On an elliptic system related to desertification studies

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Abstract

In this communication we consider the stationary problem of a nonlinear parabolic system of partial differential equations, which arises in the context of Dryland Vegetation. In the first part we present some multiplicity properties for a localized simplification of the system leading to an S-shaped bifurcation diagram in terms of the precipitation rate parameter. In the second part we consider the case of an idealized "oasis", $\omega \subset\subset \Omega$, where we study the transition of the surface-water height in a neighborhood of the set ω.

1 A multiplicity result

We consider a system of elliptic equations which is the stationary version of a dryland vegetation model proposed by Gilad et al. [6]. Motivated by the bifurcation diagram obtained in [6] for the uniform vegetation states of the system, here we are interested in the multiplicity properties of stationary solutions of some related problem in a more general framework of weak solutions (not necessarily spatially uniform). To be more precise, we study the following elliptic problem

$$\begin{cases} -\delta_b \Delta b = -b + G_b b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = -G_w w - \mathcal{E}_b w + \mathcal{I}_b h & \text{in } \Omega, \\ -\delta_h \Delta h^2 = -\mathcal{I}_b h + p & \text{in } \Omega, \\ \frac{\partial b}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial h}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^1 boundary and n denotes the outward pointing unit normal vector field on $\partial\Omega$.

Here, b represents the biomass, w the soil-water content and h the surface-water height after suitable non-dimensionalization. The growth rate G_b and the water uptake rate G_w are non-local terms given by

$$G_b(b,w) = \nu \int_\Omega g(x,y) w(y)\, dy \quad \text{and} \quad G_w(b) = \gamma \int_\Omega g(y,x) b(y)\, dy$$

where $g(x,y) = \frac{1}{2\pi} \exp\big[-\dfrac{|x-y|}{2(1+\eta b)^2}\big]$ for $x, y \in \Omega$. Moreover, $\mathcal{I}_b(b) = \alpha \dfrac{b + q/c}{b+q}$ represents the infiltration rate, and $\mathcal{E}_b(b) = \dfrac{\nu}{1+\rho b}$ represents the evaporation rate. In the third equation the key parameter, $p > 0$, represents the precipitation rate. The rest of the parameters are positive, where in fact $c > 1$ and we also assume that $\rho < 1$ (see [6, 8] for more information about the modeling).

Our goal is to obtain some rigorous multiplicity properties for the component b. In order to do that we consider a simplified localized system by replacing the non-local terms G_b and G_w in (1), respectively, by

$$\nu\psi(x)w \exp\big[-\frac{1}{2(1+\eta b)^2}\big]$$

and

$$\gamma\psi(x)b \exp\big[-\frac{1}{2(1+\eta b)^2}\big],$$

where $\psi(x)$ is a continuous function such that $\psi_1 \leq \psi(x) \leq \psi_2$ for some constants $\psi_1, \psi_2 > 0$.

Remark 1. *The above simplification relies on the fact that, for instance, the function $z(y) = \frac{-|x-y|}{2(1+\eta b)^2}$, that appears in G_b, attains its maximum at x. Therefore, the main contribution of w in the integral occurs around the point x. Using the same idea we may also simplify the term G_w. Finally, we note that $g(x, y) \neq g(y, x)$, however, since this does not affect our result, for convenience in the exposition, we suppose that these terms are equal.*

Now, letting $\delta_w = \delta_h = 0$, and adding the second and third equation of the simplified system we may solve the resulting equation with respect to w. We substitute w in the first equation arriving to the following elliptic problem :

$$\begin{cases} -\Delta b + b = pf(x, b) & \text{in } \Omega, \\ \dfrac{\partial b}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_p$}$$

for

$$f(x, b) = \frac{\nu\psi(x)b(1-b)(1+\rho b)}{\gamma\psi(x)b(1+\rho b) + \nu \exp[\frac{1}{2(1+\eta b)^2}]},$$

where without loss of generality we have taken $\delta_b = 1$ (if this is not the case we change coordinates and rescale the domain by a factor $1/\sqrt{\delta_b}$). Clearly, the function

$$f(x, \cdot) \in C^1(\mathbb{R}^+)$$

for every fixed $x \in \Omega$. In fact, $f(x, s) = 0$ for $s \in \mathbb{R}^+$ if and only if $s = 0$ or $s = 1$. Finally, using the upper bound of $\psi(x)$ we also have that

$$0 \leq f(x, s) \leq M \quad \text{for all} \quad s \in [0, 1] \text{ and } x \in \Omega.$$

We shall study ($\mathcal{P}_p$) seeking for multiplicity properties of non-negative weak solutions depending on the parameter p. Before, starting looking for solutions in the general framework of weak solutions we shall first consider a subclass of weak solutions, namely, the so-called variational solutions. So, let us consider the set

$$K = \{v \in H^1(\Omega) \,|\, 0 \leq u \leq 1 \,\mathrm{in}\, \Omega\}$$

and let

$$F_p(x, v) = p \int_0^v f(x, s)\, ds\,.$$

We define the variational functional

$$J_p(v) = \frac{1}{2} \int_\Omega \left(|\nabla v|^2 + v^2\right) dx - \Phi_p(v),$$

where

$$\Phi_p(v) := \int_\Omega F_p(x, v(x))\, dx$$

Definition 1. *We shall call a function $u \in H^1(\Omega)$ a variational solution of ($\mathcal{P}_p$), if u is a minimum of the functional J_p on the set K.*

Remark 2. *It can be easily verified that any variational solution is a weak solution (it suffices to consider the Euler-Lagrance equation associated to the functional J_p).*

We have:

Theorem 1. *For each $p > 0$, there exists at least one variational solution of ($\mathcal{P}_p$).*

Proof. Since K is a convex and closed subset of $H^1(\Omega)$, in order to show that J attains a minimum (due to a version of the Weierstrass theorem), it suffices to show that J is weakly lower semicontinuous and coercive defined on K.

(i) J_p is weakly lower semicontinuous. Indeed, the norm of $H^1(\Omega)$ is weakly lower semicontinuous. On the other hand, the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \leq q < \infty$, since $N = dim(\Omega) = 2$. Therefore, if v_n is a sequence in K that converges weakly in $H^1(\Omega)$ to a function v, we know that (up to a subsequence) $v_n \to v$ strongly in $L^q(\Omega)$. This actually implies that

$$\Phi_p(v_n) \to \Phi_p(v)$$

and so the map $\Phi : K \subset H^1(\Omega) \to R$ is weakly continuous. Thus $J_p(u)$ is weakly lower semicontinuous.

(ii) J_p is coercive. Indeed, for $u \in K$ we have $\Phi(v) \leq pM\|v\|_{L^1(\Omega)} \leq pM|\Omega|$ so for some constant $C(p, \Omega) > 0$, $J(v) \geq \dfrac{1}{2}\|v\|^2 - C(p, \Omega)$ which implies that $J(u) \to \infty$ as $\|v\| \to \infty$. This ends the proof of the Theorem 1. ■

We now proceed to consider weak solutions of ($\mathcal{P}_p$) which are not necessarily variational solutions. Our study is inspired by a previous one arising in a completely different context: some simple climate models(see [5] and [3]). Before stating our main result we need to introduce some auxiliary notation. In particular, it is useful to consider the real valued functions

$$f_1(s) = \frac{\nu\psi_2 s(1-s)(1+\rho s)}{\gamma\psi_1 s(1+\rho s) + \nu\exp[\frac{1}{2(1+\eta s)^2}]}$$

and

$$f_2(s) = \frac{\nu\psi_1 s(1-s)(1+\rho s)}{\gamma\psi_2 s(1+\rho s) + \nu\exp[\frac{1}{2(1+\eta s)^2}]},$$

as well as the auxiliary algebraic equations

$$s = pf_1(s), \quad s \in \mathbb{R}^+, \tag{E_M}$$

and

$$s = pf_2(s), \quad s \in \mathbb{R}^+. \tag{E_m}$$

Obviously, $s = 0$ satisfies both equations for all $p > 0$. Moreover, each of the functions, f_1 and f_2, has a unique positive critical point lying in $(0,1)$. We shall denote by Γ_M and Γ_m the (bifurcation) curves of nontrivial solutions corresponding to the algebraic equations E_M and E_m, respectively. We define

$$T_i(p,s) = s - pf_i(s).$$

We denote by $(p_{f_1}, 0)$ (respectively $(p_{f_2}, 0)$) the point where Γ_M(respectively Γ_m) bifurcates from the line of trivial solutions. Thus, for i=1,2, p_{f_i} are such that $\frac{\partial}{\partial s}T_i(p_{f_i}, 0) = 0$ ($p_{f_1} = \sqrt{e}/\psi_2$, $p_{f_2} = \sqrt{e}/\psi_1$).

Moreover, we shall need to assume some more specific properties about the functions $T_i(s)$ that we state below.

(P_1) For $i = 1,2$, the inequality $\frac{\partial^2}{\partial s^2}T_i(p,0) < 0$ holds, and so there exist "turning points"(s_i, p_i), for unique $s_i, p_i > 0$ such that

$$T_i(p_i, s_i) = \frac{\partial}{\partial s}T_i(p_i, s_i) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial s^2}T_i(p_i, s_i) > 0\,.$$

(P_2) The inequality $0 < p_2 < p_{f_1}$ holds.

Theorem 2. *Let p_{f_1}, p_{f_2} be the bifurcation points of E_M, E_m and assume that p_1, p_2 satisfy (P_1). Then,*

(i) if $p \in (0, p_1)$, the trivial solution $b \equiv 0$ is the only possible non-negative solution of ($\mathcal{P}_p$).

(ii) if we also assume (P_2) then for any $p \in (p_2, p_{f_1})$, ($\mathcal{P}_p$) has at least two positive weak solutions, besides the trivial solution $b \equiv 0$.

(iii) if $p \in (p_{f_2}, \infty)$ then, besides the trivial solution, $(\mathcal{P}_p)$ has at least one positive weak solution. In fact, for p large enough there exist $\xi \in (0,1)$ and a unique non-trivial solution of $(\mathcal{P}_p)$ satisfying that $\xi \leq b(x) < 1$ in Ω. Moreover, this unique solution is also a variational solution of $(\mathcal{P}_p)$.

Remark 3. *The property (P_1) is satisfied when $\sqrt{e}(\rho+\eta-1)\frac{\nu}{\gamma} > \psi_2$ holds.*

Remark 4. *For the given functions f_1, f_2, the property (P_2) is satisfied when the distance between ψ_1 and ψ_2 is sufficiently small.*

Proof. (i) By (P_1) there exists a unique pair $(p_1, s_1) \in \Gamma_M$ such that $f_1'(s_1) = \frac{1}{p_1} > 0$. In fact, $f_1(s) \leq s f_1'(s_1)$ for all $s > 0$ and since also $f(x,s) < 0$ for all $s > 1$ and $x \in \Omega$, we have that $f(x,s) \leq s f_1'(s_1)$ for all $s \geq 0$. Therefore, if $u \in H^1(\Omega)$ is a non-negative solution of $(\mathcal{P}_p)$, by multiplying by u and integrating over Ω we have that

$$\int_\Omega \left(|\nabla u|^2 + u^2\right) dx \leq p/p_1 \int_\Omega u^2\, dx$$

or

$$\int_\Omega \left(|\nabla u|^2 + (1 - p/p_1)u^2\right) dx \leq 0.$$

So, for all $0 < p < p_1$, the left hand side of the above inequality is also greater than or equal to zero, which implies $\|u\|_{H^1} = 0$ and so $u = 0$.
In order to obtain (ii) and (iii), we now focus on positive constant super and sub-solutions of $(\mathcal{P}_p)$. We observe that for any $p > 0$, any positive constant solution of E_M and E_m is strictly smaller than one. Moreover, letting $f_M(s) = \max\{f_1(s), f_2(s)\}$ and $f_m(s) = \min\{f_1(s), f_2(s)\}$ we have that

$$f_m(s) \leq f(x,s) \leq f_M(s) \quad \text{for all} \quad s \geq 0.$$

Then clearly for $p > 0$ any positive solutions of the following problems

$$\begin{cases} -\Delta U_p + U_p = p f_M(U_p) & \text{in } \Omega, \\ \frac{\partial U_p}{\partial n} \geq 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta V_p + V_p = p f_m(V_p) & \text{in } \Omega, \\ \frac{\partial V_p}{\partial n} \leq 0 & \text{on } \partial\Omega, \end{cases}$$

is respectively, a sup or sub-solutions of $\mathcal{P}_p$. So, for every $p > 0$ solutions of E_M and E_m form two families of positive constant super and sub-solutions of $(\mathcal{P}_p)$ $\big($solutions of $s = p f_1(s)$(respectively $s = p f_2(s)$) coincide with those of $s = p f_M(s)$ (respectively $s = p f_m(s)$)$\big)$.

(ii) By (P_1) and (P_2), for each $p \in (p_2, p_{f_1})$, there exist two constant sup-solutions U_p^1, U_p^2, and a constant sub-solution V_p of $\mathcal{P}_p$ such that $0 < U_p^1 < V_p < U_p^2 < 1$. Now, we denote $I = [0,1]$ and

$$D_I = \{u \in L^\infty(\Omega) : u(x) \in I \text{ for almost every } x \in \Omega\},$$

which is an ordered convex subset of $L^\infty(\Omega)$. Moreover, for $p \in \mathbb{R}^+$ and any positive real number m_0, we define

$$f_{\mu_0}(x,s) := pf(x,s) + \mu_0 s$$

Thus, we define the continuous map:

$$G : D_I \subset L^\infty(\Omega) \to L^2(\Omega)$$

given by $G(u)(x) = f_{\mu_0}(p, x, u(x))$ for $u \in D_I$. Since, there exist positive constant μ such that $f(x,s_1) - f(x,s_2) > -\mu(s_1 - s_2)$ for all $s_1, s_2 \in [0,1]$ such that $s_1 > s_2$, it follows that, for any fixed $p_0 > 0$ letting $\mu_0 = p_0\mu$, the map $G : D_I \to L^2(\Omega)$ is also strictly increasing and continuous for all $p \in (0, p_0)$. Now, we denote by K_{μ_0} the inverse operator of the elliptic operator $-\Delta + (\mu_0 + 1)$ for the homogeneous Neumman boundary conditions. Then, it is well known that K_{μ_0} is positive compact linear operator from $L^2(\Omega)$ to $L^\infty(\Omega)$. So, for the ordered interval $[0, U_p^2] \subset D_I$ we define the composition

$$\mathcal{F} := K_{\mu_0} \circ G : [0, U_p^2] \to L^\infty(\Omega)$$

which is also a compact operator and in fact a strongly increasing operator. Finally, by the comparison principle $\mathcal{F}(U_p^1) < U_p^1$, $\mathcal{F}(V_p) > V_p$, $\mathcal{F}(U_p^2) < U_p^2$ and (ii) follows from the results in [1].

(iii) For each $p \in (p_2, \infty)$ there exist a constant sup-solution V_p and a constant sub-solution U_p of $(\mathcal{P}_p)$ such that $V_p < U_p$. It is easy to check that the conditions of the results in [1] hold true and so there exist at least one positive weak solution u_p such that $V_p < u_p < U_p$. Moreover, for p large enough any such positive weak solution takes values in an interval $[\xi, 1)$ where $f(x, \cdot)$ is decreasing which implies the uniqueness of any possible weak solution taking values in that interval. Finally, the energy of such weak solution is less than zero. Therefore, we deduce that for p large enough u_p is also a variational solution of $(\mathcal{P}_p)$. ■

2 Estimate on the location of the null set of the surface water height solution

In this section we study the last equation of the original system, but assuming that $\delta_h > 0$ and that p is not completely constant in Ω but only on a closed subset $\omega \subset\subset \Omega$ and so that p vanishes outside ω. In that case we can think of p as a distributed water resource ($p(x) = p\chi_\omega(x)$ on Ω) which is limited to be non neglected on a sub-region ω of the domain Ω. We recall that the non-linear term of the equation involves the so called infiltration contrast parameter $c > 1$.

Now, supposing that b is a given non-negative solution of the corresponding equation of the system (1), for the given boundary conditions, we set

$$\theta(x) := \alpha \frac{b(x) + q/c}{b(x) + q} \quad \text{in } \Omega,$$

and obviously we have that $\frac{\alpha}{c} \leq \theta(x) \leq \alpha$ on Ω. On the other hand letting $\tilde{h} = h^2$, if $h \geq 0$ and $\delta_h > 0$, then the third equation can be written as

$$\begin{cases} -\Delta \tilde{h} + \dfrac{\theta(x)}{\delta_h} \sqrt{\tilde{h}} = \phi(x) & \text{in } \Omega, \\ \\ \dfrac{\partial \tilde{h}}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

with $\phi(x) := \dfrac{p}{\delta_h} \chi_\omega(x)$, where $\chi_\omega(x)$ is the indicator function of the subset ω.

We point out that, in general, we cannot ensure the uniqueness of function $\tilde{h}$ (in fact, in the preceding section we exhibit a case of multiplicity of b and so of h). Nevertheless, by the maximum principle we know that any possible solution $\tilde{h}$ must satisfy that

$$\|\tilde{h}\|_{L^\infty(\Omega)} \leq \left(\frac{pc}{\alpha}\right)^2.$$

The following theorem provides an estimate on the location of the null set of a solution h. This will depend on c, α, δ_h and p.

Theorem 3. *Let h be the third component of any solution of the system* (1). *Then, necessarily, $h(x) = 0$ for any $x \in \Omega - \omega$ such that*

$$d(x, \partial\Omega \cup \partial\omega) > 4\sqrt{p} \frac{c\sqrt{\delta_h}}{\alpha}.$$

In fact, at least one of those possible solutions verifies that $h(x) = 0$ for any $x \in \Omega - \omega$ such that $d(x, \partial\omega) > 4\sqrt{p}\dfrac{c\sqrt{\delta_h}}{\alpha}$.

Proof. We set $m = \dfrac{\alpha/c}{\delta_h}$. Following [4] we look for a local comparison function $\tilde{h}_m$ such that $\tilde{h} \leq \tilde{h}_m$ on the ball $B_R(x_0)$ and $\tilde{h}_m(x_0) = 0$, where $R \geq 4\sqrt{p}\dfrac{c\sqrt{\delta_h}}{\alpha}$ so that $B_R(x_0) \subset \Omega - \omega$. Then, since $\tilde{h} \geq 0$ clearly $\tilde{h}(x_0) = 0$ (and in a weak sense if $\tilde{h}$ is not continuous). In fact, if $\tilde{h}_m \in H^1(\Omega)$ satisfies

$$-\Delta \tilde{h}_m + m\sqrt{\tilde{h}_m} \geq 0 \qquad \text{in } B_R(x_0), \tag{3}$$

$$\tilde{h}_m \geq \left(\frac{pc}{\alpha}\right)^2 \qquad \text{on} \partial B_R(x_0), \tag{4}$$

then, since

$$-\Delta\tilde{h} + m\sqrt{\tilde{h}} = \left(m - \frac{\theta(x)}{\delta_h}\right)\sqrt{\tilde{h}} \leq 0 \leq -\Delta\tilde{h}_m + m\sqrt{\tilde{h}_m} \text{ in } B_R(x_0),$$

by the comparison principle, we have that $\tilde{h} \leq \tilde{h}_m$.
Now, for such $x_0 \in \Omega - \omega$, we consider the function $\tilde{h}_m(x) = C_m|x - x_0|^4$ where $C_m = \left(\frac{m}{16}\right)^2$. Then it is not difficult to check (see [4]) that

$$-\Delta\tilde{h}_m + m\sqrt{\tilde{h}_m} \geq 0 \quad \text{in} \quad B_R(x_0)$$

and so the first conclusion holds. The second assertion holds merely by extending by zero some of those solution on the set of $x \in \Omega - \omega$ such that $d(x, \partial\omega) > 4\sqrt{p}\dfrac{c\sqrt{\delta_h}}{\alpha}$ (since, obviously it also satisfies the Neumann boundary condition). ■

Remark 5. *In fact it is possible to give a sharper estimate (near $\partial\Omega$) depending on the geometry of the domain Ω (see Chapter 2 in [4]) but we shall not detail it here.*

Remark 6. *From the estimate of the preceding theorem we deduce that the distance of the free boundary from the "oasis"set ω increases when one of the parameters p, δ_h or c increases or when α decreases. Moreover, the same answer remains true when the variation of the parameters is not necessarily monotone in each of them but the combination of them given by the expression $\frac{\sqrt{p}c\sqrt{\delta_h}}{\alpha}$ increases.*

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