convex in it. Then (3.2) implies

$$\begin{aligned} 0 < (\gamma\varepsilon)^{\mathrm{N}}|\mathbf{B}_1(0)| &\leq \mu[u_2]\big(\mathbf{B}_\varepsilon(x_0)\big) - \mu[u_1]\big(\mathbf{B}_\varepsilon(x_0)\big) \\ &\leq \int_{\mathbf{B}_\varepsilon(x_0)} \big[g\big(|\mathrm{D}u_2|\big)f\big(u_2 - h_2\big) - g\big(|\mathrm{D}u_1|\big)f\big(u_1 - h_1\big)\big]dx. \end{aligned}$$

Since $g\big(|\mathrm{D}u_1(x_0)|\big) = g\big(|\mathrm{D}u_2(x_0)|\big) > 0$ (see Remark 3.2 below), by letting $\varepsilon \to 0$, the Lebesgue differentiation theorem implies

$$0 \leq g\big(|\mathrm{D}u_2(x_0)|\big)f\big(u_2(x_0) - h_2(x_0)\big) - g\big(|\mathrm{D}u_1(x_0)|\big)f\big(u_1(x_0) - h_1(x_0)\big),$$

whence

$$\big(u_1 - u_2\big)(x_0) < \big(h_1 - h_2\big)(x_0) \leq \sup_{\partial\Omega}\big[u_1 - u_2\big]_+ + \sup_{\Omega}\big[h_1 - h_2\big]_+$$

and the estimate holds. □

Remark 3.2 The above proof requires a simple fact: any convex function ψ in a convex open set $\mathcal{O} \subset \mathbb{R}^{\mathrm{N}}$ achieving a local interior maximum at some $z_0 \in \mathcal{O}$ verifies $\mathrm{D}\psi(z_0) = \mathbf{0}$. Indeed, for any $\mathbf{p} \in \partial\psi(z_0)$ (the sub–differential set of ψ at z_0) one has

$$\psi(x) \geq \psi(z_0) + \langle \mathbf{p}, x - z_0 \rangle \geq \psi(x) + \langle \mathbf{p}, x - z_0 \rangle \quad \text{with } x \text{ near } z_0,$$

and

$$0 \geq \langle \mathbf{p}, x - z_0 \rangle.$$

Then if $\tau > 0$ is small enough we may choose $x - z_0 = \tau \mathbf{p} \in \mathcal{O}$ and to deduce

$$0 \leq \tau |\mathbf{p}|^2 \leq 0.$$

□

A first consequence of the general theory for (1.2) and the Weak Maximum Principle is the following existence result

Theorem 3.2 *Suppose that* Ω *is* $(k-1)$*–convex. Let* $\varphi \in \mathcal{C}(\partial\Omega)$ *and assume the compatibility condition* (1.4)

$$h \text{ is locally } k\text{-admissible on } \overline{\Omega} \text{ and } h \leq u \text{ on } \partial\Omega.$$

Then there exists a unique locally k*–admissible function verifying*

$$\begin{cases} \mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] = g\big(|\mathrm{D}u|\big) f(u-h) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

in the generalized sense. In fact, one verifies

$$h(x) \leq u(x) \leq \mathrm{U}_\varphi(x), \quad x \in \overline{\Omega}, \tag{3.7}$$

where U_φ *is the unique harmonic function in* Ω *such that* $\mathrm{U}_\varphi = \varphi$ *on* $\partial\Omega$.

Proof. First we consider the generalized solution of the problem

$$\begin{cases} -\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 u\big)\big] + g\big(|\mathrm{D}u|\big)\big[f(u-h)\big]_+ = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Since $\mathrm{H}\big(\mathrm{D}u, u, x\big) = g\big(|\mathrm{D}u|\big)\big[f(u-h)\big]_+ \geq 0$ we can apply well known results in the literature. In particular, from [21], it follows the existence and uniqueness of the solution u. The second point is to note that, by construction, the locally k–admissible function h verifies

$$-\mathcal{S}_k\big[\lambda\big(\mathrm{D}^2 h\big)\big] + g\big(|\mathrm{D}u|\big)\big[f(h-h)\big]_+ \leq 0 \quad \text{in } \Omega.$$

Therefore, by the Weak Maximum Principle and the assumption $h \leq \varphi$ on $\partial\Omega$ we get that

$$h \leq u \quad \text{in } \Omega,$$

whence

$$\big[f(u-h)\big]_+ = f(u-h)$$

which proves that u solves (3.6). The uniqueness also follows from the Weak Maximum Principle. Finally, since u is locally k–admissible it is also sub–harmonic in Ω and so the estimate

$$h(x) \leq u(x) \leq \mathrm{U}_\varphi(x), \quad x \in \overline{\Omega}$$

holds by the weak maximum principe for harmonic functions. □

Remark 3.3 i) As it was pointed out in the Introduction, no sign assumption on h is required in Theorem 3.2. The simple structural assumption (1.4) implies that $h \leq u$ on $\overline{\Omega}$ and therefore the ellipticity, eventually degenerate, of the equation holds. Thus, the ellipticity holds once h behaves as a lower "obstacle" for the solution u. We note that these compatibility conditions are not a priori required in the Weak Maximum Principle because there we are working with functions whose existence is a priori assumed.
ii) Since u is locally k–admissible on $\overline{\Omega}$, we can prove

$$\sup_{\Omega} |\mathrm{D}u| = \sup_{\partial\Omega} |\mathrm{D}u|,$$

and then inequality (3.7) gives a priori bounds on $|\mathrm{D}u|$ on $\overline{\Omega}$, provided $h = \varphi$ on $\partial\Omega$ and $\mathrm{D}h$ is defined on $\partial\Omega$. The proof of a second derivative estimate is based on the inequality

$$\operatorname*{ess\,sup}_{\Omega} |\mathrm{D}^2 u| \leq \mathrm{C}\left(1 + \sup_{\partial\Omega} |\mathrm{D}^2 u|\right) \tag{3.8}$$

for some constant C independent on u. It will be the object of a separated article. □

In Section 5 we shall prove a kind of Strong Maximum Principle which under suitable assumptions will avoid the appearance of the mentioned free boundary.

4. Flat regions

In this section we focus our attention on a lower "obstacle" function h locally k–admissible on $\overline{\Omega}$ which is locally flat. We define

$$\mathrm{Flat}(h) = \bigcup_{\alpha} \mathrm{Flat}_\alpha(h)$$

where

$$\mathrm{Flat}_\alpha(h) = \{x \in \overline{\Omega} : h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha, \text{ for some } \mathbf{p}_\alpha \in \mathbb{R}^{\mathrm{N}} \text{ and } a_\alpha \in \mathbb{R}\}. \tag{4.1}$$

Since

$$u(y) - \big(\langle \mathbf{p}_\alpha, y\rangle + a_\alpha\big) \geq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) + \langle \mathbf{p} - \mathbf{p}_\alpha, y - x\rangle,$$

thus

$$\mathbf{p} \in \partial u(x) \quad \Leftrightarrow \quad \mathbf{p} - \mathbf{p}_\alpha \in \partial\left(u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\right),$$

and the equation (1.2) becomes

$$-\mathcal{S}_k[\lambda(\mathrm{D}^2 u_\alpha)] = \eta g\big(|\mathrm{D}u|\big) f\big(u_\alpha\big), \quad x \in \mathrm{Flat}_\alpha(h), \tag{4.2}$$

for $u_\alpha \doteq u - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$. Remember that $u_\alpha \geq 0$ in an open set $\mathcal{O} \subseteq \Omega$, if $u_h \geq 0$ on $\partial\mathcal{O}$. Assumption

$$g(|\mathbf{p}|) \geq 1 \tag{4.3}$$

leads us to study the auxiliar boundary problem

$$\begin{cases} \mathcal{S}_k[\lambda(\mathrm{D}^2\mathrm{U})] = f(\mathrm{U}) & \text{in } \mathbf{B}_{\mathrm{R}}(0), \\ \mathrm{U} \equiv \mathrm{M} > 0 & \text{on } \partial\mathbf{B}_{\mathrm{R}}(0), \end{cases} \tag{4.4}$$

for any $\mathrm{M} > 0$. From the uniqueness of solutions, it follows that U is radially symmetric, because by rotating it we would find other solutions. Moreover, by the comparison results U is nonnegative. Therefore, the solution U is governed by a non–negative radial profile function $\mathrm{U}(x) = \widehat{\mathrm{U}}(|x|)$ for which some straightforward computations leads to

$$\begin{aligned} \mathcal{S}_k[\lambda(\mathrm{D}^2\mathrm{U})](r) &= \mathrm{C}_{\mathrm{N}-1,k-1}\widehat{\mathrm{U}}''(r)\left(\frac{\widehat{\mathrm{U}}'(r)}{r}\right)^{k-1} + \mathrm{C}_{\mathrm{N}-1,k}\left(\frac{\widehat{\mathrm{U}}'(r)}{r}\right)^{k} \\ &= \mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}}\left[\frac{r^{\mathrm{N}-k}}{k}\big(\widehat{\mathrm{U}}'(r)\big)^k\right]', \end{aligned} \tag{4.5}$$

where we use the notation

$$\mathrm{C}_{m,n} = \binom{m}{n} = \frac{m!}{(m-n)!n!}, \quad 0 \leq n \leq m.$$

We summarize the well known properties:

$$\begin{cases} \mathrm{C}_{m,0} = \mathrm{C}_{m,m} & \text{(initial/boundary values)}, \\ \mathrm{C}_{m,n} = \mathrm{C}_{m,m-n} & \text{(symmetry)}, \\ \mathrm{C}_{m,k} + \mathrm{C}_{m+k+1} = \mathrm{C}_{m+1,k+1} & \text{(recursive Pascal rule)}. \end{cases}$$

Remark 4.1 For $\mathrm{N}=1$, the equation (4.5) becomes the semi linear ODE

$$\widehat{\mathrm{U}}''(r)=\lambda f\big(\widehat{\mathrm{U}}\big)$$

studied in [9]. Notice that for $\mathrm{N}>1$ the k–radial Hessian operator *is not* exactly the radial p-Laplacian operator with $p=k+1$, although there is a great resemblance among them. □

We start this section by considering the initial value problem

$$\begin{cases} \mathrm{C}_{\mathrm{N}-1,k-1}r^{1-\mathrm{N}}\left[\dfrac{r^{\mathrm{N}-k}}{k}\big(\mathrm{U}'(r)\big)^{k}\right]'=\eta f\big(\mathrm{U}(r)\big), \quad \eta>0, \\ \mathrm{U}(0)=\mathrm{U}'(0)=0. \end{cases} \tag{4.6}$$

Obviously, $\mathrm{U}(r)\equiv 0$ is always a solution, but we are interested in the existence of nontrivial and non–negative solutions. The general reasoning in this section assumes the existence of an increasing function $\mathbb{U}:[0,\mathrm{R}_{\mathbb{U}}[\to\overline{\mathbb{R}}_{+}$ solving

$$\begin{cases} \mathrm{C}_{\mathrm{N}-1,k-1}r^{1-\mathrm{N}}\left[\dfrac{r^{\mathrm{N}-k}}{k}\big(\mathbb{U}'(r)\big)^{k}\right]'=\eta_{\mathbb{U}} f\big(\mathbb{U}(r)\big), \quad 0<r<\mathrm{R}_{\mathbb{U}}, \\ \mathbb{U}(0)=\mathbb{U}'(0)=0, \end{cases} \tag{4.7}$$

for some $\eta_{\mathbb{U}}>0$ and $0<\mathrm{R}_{\mathbb{U}}\leq\infty$. We shall return to this assumption later.

By scaling by $\mathrm{A}>0$, (4.7) becomes

$$\begin{cases} -\mathrm{C}_{\mathrm{N}-1,k-1}r^{1-\mathrm{N}}\left[\dfrac{r^{\mathrm{N}-k}}{k}\big(\widehat{\mathbb{U}}'(\mathrm{A}r)\big)^{k}\right]'+\eta f\big(\mathbb{U}(\mathrm{A}r)\big)=\left[\eta-\eta_{\mathbb{U}}\mathrm{A}^{2k}\right]f\big(\mathbb{U}(\mathrm{A}r)\big) \\ \mathbb{U}(0)=\mathbb{U}'(0)=0, \end{cases} \tag{4.8}$$

$0<r<\dfrac{\mathrm{R}_{\mathbb{U}}}{\mathrm{A}}$, whence it follows

1. if $\mathrm{A}<\left(\dfrac{\eta}{\eta_{\mathbb{U}}}\right)^{\frac{1}{2k}}$ the function $\mathbb{U}(\mathrm{A}r)$ is a super–solution of the equation (4.6),
2. if $\mathrm{A}=\left(\dfrac{\eta}{\eta_{\mathbb{U}}}\right)^{\frac{1}{2k}}$ the function $\mathbb{U}(\mathrm{A}r)$ is the solution of the equation (4.6),
3. if $\mathrm{A}>\left(\dfrac{\eta}{\eta_{\mathbb{U}}}\right)^{\frac{1}{2k}}$ the function $\mathbb{U}(\mathrm{A}r)$ is a sub–solution of the equation (4.6).

Moreover, the function

$$v_{\tau}(x)\doteq\mathbb{U}\left(\left(\frac{\eta}{\eta_{\mathbb{U}}}\right)^{\frac{1}{2k}}\left[|x|-\tau\right]_{+}\right), \quad x\in\mathbf{B}_{\tau+\mathrm{R}_{\mathbb{U},\eta}}(0),\ \mathrm{R}_{\mathbb{U},\eta}=\mathrm{R}_{\mathbb{U}}\left(\frac{\eta_{\mathbb{U}}}{\eta}\right)^{\frac{1}{2k}} \tag{4.9}$$

solves

$$-\mathcal{S}_k[\lambda(\mathrm{D}^2 v_\tau(x)] + \eta f\big(v_\tau(x)\big) = 0, \quad x \in \mathbf{B}_{\tau+\mathrm{R}_{\mathbb{U},\eta}}(0).$$

Furthermore, it verifies

$$v_\tau(x) = \mathrm{M}, \quad |x| = \mathrm{R} < \tau + \mathrm{R}_{\mathbb{U},\eta}$$

once we take

$$\tau = \mathrm{R} - \left(\frac{\eta_{\mathbb{U}}}{\eta}\right)^{\frac{1}{2k}} \mathbb{U}^{-1}(\mathrm{M}) = \left[\eta_*^{-\frac{1}{2k}} - \eta^{-\frac{1}{2k}}\right] \mathbb{U}^{-1}(\mathrm{M})\eta_{\mathbb{U}}^{\frac{1}{2k}}$$

with

$$\eta \geq \eta_* \doteq \eta_{\mathbb{U}} \left(\frac{1}{\mathrm{R}}\mathbb{U}^{-1}(\mathrm{M})\right)^{2k}. \tag{4.10}$$

Now for a solution of (1.2) we may localize a core of the flat region $\mathrm{Flat}(u)$ inside the flat subregion $\mathrm{Flat}_\alpha(h)$ of the given "obstacle".

Theorem 3 *Let h be locally k–admissible on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_{\mathrm{R}}(x_0) \subset \mathrm{Flat}_\alpha(h)$ with*

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathrm{M} \leq \max_{\overline{\Omega}}(u-h), \quad x \in \partial\mathbf{B}_{\mathrm{R}}(x_0), \tag{4.11}$$

where u is a generalized solution of (1.2), for some $\mathrm{M} > 0$. Then, if (4.7) holds and

$$\eta \geq \eta_* \doteq \eta_{\mathbb{U}} \left(\frac{1}{\mathrm{R}}\mathbb{U}^{-1}(\mathrm{M})\right)^{2k},$$

one verifies

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathbb{U}\left(\left(\frac{\eta}{\eta_{\mathbb{U}}}\right)^{\frac{1}{2k}} [|x| - \tau]_+\right), \quad x \in \mathbf{B}_{\mathrm{R}}(x_0), \tag{4.12}$$

where

$$\tau = \left[\eta_*^{-\frac{1}{2k}} - \eta^{-\frac{1}{2k}}\right] \mathbb{U}^{-1}(\mathrm{M})\eta_{\mathbb{U}}^{\frac{1}{2k}}, \tag{4.13}$$

once we assume that $\mathrm{R} < \tau + \mathrm{R}_{\mathbb{U},\eta}$ and

$$\left(\frac{\eta_{\mathbb{U}}}{\eta}\right)^{\frac{1}{2k}} \mathbb{U}^{-1}(\mathrm{M}) < \mathrm{R} \leq \mathrm{dist}(x_0, \partial\Omega). \tag{4.14}$$

In particular, the function u is flat on $\overline{\mathbf{B}}_\tau(x_0)$. More precisely,

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \textit{for any } x \in \overline{\mathbf{B}}_\tau(x_0).$$

Proof. The result is a direct consequence of previous arguments. Indeed, for simplicity we can assume $x_0 = 0$. Since $g(|\mathbf{p}|) \geq 1$, by the comparison results we get that

$$0 \leq u_\alpha(x) \leq v_\tau(x), \quad x \in \mathbf{B}_{\mathrm{R}}(0)$$

(see (4.2) and (4.9)) and so the conclusions hold. $\square$

Remark 1 We have proved that under the above assumptions the flat region of u is a non–empty set. Obviously, $\mathrm{Flat}(h) \subset \mathrm{Flat}(u)$ whenever (4.11) fails, even if (4.7) holds. $\square$

Remark 2 We point out that the above result applies to the case in which $\varphi \equiv 1$ and $h \equiv 0$ (the so called "dead core" problem) as well as to cases in which u is flat only near $\partial\Omega$ (take for instance, $h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha$ in Ω and $\varphi \equiv h$ on $\partial\Omega$). $\square$

In order to study the assumption (4.7) we note that the function $\mathbb{U}(r)$ satisfies the inequality

$$\mathrm{C}_{\mathrm{N}-1,k-1}\mathbb{U}''(r)\left(\frac{\mathbb{U}'(r)}{r}\right)^{k-1} \leq \eta_{\mathbb{U}} f\big(\mathbb{U}(r)\big), \quad 0 < r < \mathrm{R}_{\mathbb{U}}, \tag{4.15}$$

(see (4.5)) whence

$$\left(\big(\mathbb{U}'(r)\big)^{k+1}\right)' \leq \eta_{\mathbb{U}}(k+1)\mathrm{C}^{-1}_{\mathrm{N}-1,k-1} r^{k-1}\left(\mathrm{F}\big(\mathbb{U}(r)\big)\right)', \quad 0 < r < \mathrm{R}_{\mathbb{U}} \quad \mathrm{F}' = f,$$

and

$$\big(\mathbb{U}'(r)\big)^{k+1} \leq \eta_{\mathbb{U}}(k+1)\mathrm{C}^{-1}_{\mathrm{N}-1,k-1} r^{k-1}\mathrm{F}\big(\mathbb{U}(r)\big), \quad 0 < r < \mathrm{R}_{\mathbb{U}}.$$

So, we deduce that (4.7) requires

$$\int_0^{\mathbb{U}(r)} \frac{ds}{\mathrm{F}(s)^{\frac{1}{k+1}}} = \int_0^r \frac{\mathbb{U}'(s)ds}{\mathrm{F}\big(\mathbb{U}(s)\big)^{\frac{1}{k+1}}} \leq \left(\eta_{\mathbb{U}}(k+1)\mathrm{C}^{-1}_{\mathrm{N}-1,k-1}\right)^{\frac{1}{k+1}} \frac{\mathrm{N}+1}{2k} r^{\frac{2k}{k+1}}$$

for $0 < r < \mathrm{R}_{\mathbb{U}}$. Therefore (1.6) is a necessary condition in order to (4.7) holds.

The reasoning in proving that (1.6) is a sufficient condition for the assumption (4.7) is very laborious and follows from some adaptations of the results of [9]. Here we only construct a function verifying a similar property for (4.15) useful to our interest

Theorem 4 *Assume* (1.6)*. Then the function* $\phi(r)$ *given implicitly by*

$$\int_0^{\phi(r)} \mathrm{F}(s)^{-\frac{1}{k+1}} ds = r^{\frac{2k-1}{k}}, \quad 0 \leq r \tag{4.16}$$

satisfies, the property

$$\begin{cases} \mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}}\left[\dfrac{r^{\mathrm{N}-k}}{k}\big(\phi'(r)\big)^k\right]' \leq \eta_{\phi,\widehat{\mathrm{R}}} f\big(\phi(r)\big), \quad 0 < r < \widehat{\mathrm{R}}, \\ \phi(0) = \phi'(0) = 0, \end{cases} \tag{4.17}$$

where

$$\begin{cases} \widehat{\mathrm{R}} < \displaystyle\int_0^\infty \mathrm{F}(s)^{-\frac{1}{k+1}}\,ds \le +\infty, \\ \eta_{\phi,\widehat{\mathrm{R}}} = \mathrm{C}_{\mathrm{N}-1,k-1}\left(\dfrac{2k-1}{k}\right)^{k+1}\dfrac{\mathrm{N}}{k+1}\widehat{\mathrm{R}}^{\frac{k-1}{k}}. \end{cases} \tag{4.18}$$

Proof. Since the function

$$\psi(t) = \int_0^t \mathrm{F}(s)^{-\frac{1}{k+1}}\,ds, \quad t \ge 0,$$

is increasing from $\overline{\mathbb{R}}_+$ to $[0, \psi(\infty)[$ and $\psi(0) = 0$, we may consider the function $\phi(r)$ given implicitly by

$$\int_0^{\phi(r)} \mathrm{F}(s)^{-\frac{1}{k+1}}\,ds = r^a, \quad 0 \le r < \psi(\infty) \le +\infty,$$

where a is a positive constant to be chosen. Then

$$\phi'(r) = a\mathrm{F}\big(\phi(r)\big)^{\frac{1}{k+1}} r^{a-1},$$

and

$$\begin{aligned} r^{1-\mathrm{N}}\left[\frac{r^{\mathrm{N}-k}}{k}\big(\phi'(r)\big)^k\right]' &= \frac{a^k r^{1-\mathrm{N}}}{k}\left[r^{\mathrm{N}-2k+ka}\mathrm{F}\big(\phi(r)\big)^{\frac{k}{k+1}}\right]' \\ &= \frac{a^k r^{1-2k+ka}}{k}\left[\frac{\mathrm{N}-2k+ka}{r}\mathrm{F}\big(\phi(r)\big)^{\frac{k}{k+1}} + \frac{ak}{k+1}r^{a-1}f\big(\phi(r)\big)\right] \end{aligned}$$

hold. Next, we choose

$$a = \frac{2k-1}{k},$$

and $\Phi(r) = \big(\mathrm{F}\big(\phi(r)\big)\big)^{\frac{k}{k+1}}$. Since $\Phi(0) = 0$ and

$$\Phi'(r) = \frac{2k-1}{k+1} f\big(\phi(r)\big) r^{\frac{k-1}{k}}$$

is increasing, the convexity inequality

$$\Phi(r) \le \Phi'(r) r$$

gives

$$\mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}}\left[\frac{r^{\mathrm{N}-k}}{k}\big(\phi'(r)\big)^k\right]' \le \mathrm{C}_{\mathrm{N}-1,k-1}\left(\frac{2k-1}{k}\right)^{k+1}\frac{\mathrm{N}}{k+1} r^{\frac{k-1}{k}} f\big(\phi(r)\big).$$

Finally, since $a \geq 1$ one has $\phi(0) = \phi'(0) = 0$. □

So that, fixed $\widehat{\mathrm{R}} < \psi(\infty)$ we have

$$\mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}} \left[\frac{r^{\mathrm{N}-k}}{k} \left(\phi'(\mathrm{A}r)\right)^k \right]' + \eta f\big(\phi(\mathrm{A}r)\big) \geq \left[\eta - \eta_{\phi,\widehat{\mathrm{R}}} \mathrm{A}^{2k} \right] f\big(\phi(\mathrm{A}r)\big), \quad (4.19)$$

for $0 < r < \widehat{\mathrm{R}}$ and $\phi(0) = \phi'(0) = 0$ (see (4.8)), whence the function

$$v_\tau(x) \doteq \phi\left(\left(\frac{\eta}{\eta_{\phi,\widehat{\mathrm{R}}}} \right)^{\frac{1}{2k}} [|x| - \tau]_+ \right), \quad x \in \mathbf{B}_{\tau+\mathrm{R}_{\phi,\eta}}(0), \quad (4.20)$$

solves

$$-\mathcal{S}_k[\lambda(\mathrm{D}^2 v_\tau(x)] + \eta f\big(v_\tau(x)\big) \geq 0, \quad x \in \mathbf{B}_{\tau+\mathrm{R}_{\mathbb{U}_{\phi,\eta,\widehat{\mathrm{R}}}}}(0),$$

for

$$\mathrm{R}_{\phi,\eta,\widehat{\mathrm{R}}} = \left(\frac{\eta_{\phi,\widehat{\mathrm{R}}}}{\eta} \right)^{\frac{1}{2k}} \widehat{\mathrm{R}}.$$

The reasonings of Theorem 3 apply and enable us to localize again a core of the flat region $\mathrm{Flat}(u)$ but now using the function ϕ given by (4.16) instead to use the function $\mathbb{U}$ given by (4.7).

Corollary 1 *Let h be locally k–admissible on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_\mathrm{R}(x_0) \subset \mathrm{Flat}_\alpha(h)$ with*

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x \rangle + a_\alpha\big) \leq \mathrm{M} \leq \max_{\overline{\Omega}}(u - h), \quad x \in \partial \mathbf{B}_\mathrm{R}(x_0), \quad (4.21)$$

where u is a generalized solution of (1.2), for some $\mathrm{M} > 0$. Then, if (1.6) holds and

$$\eta \geq \eta_* \doteq \eta_{\phi,\widehat{\mathrm{R}}} \left(\frac{1}{\mathrm{R}} \phi^{-1}(\mathrm{M}) \right)^{2\mathrm{N}},$$

one verifies

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x \rangle + a_\alpha\big) \leq \phi\left(\left(\frac{\eta}{\eta_{\phi,\widehat{\mathrm{R}}}} \right)^{\frac{1}{2k}} [|x| - \tau]_+ \right), \quad x \in \mathbf{B}_\mathrm{R}(x_0), \quad (4.22)$$

where

$$\tau = \left[\eta_*^{-\frac{1}{2k}} - \eta^{-\frac{1}{2k}} \right] \phi^{-1}(\mathrm{M}) \eta_{\phi,\widehat{\mathrm{R}}}^{\frac{1}{2k}}, \quad (4.23)$$

once we assume that $\mathrm{R} < \tau + \mathrm{R}_{\phi,\eta,\widehat{\mathrm{R}}}$ and

$$\left(\frac{\eta_{\phi,\widehat{\mathrm{R}}}}{\eta} \right)^{\frac{1}{2k}} \phi^{-1}(\mathrm{M}) < \mathrm{R} \leq \mathrm{dist}(x_0, \partial\Omega). \quad (4.24)$$

In particular, the function u is flat on $\overline{\mathbf{B}}_\tau(x_0)$. More precisely,

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \textit{for any } x \in \overline{\mathbf{B}}_\tau(x_0).$$

□

Remark 3 Corollary 1 is the relative version of Theorem 3. Consequently, the comments of Remarks 1 and 2 apply. □

In the particular case $f(t) = t^{\mathrm{q}}$, the condition (1.6) holds if and only if $k > \mathrm{q}$. Moreover, the assumption (4.7) is verified for

$$\mathbb{U}_{\mathrm{q},k}(r) = r^{\frac{2k}{k-\mathrm{q}}}, \quad \eta_{\mathrm{q},k} = \mathrm{C}_{\mathrm{N}-1,k-1}\left(\frac{2k}{k-\mathrm{q}}\right)^k \frac{2k\mathrm{q}+\mathrm{N}(k-\mathrm{q})}{k(k-\mathrm{q})}, \quad r \geq 0, \tag{4.25}$$

consequently all above results apply. If we scale by $\mathrm{A}^{\frac{k-\mathrm{q}}{2k}}$ for the function

$$\mathrm{U}(r) = \mathrm{A}\mathbb{U}_{\mathrm{q},k}(r), \quad r \geq 0,$$

the property (4.8) becomes

$$-\mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}}\left[\frac{r^{\mathrm{N}-k}}{k}\big(\mathrm{U}'(r)\big)^k\right]' + \eta \mathrm{U}(r)^{\mathrm{q}} = \eta\left[1 - \frac{\eta_{\mathrm{q},k}}{\eta}\mathrm{A}^{k-\mathrm{q}}\right]\mathrm{U}(r)^{\mathrm{q}} \tag{4.26}$$

fot $r > 0$. Now,

1. if $\mathrm{A} < \left(\frac{\eta}{\eta_{\mathrm{q},k}}\right)^{\frac{1}{k-\mathrm{q}}}$ the function $\mathrm{U}(r)$ is a super–solution of the equation (4.26),
2. if $\mathrm{A} = \left(\frac{\eta}{\eta_{\mathrm{q},k}}\right)^{\frac{1}{k-\mathrm{q}}}$ the function $\mathrm{U}(r)$ is the solution of the equation (4.26),
3. if $\mathrm{A} > \left(\frac{\eta}{\eta_{\mathrm{q},k}}\right)^{\frac{1}{k-\mathrm{q}}}$ the function $\mathrm{U}(r)$ is a sub–solution of the equation (4.26).

So that, the particular choice

$$\mathrm{U}(r) = \left(\frac{\eta}{\eta_{\mathrm{q},k}}\right)^{\frac{1}{k-\mathrm{q}}} \mathbb{U}_{\mathrm{q},k}(r), \quad r \geq 0, \tag{4.27}$$

enables us to construct the function

$$v_\tau(x) \doteq \mathrm{U}\big([|x| - \tau]_+\big), \quad x \in \mathbb{R}^{\mathrm{N}}, \tag{4.28}$$

vanishing in a ball $\mathbf{B}_\tau(0)$ and solving

$$-\mathcal{S}_k[\lambda(\mathrm{D}^2 v_\tau(x))] + \eta v_\tau(x)^{\mathrm{q}} = 0, \quad x \in \mathbb{R}^{\mathrm{N}}.$$

Moreover, given $\mathrm{M} > 0$, it verifies

$$v_\tau(x) = \mathrm{M}, \quad |x| = \mathrm{R}$$

once we take

$$\tau = \mathrm{R} - \mathrm{U}^{-1}(\mathrm{M}) = \eta_{\mathrm{q},k}^{\frac{1}{2k}} \mathrm{M}^{\frac{k-\mathrm{q}}{2k}} \left[\eta_*^{-\frac{1}{2k}} - \eta^{-\frac{1}{2k}}\right]$$

with

$$\eta \geq \eta_* \doteq \frac{\eta_{\mathrm{q},k}\mathrm{M}^{k-\mathrm{q}}}{\mathrm{R}^{2k}}.$$

The localization of a core of the flat region $\mathrm{Flat}(u)$ inside the flat subregion $\mathrm{Flat}_\alpha(h)$ of the "obstacle" is estimated by

Theorem 5 *Let $f(t) = t^{\mathrm{q}}$, $\mathrm{q} < k$. Let h be locally k–admissible on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_\mathrm{R}(x_0) \subset \mathrm{Flat}_\alpha(h)$ with*

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathrm{M} \leq \max_{\overline{\Omega}}(u - h), \quad x \in \partial \mathbf{B}_\mathrm{R}(x_0), \tag{4.29}$$

where u is a generalized solution of (1.2), for some $\mathrm{M} > 0$. Then, if $k > \mathrm{q}$ and

$$\eta \geq \eta_* \doteq \frac{\eta_{\mathrm{q},k}\mathrm{M}^{k-\mathrm{q}}}{\mathrm{R}^{2k}}. \tag{4.30}$$

one verifies

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \left(\frac{\eta}{\eta_{\mathrm{q},k}}\right)^{\frac{1}{k-\mathrm{q}}} \big[|x - x_0| - \tau\big]_+^{\frac{2k}{k-\mathrm{q}}}, \quad x \in \mathbf{B}_\mathrm{R}(x_0), \tag{4.31}$$

where

$$\tau = \eta_{\mathrm{q},k}^{\frac{1}{2k}} \mathrm{M}^{\frac{k-\mathrm{q}}{2k}} \left[\eta_*^{-\frac{1}{2k}} - \eta^{-\frac{1}{2k}}\right] \tag{4.32}$$

provided

$$\left(\frac{\eta_{\mathrm{q},k}}{\eta}\right)^{\frac{1}{2k}} \mathrm{M}^{\frac{k-\mathrm{q}}{2k}} < \mathrm{R} \leq \mathrm{dist}(x_0, \partial\Omega). \tag{4.33}$$

In particular, the function u is flat on $\overline{\mathbf{B}}_\tau(x_0)$. More precisely,

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \textit{for any } x \in \overline{\mathbf{B}}_\tau(x_0).$$

□

Remark 4 Theorem 5 is a new version of Theorem 3 but now with more explicit data. Therefore, once more the comments of Remarks 1 and 2 apply also to this power like case $f(t) = t^{\mathrm{q}}$, $k > \mathrm{q}$. □

Theorem 5 gives some estimates on the localization of the points inside $\mathrm{Flat}(h)$ where u becomes flat too. The following result shows that if h decays in a suitable way at the boundary points of $\mathrm{Flat}(h)$ then the solution u becomes also flat in those points of the boundary of $\mathrm{Flat}(h)$. In this result the parameter η is irrelevant, therefore with no loss of generality we shall assume that $\eta = 1$.

Theorem 6 *Let $f(t) = t^{\mathrm{q}}$, $\mathrm{q} < k$. Let $x_0 \in \partial \mathrm{Flat}_\alpha(h)$ such that*

$$h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathrm{K}|x - x_0|^{\frac{2k}{k-\mathrm{q}}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0) \cap \big(\mathbb{R}^{\mathrm{N}} \setminus \mathrm{Flat}(h)\big), \tag{4.34}$$

and

$$0 \leq \max_{|x-x_0|=\mathrm{R}} \big\{u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big\} \leq \mathrm{A}\mathrm{R}^{\frac{2k}{k-\mathrm{q}}} \tag{4.35}$$

for some suitable positive constants K *and* A (*see* (4.37) below) *and u is a generalized solution of* (1.2)*. Then*

$$u(x_0) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha. \tag{4.36}$$

Proof. Define the function

$$\mathrm{V}(x) = u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big),$$

which by construction is non–negative in $\partial \mathbf{B}_{\mathrm{R}}(x_0)$ (see (4.35)). In fact, the Weak Maximum Principle implies that V is non–negative on $\overline{\mathbf{B}}_{\mathrm{R}}(x_0)$. Then

$$\begin{aligned}
-\big(\mathcal{S}_k[\lambda(\mathrm{D}^2\mathrm{V}(x))]\big)^{\frac{1}{k}} + \mathrm{V}(x)^{\frac{\mathrm{q}}{k}} &= -\big(\mathcal{S}_k[\lambda(\mathrm{D}^2 u(x))]\big)^{\frac{1}{k}} + \big(u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big)^{\frac{\mathrm{q}}{k}} \\
&= -\big(u(x) - h(x)\big)^{\frac{\mathrm{q}}{k}} + \big(u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big)^{\frac{\mathrm{q}}{k}} \\
&\leq \big(h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big)^{\frac{\mathrm{q}}{k}} \\
&\leq \mathrm{K}^{\frac{\mathrm{q}}{k}}|x - x_0|^{\frac{2k}{k-\mathrm{q}}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0),
\end{aligned}$$

where we have used a kind of Minkovsky inequality

$$(a+b)^{\frac{1}{\mathrm{p}}} \leq a^{\frac{1}{\mathrm{p}}} + b^{\frac{1}{\mathrm{p}}}, \text{ for any } a, b \geq 0, \text{ where } \mathrm{p} > 1,$$

for the special case $\mathrm{p} = \frac{k}{\mathrm{q}} > 1$, as well as (4.34). On the other hand, from (4.25) we have

$$\left(\mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}} \left[\frac{r^{\mathrm{N}-k}}{k}\big(\mathbb{U}'_{\mathrm{q},k}(r)\big)^k\right]'\right)^{\frac{1}{k}} = \eta_{\mathrm{q},k}^{\frac{1}{\mathrm{N}}} \mathbb{U}_{\mathrm{q},k}(r)^{\frac{\mathrm{q}}{\mathrm{N}}}, \quad 0 < r < \mathrm{R}_{\eta_{\mathrm{q},k}},$$

for

$$\mathbb{U}_{\mathrm{q},k}(r) = r^{\frac{2k}{k-\mathrm{q}}}, \quad \eta_{\mathrm{q},k} = \mathrm{C}_{\mathrm{N}-1,k-1}\left(\frac{2k}{k-\mathrm{q}}\right)^k \left(\frac{2k\mathrm{q} + \mathrm{N}(k-\mathrm{q})}{k(k-\mathrm{q})}\right), \quad \mathrm{R}_{\mathrm{q},k} = +\infty.$$

Then since the function $\mathrm{U}(r) = \mathrm{A}\mathbb{U}_{\mathrm{q},k}(r)$ verifies

$$-\mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}} \left[\frac{r^{\mathrm{N}-k}}{k} \big(\mathrm{U}'(r) \big)^k \right]' + \mathrm{U}(r)^{\mathrm{q}} = \left[1 - \eta_{\mathrm{q},k} \mathrm{A}^{k-\mathrm{q}} \right] \mathrm{U}(r)^{\mathrm{q}}, \quad 0 < r.$$

we may take $\mathrm{A} < \eta_{\mathrm{q},k}^{-\frac{1}{k-\mathrm{q}}}$ and then K such that

$$\mathrm{K}^{\frac{\mathrm{q}}{k}} \le \mathrm{A}^{\frac{\mathrm{q}}{k}} \left[1 - \eta_{\mathrm{q},k} \mathrm{A}^{k-\mathrm{q}} \right]. \tag{4.37}$$

Then we obtain

$$-\big(\mathcal{S}_k[\lambda(\mathrm{D}^2\mathrm{V}(x)]\big)^{\frac{1}{k}} + \big(\mathrm{V}(x)\big)^{\frac{\mathrm{q}}{k}} \le -\big(\mathcal{S}_k[\lambda(\mathrm{D}^2\mathrm{U}(|x|)]\big)^{\frac{1}{k}} + \mathrm{U}(|x|)^{\frac{\mathrm{q}}{k}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0).$$

Finally, by choosing R satisfying (4.35) one has

$$\mathrm{V}(x) \le \mathrm{U}(|x|), \quad x \in \partial \mathbf{B}_{\mathrm{R}}(x_0),$$

whence the comparison principle concludes

$$0 \le \mathrm{V}(x) \le \mathrm{A}|x - x_0|^{\frac{2k}{k-\mathrm{q}}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0),$$

and so $u(x_0) = \big(\langle \mathbf{p}_\alpha, x_0 \rangle + a_\alpha \big)$. □

Remark 5 The assumption (4.35) is satisfied if we know that the ball $\mathbf{B}_{\mathrm{R}}(x_0)$ where (4.34) holds is assumed large enough. The above result is motivated by [9, Theorem 2.5]. By adapting the reasoning used in previous results of the literature (see [8, Remark 10]) it can be shown that the decay of $h(x) - \big(\langle \mathbf{p}_\alpha, x \rangle + a_\alpha \big)$ near the boundary point x_0 is optimal in the sense that if

$$h(x) - \big(\langle \mathbf{p}_\alpha, x \rangle + a_\alpha \big) > \mathrm{A}|x - x_0|^{\frac{2k}{k-\mathrm{q}}} \quad \text{on a neighbourhood of } x_0$$

then it can be shown that

$$u(x_0) - \big(\langle \mathbf{p}_\alpha, x_0 \rangle + a_\alpha) \big) > \mathrm{A}|x - x_0|^{\frac{2k}{k-\mathrm{q}}} \quad \text{for } x \text{ near } x_0.$$

This type of results gives very rich information on the non–degeneracy behavior of the solution near the free boundary. This is very useful to the study of the continuous dependence of the free boundary with respect to the data h and φ (see [10]). □

5. Unflat solutions

The case where the free boundary can not appear (even if a priori the diffusion operator is degenerate) is examined here. Independent on the size of the domain, it requires the condition

$$k \le \mathrm{q} \qquad \text{for } f(t) = t^{\mathrm{q}}$$

or the more general assumption (1.7). We shall obtain here a version of the Strong Maximum Principles inspired on the classical reasoning by E. Höpf (see *e.g.* [12, 16, 20]). Among other consequences, we shall deduce that the solution can not be flat. Again, since the parameter η is irrelevant, in this section, with no loss of generality, we assume $\eta = 1$. So, we begin with

Lemma 5.1 (Höpf boundary point lemma) *Assume (1.7). Let u be a non–negative viscosity solution of*

$$-\mathcal{S}_k[\lambda(\mathrm{D}^2 u)] + f(u) \geq 0 \quad \textit{in } \Omega.$$

Let $x_0 \in \partial\Omega$ be such that $u(x_0) \doteq \liminf\limits_{\substack{x \to x_0 \\ x \in \Omega}} u(x)$ and

$$\begin{cases} \textit{i)} & u \textit{ achieves a strict minimum on } \Omega \cup \{x_0\}, \\ \textit{ii)} & \exists\ \mathbf{B}_{\mathrm{R}}(x_0 - \mathrm{R}\mathbf{n}(x_0)) \subset \Omega \quad (\partial\Omega \textit{ satisfies an interior sphere condition at } x_0). \end{cases}$$

Then there exists a positive constant C such that

$$\liminf_{\tau \to 0} \frac{u(x_0 - \tau\mathbf{n})}{\tau} \geq C > 0, \tag{5.1}$$

where $\mathbf{n}$ stands for the outer normal unit vector of $\partial\Omega$ at x_0.

Proof. Let $y = x_0 - \mathrm{R}\mathbf{n}(x_0)$ and $\mathbf{B}_{\mathrm{R}} \doteq \mathbf{B}_{\mathrm{R}}(y)$. As it was pointed out before, the equation (1.2) leads to the study of the differential equation

$$\mathrm{C}_{\mathrm{N}-1,k-1} \left[\frac{r^{\mathrm{N}-k}}{k} \big(\Phi'(r)\big)^k \right]' = f\big(\Phi(r)\big), \quad r > 0,$$

for radially symmetric solutions. We consider now the classical solution of the boundary value problem

$$\begin{cases} \mathrm{C}_{\mathrm{N}-1,k-1} r^{1-\mathrm{N}} \left[\dfrac{r^{\mathrm{N}-k}}{k} \big(\Phi'(r)\big)^k \right]' = f\big(\Phi(r)\big), \quad 0 < r < \dfrac{\mathrm{R}}{2}, \\ \Phi(0) = 0, \quad \Phi\left(\dfrac{\mathrm{R}}{2}\right) = \Phi_1 > 0. \end{cases} \tag{5.2}$$

The existence of solution follows from standard arguments and the uniqueness of solution can be proved as in Theorem 2, whence

$$\Phi'(0) \geq 0 \quad \Rightarrow \quad \Phi'(r) > 0 \quad \Rightarrow \quad \Phi''(r) > 0.$$

Then

$$0 \leq \Phi(r) \leq \Phi_1, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

Obviously, the singularity at $r = 0$ must be removed since we have

$$\lim_{r \to 0} r^{1-\mathrm{N}} \left[r^{\mathrm{N}-k} \big(\Phi'(r)\big)^k \right]' = 0. \tag{5.3}$$

Let r_0 be the largest r for which $\Phi(r) = 0$. We want to prove that $r_0 = 0$ by proving that $r_0 > 0$ leads to a contradiction. In order to do that we note

$$\mathrm{C}_{\mathrm{N}-1,k-1}\Phi''(r)\left(\frac{\Phi'(r)}{r}\right)^{k-1} \leq f\big(\Phi(r)\big), \quad 0 < r < \frac{\mathrm{R}}{2}$$

(see (4.5)). So, we multiply by $r^{k-1}\Phi'(r)$ to get

$$\Big[\big(\Phi'(r)\big)^{k+1}\Big]' \leq \mathrm{C}^{-1}_{\mathrm{N}-1,k-1}(k+1) f\big(\Phi(r)\big)\Phi'(r) r^{k-1}, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

Next, since $\Phi'(r_0) = 0 = \Phi(r_0)$, an integration between r_0 and r leads to

$$\begin{aligned}\big(\Phi'(r)\big)^{k+1} &\leq \mathrm{C}^{-1}_{\mathrm{N}-1,k-1}(k+1)\mathrm{F}\big(\Phi(r)\big) r^{k-1} \\ &\quad -\mathrm{C}^{-1}_{\mathrm{N}-1,k-1}(k+1)(k-1)\int_{r_0}^{r} \mathrm{F}\big(\Phi(s)\big) r^{k-2} ds \\ &\leq \mathrm{C}^{-1}_{\mathrm{N}-1,k-1}(k+1)\mathrm{F}\big(\Phi(r)\big) r^{k-1}, \quad r_0 < r < \frac{\mathrm{R}}{2}.\end{aligned}$$

Since we assume (1.7), a new integration between r_0 and $\dfrac{\mathrm{R}}{2}$ yields

$$\begin{aligned}\infty = \int_0^{\Phi_1} \frac{ds}{\mathrm{F}(s)^{\frac{1}{k+1}}} &= \int_{r_0}^{\frac{\mathrm{R}}{2}} \frac{\Phi'(r)}{\mathrm{F}\big(\Phi(r)\big)^{\frac{1}{k+1}}} dr \\ &\leq \Big(\mathrm{C}^{-1}_{\mathrm{N}-1,k-1}(k+1)\Big)^{\frac{1}{k+1}} \int_{r_0}^{\frac{\mathrm{R}}{2}} r^{\frac{k-1}{k+1}} dr < \infty\end{aligned}$$

and the conjectured contradiction follows. So that, we have proved $\Phi'(0) > 0$ and also

$$0 < \Phi(r) < \Phi_1, \;\; \Phi'(r) > 0, \quad 0 < r < \frac{\mathrm{R}}{2},$$

as well as $\Phi''(0) = 0$ (see (5.3)). Hence, straightforward computations on the $\mathcal{C}^2$ convex function $w(x) = \Phi(\mathrm{R} - |x - y|)$, defined in the annulus $\mathcal{O} \doteq \mathbf{B}_{\mathrm{R}} \setminus \overline{\mathbf{B}}_{\frac{\mathrm{R}}{2}}$, prove

$$\begin{cases} \mathcal{S}_k[\lambda\big(\mathrm{D}^2 w(x)\big)] = f\big(w(x)\big), & x \in \mathcal{O}, \\ w(x) = \Phi_1, \quad x \in \partial\mathbf{B}_{\frac{\mathrm{R}}{2}}, \\ w(x) = 0, \quad x \in \partial\mathbf{B}_{\mathrm{R}}. \end{cases}$$

Moreover, by construction

$$u(x) > 0, \quad x \in \partial\mathbf{B}_{\frac{\mathrm{R}}{2}} \quad \Rightarrow \quad u(x) \geq w(x), \quad x \in \partial\mathbf{B}_{\mathrm{R}},$$

for Φ_1 small enough. Then the Weak Maximum Principle of Theorem 2 implies

$$(u - w)(x) \geq 0, \quad x \in \overline{\mathcal{O}}.$$

This leads to
$$\frac{u(x_0-\tau\mathbf{n})}{\tau} \geq \frac{\Phi(\mathrm{R}-\mathrm{R}(1-\tau))}{\tau}, \qquad (\tau \ll 1)$$
whence
$$\liminf_{\tau\to 0} \frac{u(x_0-\tau\mathbf{n})}{\tau} \geq \Phi'(0) > 0.$$
□

Remark 6 In fact, the above result implies
$$\liminf_{\substack{x\to x_0\\ x\in\Omega}} \frac{u(x)}{|x-x_0|} \geq \Phi'(0) > 0.$$
□

If $\partial\Omega$ satisfies an uniform interior sphere condition one has
$$u(x) \geq \Phi'(0)\mathrm{dist}(x,\partial\Omega) \quad \text{near } \partial\Omega.$$
In particular, for $\Omega = \mathbf{B}_\mathrm{R}(y_0)$ we have
$$u(x) \geq \Phi'(0)(\mathrm{R}-|x-y_0|), \quad x \in \overline{\mathbf{B}}_\mathrm{R}(y_0). \tag{5.4}$$
This property can be improved

Theorem 7 (Locally lower bound) *Assume* (1.7)*. Let u a positive viscosity solution u of*
$$-\mathcal{S}_k\left[\lambda\left(\mathrm{D}^2u\right)\right] + f(u) \geq 0 \quad \textit{in } \Omega.$$
Then, for each compact subset $\mathcal{K} \subset\subset \Omega$ there exists a positive constant $c_\mathcal{K}$ such that
$$u(x) \geq c_\mathcal{K}, \quad x \in \mathcal{K},$$

Proof. From the property (5.4) the conclusion holds for $\mathcal{K} \subset \mathbf{B}_\mathrm{R}(y_0) \subset\subset \Omega$. Next, the reasoning applies to every ball intersecting $\mathbf{B}_\mathrm{R}(y_0)$ and then to every ball that intersects one of those balls and so on. Finally, the conclusion holds for any compact $\mathcal{K} \subset\subset \Omega$, by means of suitable finite covering. □

Our main result proving the absence of the free boundary is the following

Theorem 8 (Höpf Strong Maximum Principle) *Assume* (1.7)*. Let u be a nonnegative viscosity solution of*
$$-\mathcal{S}_k[\lambda(\mathrm{D}^2u)] + f(u) \geq 0 \quad \textit{in } \Omega.$$
Then u can not vanish at some $x_0 \in \Omega$ unless u is constant in a neighborhood of x_0.

Proof. Assume that u is non–constant and achieves the minimum value $u(x_0) = 0$ on some ball $\mathbf{B} \subset \Omega$. Then we consider the semi-concave approximation of u, *i.e.*

$$u^\varepsilon(x) \doteq \inf_{y\in\Omega} \left\{ u(y) + \frac{|x-y|^2}{2\varepsilon^2} \right\}, \quad x \in \mathbf{B}_\varepsilon \qquad (\varepsilon > 0), \tag{5.5}$$

where $\mathbf{B}_\varepsilon \doteq \{x \in \mathbf{B} : \operatorname{dist}(x, \partial\mathbf{B}) > \varepsilon\sqrt{1 + 4\sup_{\mathbf{B}} |u|}\}$. For ε small enough we can assume $x_0 \in \mathbf{B}_\varepsilon$. Then u^ε achieves the minimum value in $\mathbf{B}_\varepsilon$, with $u(x_0) = u^\varepsilon(x_0) = 0$. Moreover, from well known reasoning for general fully nonlinear equations (see, for instance [19, Proposition 2.3] or [1], [6]) one deduces that u^ε satisfies

$$-\mathcal{S}_k[\lambda(\mathrm{D}^2 u_\varepsilon)] + f\big(u_\varepsilon\big) \geq 0 \quad \text{on } \mathbf{B}_\varepsilon. \tag{5.6}$$

Denote

$$\mathbf{B}_\varepsilon^+ \doteq \{x \in \mathbf{B}_\varepsilon : \ u^\varepsilon(x) > 0\},$$

by geometric classical arguments there exists the largest ball $\mathbf{B}_{\mathrm{R}}(y) \subset \mathbf{B}_\varepsilon^+$ (see [12]). Certainly there exists some $z_0 \in \partial\mathbf{B}_{\mathrm{R}}(y) \cap \mathbf{B}_\varepsilon$ for which $u^\varepsilon(z_0) = 0$ is a local minimum. Then, Lemma 5.1 implies

$$\mathrm{D}u^\varepsilon(z_0) \neq \mathbf{0}$$

contrary to

$$\mathrm{D}u^\varepsilon(z_0) = \mathbf{0}, \tag{5.7}$$

(see Lemma 5.2 below). Therefore, u^ε is constant on $\mathbf{B} \subset \Omega$, *i.e.*

$$u^\varepsilon(y) = u^\varepsilon(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

Finally, for every $y \in \mathbf{B}$ we denote by $\widehat{y}$ the point of Ω such that

$$u^\varepsilon(y) = u(\widehat{y}) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2$$

whence

$$u(x_0) = u^\varepsilon(x_0) = u^\varepsilon(y) = u(y) + \frac{1}{2\varepsilon^2}|y-\widehat{y}|^2 \geq u(x_0) + \frac{1}{2\varepsilon^2}|y-\widehat{y}|^2 \geq u(x_0) \Rightarrow \widehat{y} = y.$$

So that, one concludes

$$u(y) = u^\varepsilon(y) = u^\varepsilon(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

□

Corollary 2 *Assume* (1.7)*. Let u be a generalized solution u of* (1.2)*. Then if $u(x_0) > h(x_0)$ or $\mathcal{S}_k[\lambda(\mathrm{D}^2 h(x_0))] > 0$ at some point x_0 of a ball $\overline{\mathbf{B}} \subseteq \overline{\Omega}$ then $u > h$ on $\overline{\mathbf{B}}$, consequently the equation* (1.2) *is elliptic in $\overline{\mathbf{B}}$. In particular, if $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\mathcal{S}_k[\lambda(\mathrm{D}^2 h(x_0))] > 0$ at some point $x_0 \in \Omega$ the problem* (3.6) *is elliptic non degenerate in path-connected open sets Ω, provided the compatibility condition* (1.4) *holds.*

Proof. From Theorem 8, both cases imply $u > h$ on $\overline{\mathbf{B}}$. Finally, a continuity argument concludes the proof. □

We end this section by proving property (5.7) used in the proof of Theorem 8

Lemma 5.2 *Let ψ be a function achieving a local minimum at some $z_0 \in \mathcal{O}$. Assume that there exists a function $\widehat{\psi}$ defined in $\mathcal{O}$ such that $\widehat{\psi}(z_0) = 0$, $\Psi = \psi + \widehat{\psi}$ is concave on $\mathcal{O}$ and*

$$\widehat{\psi}(x) \geq -\mathrm{K}|x - z_0|^2, \quad x \in \mathcal{O} \text{ with } |x - z_0| \text{ small,}$$

for some constant $\mathrm{K} > 0$. Then the function ψ is differentiable at z_0 and $\mathrm{D}\psi(z_0) = \mathbf{0}$.

Proof. By simplicity we can take $z_0 = 0 \in \mathcal{O}$. By applying the convex separation theorem there exists $\mathbf{p} \in \mathbb{R}^{\mathrm{N}}$ such that

$$\Psi(x) \leq \Psi(0) + \langle \mathbf{p}, x \rangle = \psi(0) + \langle \mathbf{p}, x \rangle, \quad x \in \mathcal{O}, \text{ with } |x| \text{ small.}$$

Then we have

$$\begin{aligned} \psi(x) &= \Psi(x) - \widehat{\psi}(x) \leq \psi(0) + \langle \mathbf{p}, x \rangle + \mathrm{K}|x|^2 \\ &\leq \psi(x) + \langle \mathbf{p}, x \rangle + \mathrm{K}|x|^2, \quad x \in \mathcal{O} \text{ with } |x| \text{ small} \end{aligned} \tag{5.8}$$

whence

$$-\langle \mathbf{p}, x \rangle \leq \mathrm{K}|x|^2, \quad x \in \mathcal{O} \text{ with } |x| \text{ small.}$$

For $\tau > 0$ small enough we can choose $x = -\tau\mathbf{p} \in \mathcal{O}$ and $\tau\mathrm{K} < 1$, for which

$$\tau|\mathbf{p}|^2 \leq \mathrm{K}\tau^2|\mathbf{p}|^2.$$

Therefore $\mathbf{p} = 0$. Finally, (5.8) leads to

$$0 \leq \psi(x) - \psi(0) \leq \mathrm{K}|x|^2, \quad x \in \mathcal{O} \ \text{ with } |x| \text{ small,}$$

and the result follows.

Remark 5.1 The result is immediate if ψ is concave (in this case we can choose $\widehat{\psi} \equiv 0$). The convex version follows by changing ψ and $\widehat{\psi}$ by $-\psi$ and $-\widehat{\psi}$, respectively (see Remark 3.2 above). □

Note that since the function u^{ε} defined in (5.5) is semi concave, the property (5.7) holds.

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