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Stopping a Viscous Fluid by a Feedback Dissipative Field: Thermal Effects without Phase Changing

S.N. Antontsev, J.I. Díaz and H.B. de Oliveira

Dedicated to Professor V.A. Solonnikov on the occasion of his 70th birthday.

Abstract. We show how the action on two simultaneous effects (a suitable coupling about velocity and temperature and a low range of temperature but upper that the phase changing one) may be responsible of stopping a viscous fluid without any changing phase. Our model involves a system, on an unbounded pipe, given by the planar stationary Navier-Stokes equation perturbed with a sublinear term $\mathbf{f}(\mathbf{x}, \theta, \mathbf{u})$ coupled with a stationary (and possibly nonlinear) advection diffusion equation for the temperature θ.

After proving some results on the existence and uniqueness of weak solutions we apply an energy method to show that the velocity $\mathbf{u}$ vanishes for $\mathbf{x}$ large enough.

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1. Introduction

It is well known (see, for instance, [6, 8, 14]) that in phase changing flows (as the Stefan problem) usually the solid region is assumed to remain static and so we can understand the final situation in the following way: the thermal effect are able to stop a viscous fluid.

The main contribution of this paper is to show how the action on two simultaneous effects (a suitable coupling about velocity and temperature and a low range of temperature but upper the phase changing one) may be responsible of stopping a viscous fluid without any changing phase. This philosophy could be useful in the monitoring of many flows problems, specially in metallurgy.

We shall consider a, non-standard, Boussinesq coupling among the temperature θ and the velocity $\mathbf{u}$. Motivated by our previous works (see [1, 2, 3, 4]), we assume the body force field is given in a non-linear feedback form, $\mathbf{f} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f} = (f_1(\mathbf{x}, \theta, \mathbf{u}), f_2(\mathbf{x}, \theta, \mathbf{u}))$, where $\mathbf{f}$ is a Carathéodory function (*i.e.*, continuous on θ and $\mathbf{u}$ and measurable in $\mathbf{x}$) such that, for every $\mathbf{u} \in \mathbb{R}^2$, $\mathbf{u} = (u, v)$, for any $\theta \in [m, M]$, and for almost all $\mathbf{x} \in \Omega$

$$-\mathbf{f}(\mathbf{x}, \theta, \mathbf{u}) \cdot \mathbf{u} \geq \delta\, \chi_{\mathbf{f}}(\mathbf{x})\, |u|^{1+\sigma(\theta)} - g(\mathbf{x}, \theta) \tag{1.1}$$

for some $\delta > 0$, σ a Lipschitz continuous function such that

$$0 < \sigma^- \leq \sigma(\theta) \leq \sigma^+ < 1, \quad \theta \in [m, M], \tag{1.2}$$

and

$$g \in \mathrm{L}^1\left(\Omega^{x_g} \times \mathbb{R}\right), \quad g \geq 0,\ g(\mathbf{x}, \theta) = 0 \text{ a.e. in } \Omega^{x_g} \text{ for any } \theta \in [m, M], \tag{1.3}$$

for some $x_{\mathbf{f}}$, x_g, with $0 \leq x_g < x_{\mathbf{f}} \leq \infty$ and $x_{\mathbf{f}}$ large enough, where $\Omega^{x_g} = (0, x_g) \times (0, L)$ and $\Omega_{x_g} = (x_g, \infty) \times (0, L)$. The function $\chi_{\mathbf{f}}$ denotes the characteristic function of the interval $(0, x_{\mathbf{f}})$, *i.e.*, $\chi_{\mathbf{f}}(\mathbf{x}) = 1$, if $x \in (0, x_{\mathbf{f}})$ and $\chi_{\mathbf{f}}(\mathbf{x}) = 0$, if $x \notin (0, x_{\mathbf{f}})$. We shall not need any monotone dependence assumption on the function $\sigma(\theta)$.

It seems interesting to notice that the term $\mathbf{f}(\mathbf{x}, \theta, \mathbf{u})$ plays a similar role to the one in the penalized changing phase problems (see equation (3.13) of [14]), although our formulation and our methods of proof are entirely different. We shall prove that the fluid is stopped at a finite distance of the semi-infinite strip entrance by reducing the nonlinear system to a fourth order non-linear scalar equation for which the localization of solutions is obtained by means of a suitable energy method (see [5]).

2. Statement of the problem

In the domain $\Omega = (0, \infty) \times (0, L)$, $L > 0$, we consider a planar stationary thermal flow of a fluid governed by the following system

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \triangle \mathbf{u} - \nabla p + \mathbf{f}(\mathbf{x}, \theta, \mathbf{u}), \tag{2.4}$$

$$\mathrm{div}\mathbf{u} = 0, \tag{2.5}$$

$$\mathbf{u} \cdot \nabla \mathcal{C}(\theta) = \triangle \varphi(\theta), \tag{2.6}$$

where $\mathbf{u} = (u, v)$ is the vector velocity of the fluid, θ its absolute temperature, p is the hydrostatic pressure, ν is the kinematics viscosity coefficient,

$$\mathcal{C}(\theta) := \int_{\theta_0}^{\theta} C(s)\, ds \quad \text{and} \quad \varphi(\theta) := \int_{\theta_0}^{\theta} \kappa(s)\, ds,$$

with $C(\theta)$ and $\kappa(\theta)$ being the specific heat and the conductivity, respectively. Assuming $\kappa > 0$ then φ is invertible and so $\theta = \varphi^{-1}(\overline{\theta})$ for some real argument $\overline{\theta}$. Then we can define functions

$$\overline{\mathcal{C}}(\overline{\theta}) := \mathcal{C} \circ \varphi^{-1}(\overline{\theta}), \quad \overline{\mathbf{f}}(\mathbf{x}, \overline{\theta}, \mathbf{u}) := \mathbf{f} \circ \varphi^{-1}(\overline{\theta}), \quad \overline{\mu}(\overline{\theta}) := \mu \circ \varphi^{-1}(\overline{\theta}).$$