La extinción en tiempo finito de la solución de una clase de EDPs no lineales es más lenta si la derivada temporal es fraccionaria

Extinction in finite time for some fractional nonlinear evolution problems

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Nonlinear waves, fractional calculus, numerics and noise

A conference in honor of the 60+1 birthday of Prof. Luis Vazquez

Outline of Topics

General assumptions

Let us present the mathematical treatment of a model

- arising in the theory of unsaturated filtration flow in a porous medium
- under the absorption action of some plants
- and when the constitutive law on the porosity of the medium is formulated in terms of a fractional time derivative

Let $Q_T = \Omega \times (0, T)$, $\Omega = (-L, L)$ be a general real open set, $T \in R_+$. Let $v(x,t) \in [0,1]$ represent the humidity of the soil. Then our model initial-boundary value problem for a nonlinear degenerate parabolic equation with a single space variable is formulated as follows:

$$\left\{ \begin{array}{ll} a_1 \frac{\partial}{\partial t} v + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha} v - (\gamma^{1-p} |v|^{m-1} |v_x|^{p-2} v_x)_x + \lambda v |v|^{q-1} = f(x,t) & \text{in } Q_T \\ v(\pm L, t) = 0 \qquad t \in (0,T); \qquad v(x,0) = v_0(x) & \text{in } \Omega \end{array} \right. \tag{1}$$

- $\lambda > 0$, $0 < \gamma < \infty$, $1 < p < \infty$, $m > 0$, $q > 0$, $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$

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- $\partial^\alpha / \partial t^\alpha$ is the Riemann-Liouville fractional derivative:

$$\frac{\partial}{\partial t^\alpha} u(x,t) = \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau$$

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- if $a_1 = 0$ the initial condition must be understood as follows: $\lim_{t \to 0} \Gamma(\alpha) t^{1-\alpha} u(x,t) = u_0(x)$
- Equation (1) with $\alpha = 1$ is usually referred to as the *nonlinear heat equation with absorption*

Constitutive Terms

- *accumulation term*: it is usually described by the term

$$a_1(x,t)\frac{\partial}{\partial t}v$$

where $a_1(x,t)$ depends on the characteristics of the porous medium.

Constitutive Terms

- *accumulation term*: here it is replaced by [a]

$$a_1 \frac{\partial}{\partial t} v + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha} v.$$

The non locality of the fractional derivative operator tries to reproduce the complexity of the medium.

[a] C. M. Case. *Physical Principles of Flow in Unsaturated Porous Media*. Clarendon Press, Oxford, 1994.

Constitutive Terms

- *diffusion*: it is usually expressed by the term

$$-\mathrm{div}\,(K(x,t)\nabla\varphi(v))$$

  - where $K(x,t)$ is the tensor of absolute permeability and characterizes the porous medium. Here we assume, for simplicity, $K = \mathit{Identity}$.

Constitutive Terms

- *diffusion*: it is usually expressed by the term

$$-\mathrm{div}\,(K(x,t)\nabla\varphi(v))$$

  - when the velocity of the propagation of the flow trough the porous media is *slow*, then $\varphi(v) = v^2$ (and $m = p = 2$ in our model).

Constitutive Terms

- *diffusion*: it is usually expressed by the term

$$-\mathrm{div}\,(K(x,t)\nabla\varphi(v))$$

  - when such a flux is *turbulent* [a], then the diffusion term is replaced by the p-Laplacian:

$$-\mathrm{div}\,(K(x,t)|v|^{m-1}|\nabla v|^{p-2}\nabla v)$$

    where it is generally assumed that $m > 0$ and $p > 1$.

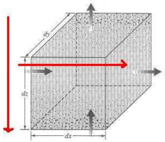
[a] J.I. Díaz, F. De Thelin, *On a nonlinear parabolic problem arising in some models related to turbulent flows*, SIAM J. Math. Anal. Vol.25, No. 4, 1085-1111, 1994

Constitutive Terms

- *convection*: is the effect of the gravity on the penetration of the flow trough the media and it is described by a derivative term of the first order along the the vertical direction z

$$\frac{\partial}{\partial z}\psi(v)$$

Here this term is omitted as we are restricted to consider just the ground layer (e.g., of an agricultural surface).



Constitutive Terms

- *absorption*: here the term $\lambda v|v|^{q-1}$, $q > 0$, represents the absorption phenomenon due to the plants on the ground layer[a]. It is a mechanism which reduces the humidity in the soil.

[a] G.Gonzalo, J. Velasco *A dynamic boundary value problem arising in the ecology of mangroves*, Nonlinear Analysis: Real World Applications, Vol. 7 (5),

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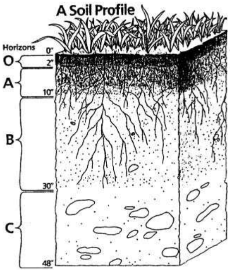

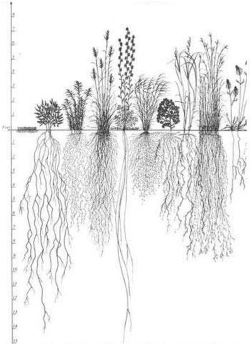

Constitutive Terms

- *external source*: the term $f(x,t)$ models an external source or sink of fluid. For example, the rain or the irrigation.

Remark

In our modelization, we include all the key elements of an unsaturated filtration flow process in a porous medium, with some modifications/generalizations which can all be justified by previous works appeared in the literature.
However, no concrete experimental model corresponds, as far as we know, to such a formulation.

The general fractional evolution boundary value problem

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a general open set.
Let $Q_\infty = \Omega \times (0, +\infty)$, $\Sigma_\infty = \partial\Omega \times (0, +\infty)$, and consider a fractional evolution boundary value problem formulated as follows:

$$\left\{ \begin{array}{rcll} a_1 \frac{\partial u}{\partial t} + a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} + Au & = & f(x,t) & \text{in } Q_\infty, \\ Bu & = & g(x,t) & \text{on } \Sigma_\infty, \\ u(x,0) & = & u_0(x) & \text{in } \Omega. \end{array} \right. \quad (2)$$

- $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$
- $\partial^\alpha / \partial t^\alpha$ is the Riemann-Liouville fractional derivative:

$$\frac{\partial}{\partial t^\alpha} u(x,t) = \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau$$

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- Au denotes a nonlinear operator (usually in terms of u and the partial differentials of u),
- Bu denotes a boundary operator
- the data $f(x,t), g(x,t)$ and $u_0(x)$ are given functions
- for simplicity, A and B are assumed to be *autonomous operators*, i.e., with time independent coefficients.

Stabilization of a solution: main task

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- it is usually assumed that:

$$f(x,t) \to f_\infty(x) \quad \text{and} \quad g(x,t) \to g_\infty(x) \quad \text{as } t \to +\infty,$$

in some functional spaces

Stabilization of a solution: main task

In the study of the stabilization of solutions:

- the main task is to prove that

$$u(x,t) \to u_\infty(x) \text{ as } t \to +\infty,$$

in some topology of a suitable functional space, with $u_\infty(x)$ solution of

$$\begin{cases} Au_\infty = f_\infty(x) & \text{in } \Omega, \\ Bu_\infty = g_\infty(x) & \text{on } \partial\Omega. \end{cases}$$

- This has been the most recurrent approach in the literature:

♠ Ph. Clément, R.C. MacCamy, J.A. Nohel, *Asymptotic Properties of Solutions of Nonlinear Abstract Volterra Equations*, J. Int. Eq., 3, 185-216, 1981.

A stronger property: extinction in finite time

- Starting by assuming that $A0 = 0$, $B0 = 0$ and

$$f(x,t) = 0 \quad \forall\, t \geq T_f,$$
$$g(x,t) = 0 \quad \forall\, t \geq T_g,$$

for some $T_f < \infty$ and $T_g < \infty$

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  for some $T_f < \infty$ and $T_g < \infty$

- we want to arrive to the following natural phenomenon of the *extinction in finite time:*

Definition

Let u be a solution of the evolution boundary value problem (2). We will say that $u(x,t)$ possesses the property of extinction in a finite time if there exists $t^* < \infty$ such that

$$u(x,t) \equiv 0 \text{ on } \Omega, \ \forall\, t \geq t^*.$$

Aims

Under suitable conditions, we shall prove that the solution to

$$\begin{cases} a_1 \frac{\partial}{\partial t} v + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha} v - (\gamma^{1-p} |v|^{m-1} |v_x|^{p-2} v_x)_x + \lambda v |v|^{q-1} = f(x,t) \text{ in } Q_T \\ v(\pm L, t) = 0 \qquad t \in (0,T); \qquad\qquad v(x,0) = v_0(x) \quad \text{in } \Omega \end{cases} \tag{3}$$

when we assume that $f(x,t) = 0\ \forall t \geq T_f$ for some $T_f < \infty$, satisfies an integral energy inequality leading to its extinction in a finite time.

Remark: In our modelization, the extinction of the solution of (3) models as the soil drain away; in fact, it means that it becomes dry in a finite time.

Aims

Concretely:

- We will first prove the occurrence of the extinction in finite time for the problem (3) with $a_1 > 0$ and $a_\alpha > 0$.

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- We will first prove the occurrence of the extinction in finite time for the problem (3) with $a_1 > 0$ and $a_\alpha > 0$.

- Then, we will pass to consider the limit problem obtained when $a_1 = 0$ and $a_\alpha > 0$.

  This is the most extraordinary case, since we prove that the finite time extinction phenomenon still appears, even with a non-smooth profile near the extinction time.

Lemma1

Lemma

Let $\alpha \in (0,1)$ *and* $u \in C^0([0,T]:\mathbb{R})$, $u' \in L^1(0,T:\mathbb{R})$ *and* u *monotone. Then*

$$2\,u(t)\frac{d^\alpha u}{dt^\alpha}(t) \geq \frac{d^\alpha u^2}{dt^\alpha}(t), \qquad \text{a.e. } t \in (0,T]. \tag{4}$$

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Remarks:

- the inequality (4) can be trivially checked if $\alpha = 1$.
- we conjecture that inequality (4) still holds true under weaker hypothesis on u (avoiding the monotonicity).

- Inequality (4) allows to conclude the monotonicity (or accretiveness) of the fractional differential operator in a very direct way.

- Inequality (4) allows to conclude the monotonicity (or accretiveness) of the fractional differential operator in a very direct way.

- The proof of the monotonicity has been already provided in the literature by means of very sophisticated arguments. See e.g:

  - G. Gripenber, *Volterra integro-differential equations with accrettive nonlinearity*, J. Differential Eqs., 60, 57-79, 1985
  - PH. Clément and J. Prüss, *Completely positive measures and Feller semigroups*, Math. Ann.,287, 73-105, 1990
  - PH. Clément and S.O. Londen, *On the sum of fractional derivatives and m-accretive operators*, in Partial Differential Equation Models in Physics and Biology, Vol.82, G. Lumer and S. Nicaise and B.W. Schulze (Eds.), Akademie Verlag, 91-100, 1994

Lemma2: a more general version of Lemma 1

Lemma

Given the Hilbert space H, let $\alpha \in (0,1)$ and $u \in L^{\infty}(0,T:H)$ such that $\frac{d^\alpha}{dt^\alpha}u \in L^1(0,T:H)$. Assume that $\|u(\cdot)\|_H$ is non-increasing (i.e. $\|u(t_2)\|_H \leq \|u(t_1)\|_H$ for a.e. $t_1, t_2 \in (0,T)$ such that $t_1 \leq t_2$). Then, there exists $k(\alpha) > 0$ such that for almost every $t \in (0,T)$ we have that

$$\left(u(t), \frac{d^\alpha}{dt^\alpha}u(t)\right)_H \geq k(\alpha)\ \frac{d^\alpha}{dt^\alpha}\|u(t)\|_H^2 . \qquad (5)$$

Lemma2: a more general version of Lemma 1

- Inequality (5) directly implies $\frac{d^\alpha}{dt^\alpha}\|u(t)\|_H^2 \in L^1(0,+\infty)$, which is not straightforward to see.

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- Inequality (5) directly implies $\frac{d^\alpha}{dt^\alpha}\|u(t)\|_H^2 \in L^1(0,+\infty)$, which is not straightforward to see.

- Alternately, the following inequality had already been proved (by many authors) in the literature:

$$\int_0^t u(t)\frac{d^\alpha}{dt^\alpha}u(t)dt \geq \int_0^t |u(t)|^2 dt$$

♠ Ph. Clément, R.C. MacCamy, J.A. Nohel, *Asymptotic Properties of Solutions of Nonlinear Abstract Volterra Equations*, J. Int. Eq., 3, 185-216, 1981.

- This inequality does not depend on α in the right term. This is the weakness of this formula.

The main Theorem: extinction in finite time

Theorem

For any $f \in L^1_{\mathrm{loc}}(0, +\infty : L^1(\Omega))$ *and* $u_0 \in L^1(\Omega)$, *there exists a weak solution of the problem* (3).

Assume also that either $p < 2$ *and* $q > 0$ *arbitrary, or* $q < 1$ *and* $p > 1$ *arbitrary, and* f *satisfying that* $\exists t_f \geq 0$ *such that* $f(x,t) \equiv 0$ *a.e.* $x \in \Omega$ *and a.e.* $t > t_f$.

Then, there exists $t_0 \geq t_f \geq 0$ *such that* $v(x,t) \equiv 0$ *for a.e.* $x \in \Omega$ *and for any* $t \geq t_0$.

Proof of the existence

- The existence of a weak solution $v \in C([0, +\infty) : L^1(\Omega))$ can be deduced from the abstract results on Volterra intregro-differential equations with accretive operators.

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♠ A. Friedman, *On integral equations of the Volterra type, J. Analyse Math.*, 11, 381-413, 1963.

♠ ⋮

♠ S. Bonaccorsi and M. Fantozzi, *Volterra integro-differential equations with accretive operators and non-autonomous perturbations*, Journal of Integral Equations and Applications, 18, 437-470, 2006

Proof of the existence

- The existence of a weak solution $v \in C([0,+\infty) : L^1(\Omega))$ can be deduced from the abstract results on Volterra intregro-differential equations with accretive operators.

- The operator $G(v(t)) = -(\gamma^{1-p}|v|^{m-1}|v_x|^{p-2}v_x)_x + \lambda v|v|^{q-1}$ is m-accretive (or, equivalently, maximal monotone) in $H = L^1(\Omega)$, as it is already known in the literature.
  - J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Elliptic equations*, Research Notes in Mathematics N.106, Vol.1, Pitman, London, 1985.

An Energy Method

To prove the occurrence of the extinction, we write problem (3) in an equivalent way. Indeed, the change of the unknown function $v = u|u|^{\gamma-1}$ with the parameters $m = 1 + \frac{(1-\gamma)(p-1)}{\gamma}$ and $q = \frac{\sigma}{\gamma}$ leads to the following formulation:

$$\begin{cases} a_1 \frac{\partial}{\partial t}\left(u|u|^{\gamma-1}\right) + a_\alpha \frac{\partial^\alpha}{\partial t^\alpha}\left(u|u|^{\gamma-1}\right) - \left(|u_x|^{p-2}u_x\right)_x + \lambda u|u|^{\sigma-1} = f \quad \text{in } Q_T \\ u(\pm L, t) = 0 \qquad t \in (0,T); \qquad u(x,0) = u_0(x) \quad \text{in } \Omega \end{cases} \tag{6}$$

where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$, $\lambda > 0$, $0 < \gamma < \infty$, $1 < p < \infty$ and $\sigma > 0$.

An Energy Method. Case $a_1 > 0$

- We define the energy functions

$$y(t) = \int_{\Omega} |u(x,t)^{1+\gamma}| dx = \| u(\cdot,t) \|_{L^{1+\gamma}(\Omega)}^{1+\gamma}$$

$$D(t) = \int_{\Omega} |u_x(x,t)^p| dx = \| u(\cdot,t) \|_{L^p(\Omega)}^{p}$$

$$A(t) = \int_{\Omega} |u(x,t)^{1+\sigma}| dx = \| u(\cdot,t) \|_{L^{1+\sigma}(\Omega)}^{1+\sigma}$$

which, are defined for almost all $t \in (0,T)$ and are in $L^1(0,T)$.

An Energy Method. Case $a_1 > 0$

- multiplying by u and integrating on Ω the equation appearing in (6), we get:

$$a_1 \frac{\gamma}{\gamma+1}\frac{d}{dt}y + a_\alpha \int_\Omega u \frac{\partial^\alpha}{\partial t^\alpha}\left(u|u|^{\gamma-1}\right) dx + D(t) + \lambda A(t) = \int_\Omega fu \, dx \tag{7}$$

and it can be shown [1] how to pass, when $\alpha = 1$ and $f(x,t) \equiv 0$, from this to the following ordinary differential inequality:

$$a_1 y' + Cy^\nu \leq 0, \tag{8}$$

where $C > 0$ and $0 < \nu < 1$ and for a.e. $t \in (t_f, +\infty)$.

[1] S.N. Antontsev, J.I. Díaz, S.I. Shmarev, *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics*, Birkhäuser, Boston, 2001.

An Energy Method. Case $a_1 > 0$

- Also, we know [2] that the operator:

$$u \mapsto a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} \tag{9}$$

generates contraction semigroups in $E = L^r(0, +\infty : L^q(\Omega))$, with $1 < r,\ q < \infty$ which are positive with respect to the usual cone E^+ of positive functions.

[2] PH. Clément, J. Prüss, *Completely positive measures and Feller semigroups*, Math. Ann. 287 (1990) 73-105.

An Energy Method. Case $a_1 > 0$

- So, since $a_1 > 0$ we get that, for any $t \geq t_f$, the application

$$t \mapsto y(t)$$

is non increasing, $y \in C([t_f, +\infty])$ and $\frac{d^\alpha y}{dt^\alpha} \in L^1(t_f, T)$.

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- Therefore, we are in conditions as to apply Lemma 2, and we get:

$$\begin{cases} a_1 \frac{dy}{dt} + a_\alpha \frac{d^\alpha y}{dt^\alpha}(t) + C\, y(t)^\nu \leq 0 \quad \text{on } (t_f, +\infty) \\ y(t_f) = Y_0. \end{cases} \tag{10}$$

An Energy Method. Case $a_1 > 0$

- Moreover, since the semigroup generated by the operator (9) is positive [although it is non local] (Clement and Prüss, 1990), we have that:

$$0 \leq y(t) \leq Y(t) \qquad \text{for any } t \in [t_f, +\infty), \tag{11}$$

where $Y(t)$ is a *supersolution*, i.e, $Y(t)$ satisfies the inequality:

$$\begin{cases} a_1 \frac{dY}{dt} + a_\alpha \frac{d^\alpha Y}{dt^\alpha}(t) + C\, Y(t)^\nu \geq 0 \quad \text{on } (t_f, +\infty) \\ Y(t_f) \geq Y_0. \end{cases} \tag{12}$$

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- Our conclusion comes from the fact that we can construct $Y(t)$ satisfying (12) and such that $Y(t) \equiv 0 \; \forall\, t \geq t_Y$, for some $t_Y > t_f$.

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- For instance,

$$Y(t) = k\,(t_Y - t)_+^{\frac{1}{1-\nu}}$$

for some $t_Y > t_f$ and some $k > 0$.

An Energy Method. Case $a_1 = 0$

- Let u_ε be the solution of $(\mathcal{P}_\varepsilon)$ when $a_1 = \varepsilon$, $\varepsilon > 0$.
- We can prove that:

$$u_\varepsilon \to u^* \quad \text{in } L^1(0, +\infty : L^1(\Omega)) \quad \text{when } \varepsilon \to 0,$$

- so the application

$$t \mapsto y^*(t) := \|u^*(\cdot, t)\|^{1+\gamma}_{L^{1+\gamma}(\Omega)}$$

is also decreasing.

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is also decreasing.

- Then, we can apply Lemma 2 and write for y^*:

$$\begin{cases} a_\alpha \frac{d^\alpha y^*}{dt^\alpha}(t) + C\, y^*(t)^\nu \leq 0 \quad \text{on } (t_f, +\infty) \\ y(t_f) = W_0. \end{cases} \tag{13}$$

An Energy Method. Case $a_1 = 0$

- The conclusion, as before, comes now from the fact that we can construct a supersolution $W(t)$ satisfying:

$$\begin{cases} a_\alpha \frac{d^\alpha W}{dt^\alpha}(t) + C\, W(t)^\nu \geq 0 \quad \text{on } (t_f, +\infty) \\ W(t_f) = W_0, \end{cases} \tag{14}$$

and such that $W(t) \equiv 0 \;\forall\; t \geq t_W$, for some $t_W > t_F$.

- Indeed, let i.e. $W(t) = h\,(t_W - t)^{\frac{\alpha}{1-\nu}}$ for some $t_W > t_f$ and some $h > 0$.

Remarks

- The decreasing behavior of the norm:

$$\|u(\cdot,t)\|^{1+\gamma}_{L^{1+\gamma}(\Omega)} \leq k\,(t_Y - t)_+^{\frac{1}{1-\nu}} \qquad \forall\; t \geq t_f\,,\;\; \nu \in (0,1) \tag{15}$$

  when $a_1 > 0$ is actually the same as when the fractional derivative is not included in the problem (6).

- It has to be highlighted that when $\nu > 1$ it is well-known that the solution to problem (6) shows an exponential decay at infinity. However our method allows to estimate trough (15) the rate of this decay, which is impossible to achieve with the stabilization methods.

Remarks

- What is more extraordinary is the decreasing behavior of the norm:

$$\|u^*(\cdot,t)\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leq h\,(t_W - t)_+^{\frac{\alpha}{1-\nu}} \qquad \forall\ t \geq t_f.$$

when $a_1 = 0$ as we are dealing with a function $W(t)$ such that $\frac{d^\alpha W}{dt^\alpha}(t) \in L^\infty(0,+\infty)$ whereas $W'(t) \notin L^\infty(0,+\infty)$ although $W'(t) \in L^1(0,+\infty)$.

Conclusions

- This work extends the application of the very fine techniques of nonlinear operators on Banach spaces to the case of nonlinear fractional partial differential equations.

Conclusions

- This work extends the application of the very fine techniques of nonlinear operators on Banach spaces to the case of nonlinear fractional partial differential equations.
- The **finite time extinction phenomenon** for certain evolution boundary values problems are still valid when the evolution in time is given by an *ordinary derivative jointly with a real order differential operator* (which is compatible with unbounded, but integrable, first time derivatives).

Thank you for your kind attention!