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On the mathematical and numerical analysis of a model for the river channel formation

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Abstract. *We consider a free boundary formulation for the river formation giving rise to a measure on the free boundary.*

1 Introduction

We present here the main results of a recent work, in collaboration with A.C. Fowler, A. I. Muñoz and E. Schiavi [1], concerning the deterministic model for the river channel formation introduced by A.C. Fowler, N. Kopteva and C. Oakley [3]. The ingredients of a suitable model are variables describing water flow and sediment transport, and the mechanism of channel formation arises through an instability, in which locally increased flow causes increased erosion, which in turn increases the flow depth and thus also the flow. This positive feedback induces instability, as was shown by Smith and Bretherton [4]. They starting point was a coupled set of partial differential equations describing $s(x, y, t)$, the hillslope elevation, and $h(x, y, t)$, the water depth

$$\boldsymbol{\nabla}.\,(h\mathbf{u}) = r, \quad s_t + \boldsymbol{\nabla}.\,\mathbf{q} = U, \tag{1.1}$$

which represents conservation of mass of water and sediment. The mean water velocity $\mathbf{u}$ is determined through a momentum balance equation, while the sediment flux $\mathbf{q}$ is usually taken as an empirically prescribed function of flow-induced bed stress and bed slope, the resulting combination (the effective bed stress) being denoted $\boldsymbol{\tau}$. The source term r represents rainfall, while U represents tectonic uplift. The time derivative in the water mass equation is ignored. We assume it has been written in dimensionless form, so that the variables are $O(1)$. One can show that suitable models for the flow speed $\mathbf{u}$ and effective bed stress $\boldsymbol{\tau}$ are

$$\mathbf{u} = h^{1/2}|\boldsymbol{\nabla}\eta|^{1/2}\mathbf{n}, \quad \boldsymbol{\tau} = \mathbf{u}|\mathbf{u}| - \beta\boldsymbol{\nabla}s, \tag{1.2}$$

where typically $\beta = O(1)$, and the down-water slope normal $\mathbf{n}$ is defined by $\mathbf{n} = -\dfrac{\boldsymbol{\nabla}\eta}{|\boldsymbol{\nabla}\eta|}$, η represents the water surface elevation, and in dimensionless terms is related to hillslope elevation s and water film thickness h by $\eta = s + \delta h$. The parameter δ is very small, a typical estimate being 10^{-5}. Finally, the sediment flux is taken to have the form $\mathbf{q} = V(\tau)\mathbf{N}$, where $\tau = |\boldsymbol{\tau}|$ and the down-sediment flow normal $\mathbf{N}$ is $\mathbf{N} = \dfrac{\boldsymbol{\tau}}{\tau}$. V is an increasing function of τ, with $V \approx \tau^{3/2}$ being a popular choice (this essentially stemming from the model of Meyer-Peter and Müller, 1948). Uniform overland flow is unstable to y–dependent perturbations of small wavelength, and we can examine the nonlinear evolution of these by directly seeking asymptotic expansions in terms of δ. To do so, we firstly suppose that the channels which form are aligned in the X direction, and (sensibly) that the perturbation to the water surface is small, comparable to the overland flow depth: $\eta = \eta_0 + \delta Z$. We may then linearize the geometry of the system, to find that $\mathbf{n} = \mathbf{i} - \dfrac{\delta Z_y}{S}\mathbf{j} + \dots$, and $\mathbf{N} = \mathbf{i} - \dfrac{\delta}{S}\left\{Z_y - \dfrac{\beta}{h+\beta}h_y\right\}\mathbf{j} + \dots$, where $\mathbf{j}$ is the unit vector in the y direction and $S(X) = |\eta_0'(X)|$ is the unperturbed downhill slope. The nonlinear channel evolution then arises from a rescaling of the hillslope evolution equation, in which we put $y = \delta^{1/2}Y, \quad h = \dfrac{H}{\delta^{1/3}}, \quad t = \delta^{7/6}T$; after some algebra, we find the leading order sediment transport equation takes the form $\dfrac{\partial H}{\partial T} = S'S^{1/2}H^{3/2} + S^{1/2}\dfrac{\partial}{\partial Y}\left[\beta H^{1/2}\dfrac{\partial H}{\partial Y}\right]$, where $S' = dS/dX$.

It is important to realize that this equation arises through conservation of sediment. Only Y derivatives are present, because the lateral length scale is so much smaller than the downslope one. The perturbation Z to the water surface is in fact then determined by quadrature of the water conservation equation, but integration of this equation in the across stream direction yields the integral constraint $\displaystyle\int_{-\infty}^{\infty} H^{3/2}\,dY = \frac{2LrX}{S^{1/2}}$, where L is the spacing (on the original hillslope length scale for y) between channels; the limits in the integral are, however, infinite because the integral is with respect to the much smaller channel width length scale. Suitable initial and boundary conditions for the channel depth are that $H \to 0 \quad \text{as} \quad Y \to \pm\infty, \quad H = H_0(Y) \quad \text{at} \quad T = 0$. The above equation, together with the integral constraint and initial/boundary conditions, forms the basis of our study. We will assume that $S' > 0$, so that the nonlinear term in the H equation is a source. We define $H = \left(\dfrac{6}{\beta}\right)^{1/3}(LrX)^{2/3}u$, $T = \left(\dfrac{\beta}{6}\right)^{1/6}\dfrac{t}{S^{1/2}S'(LrX)^{1/3}}, \quad Y = \left(\dfrac{2\beta}{3S'}\right)^{1/2}x$.

2 Mathematical analysis

We consider the problem obtained previously assuming an initial thickness perturbation $u_0(x)$ satisfying some natural physically based hypothesis, i.e., a bounded and non negative function with a compact and connected support $[-\zeta_0, \zeta_0]$ such that $\displaystyle\int_0^{+\infty} u_0^m(x)dx = M/2$, for $m > 1$ (so including the case of $m = 3/2$ as before. For the sake of simplicity of the exposition we also assume symmetric initial data. We shall be especially interested in the question of *global solvability* (in time) of the following problem: find a continuous curve

$\zeta : [0, +\infty) \to I\!R^+$ and a function $u : \mathcal{P} \to [0, +\infty)$ (regular enough) such that

$$(SL)\begin{cases} u_t = (u^m)_{xx} + u^m, & \text{in } \mathcal{D}'(\mathcal{P}),\\ u(x,0) = u_0(x) & \text{a.e. } x \in \Omega_0,\\ u(x,t) > 0, \quad \text{a.e. } (x,t) \in \mathcal{P}, \text{ and } u(x,t) \equiv 0, & \text{a.e. } (x,t) \notin \mathcal{P}\\ u(\zeta(t),t) = 0, \quad (u^m)_x(0,t) = 0 & \text{a.e. } t \in (0,+\infty),\\ \zeta(0) = \zeta_0 \text{ and } \zeta(t) > 0 \quad \displaystyle\int_0^{\zeta(t)} u^m(x,t)dx = \frac{M}{2} & \text{a.e. } t \in (0,+\infty). \end{cases}$$

where $\Omega_0 = (0, \zeta_0)$, $\Omega_t = (0, \zeta(t)) \times \{t\}$, $\mathcal{P} = \cup_{t>0}\Omega_t$. Notice that $\mathcal{D}'(\mathcal{P})$ denotes the space of *distributions* on $\mathcal{P}$ and $\mathcal{P}$ is the *positivity subset* of the solution. Later on we shall make more precise the (minimal) regularity of the solution. The function $\zeta(t)$ is called *the interface* separating the (connected) region where $u(x,t) > 0$ from the region where $u(x,t) = 0$. It is unknown and it is usually called the *free* or *moving boundary* of the problem. Due to the free boundary we shall refer to the strong formulation (SL) as the *strong-local* formulation. We emphasize that the mass conservation constraint given in (SL) prevents possible blow-up phenomena which could arise (without this condition) due to the presence of the source term u^m in the equation.

An important difficulty, in order to get a global formulation (i.e. extended to the whole domain $(x,t) \in (0,+\infty) \times (0,+\infty)$, and not only on $(x,t) \in \mathcal{P}$), is the necessity to provide a suitable description of the flux $-(u^m)_x(\zeta(t),t)$ at the free boundary. This leads to a new constrained global formulation suitable for mathematical analysis and numerical resolution. Problems of this type arise in fluid mechanics (problems of the Bernoulli type), in combustion and in plasma physics (see, e.g., Díaz et al., 2007 [2] and its references).

To prove the existence we shall use an auxiliary global formulation on the whole domain $I\!R^+ \times [0,T]$. To be precise we introduce the notation $\delta_{\partial\{u(t,\cdot)=0\}}$ to design the Dirac delta distribution located at the interface $x = \zeta(t)$ for each $t \in (0,T)$ (i.e. $\delta_{\partial\{u(t,\cdot)=0\}} = \delta_{(\zeta(t),t)}$).

The reformulation of the mass constraint requires the "zero total measure" condition. So, the global formulation is:

$$(P)\begin{cases} u_t = (u^m)_{xx} + u^m - \dfrac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}, & \mathcal{D}'(I\!R^+ \times (0,T)),\\ u(x,0) = u_0(x) & \text{a.e. } x \in (0,+\infty),\\ u_x(0,t) = 0,\ u(x,t) \to 0 \text{ as } x \to +\infty & \text{a.e. } t \in (0,T),\\ \mu(t,\cdot) := u_t(t,\cdot) - (u^m)_{xx}(t,\cdot) \text{ and } \displaystyle\int_0^{+\infty} d\mu(t,\cdot) = 0, & \text{a.e. } t \in (0,T). \end{cases} \tag{2.3}$$

We use a two steps iterative approximation. The main idea is to construct the sequence $\{u_{2n+1} : n = 0, 1, 2...\}$ as solutions of the problems

$$(P_{2n+1})\begin{cases} (u_{2n+1})_t = ((u_{2n+1})^m)_{xx} + (u_{2n})^{m-1}(u_{2n+1}) - \dfrac{M}{2}\delta_{\partial\{(u_{2n+1})(t,\cdot)=0\}}, & \mathcal{D}'(I\!R^+ \times (0,T)),\\ (u_{2n+1})(x,0) = u_0(x) & \text{a.e. } x \in (0,+\infty),\\ (u_{2n+1})_x(0,t) = 0,\ (u_{2n+1})(x,t) \to 0 \text{ as } x \to +\infty & \text{a.e. } t \in (0,T), \end{cases}$$

(where for $n = 0$ we use as u_{2n} the initial condition u_0) and then $\{u_{2n} : n = 1, 2...\}$ by

$$(P_{2n}) \begin{cases} u_{2n}(x,t) = C_{2n}(t)u_{2n-1}(x,t) & \text{for a.e. } (x,t) \in I\!R^+ \times (0,T), \\ \displaystyle\int_0^{+\infty} ((u_{2n}(x,t))^{m-1}(u_{2n-1}(x,t))dx = \frac{M}{2} & \text{for a.e. } t \in (0,T), \end{cases}$$

for some $C_{2n}(t) > 0$. For the detailed proof of the convergence of the algorithm see Díaz et al.

Theorem *There exists a function* $C^*(t) > 0$, $C^* \in L^\infty(0,T)$ *and a function* $u \in C([0,T] : L^1(I\!R^+))$ *such that*

$$\begin{cases} u_t = (u^m)_{xx} + C^*(t)^{m-1}u^m - \frac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}, & \mathcal{D}'(I\!R^+ \times (0,T)), \\ u(x,0) = u_0(x) & \textit{a.e. } x \in (0,+\infty), \\ u_x(0,t) = 0,\ u(x,t) \to 0 \textit{ as } x \to +\infty & \textit{a.e. } t \in (0,T), \end{cases}$$

and $C^*(t)^{m-1} \displaystyle\int_0^{+\infty} u(x,t)^m dx = \frac{M}{2}$.

Concerning the numerical resolution of the problem (P), for each initial condition h_0, we compute its mass, say $M/2$, and the associated stationary solution $v(x)$ to which the solution should converge when $t \to +\infty$, see Fowler et al., 2007 ([3]). In order to discretize with respect to the coordinate x, at each time level $l \cdot dt$, we will employ piecewise linear finite elements $L_{l,k} := \{\phi \in C^0([0,+\infty)) : \phi|E \in \mathbf{P}_1,\ \forall E \in \mathbf{T}_{l,k}\}$ in a uniform grid, $\mathbf{T}_{l,k}$, of step k. Also, $\mathbf{B}_{l,k} := \{\phi_i\}$ is a base of finite linear elements in $L_{l,k}$. Then, the discretized problem is formulated as follows: *Find* $(u_{l+1})_k \in L_{l,k}$, $(u_{l+1})_k = \sum_j (u_{l+1})_k^j \phi_j$, *such that*

$$\int_{\mathbf{T}_{l,k}} (u_{l+1})_k \phi_i dx = \int_{\mathbf{T}_{l,k}} (u_l)_k \phi_i dx - \frac{3dt}{2} \int_{\mathbf{T}_{l,k}} (u_{n+1})_k^{\frac{1}{2}} ((u_{l+1})_k)_x \phi_{ix} dx$$
$$+ dt \int_{\mathbf{T}_{l,k}} (u_{l+1})_k^{\frac{3}{2}} \phi_i dx - dt \int_{\mathbf{T}_{l,k}} \frac{M}{2} \delta(u_l) \phi_i dx,\ \forall\, \phi_i \in \mathbf{B_{l,k}}. \tag{2.4}$$

In order to deal with the nonlinearities, we consider the iterative scheme: for p=2n+1 from 1 to N, $n = 0, 1, 2...$, and N an odd number to be fixed, we consider the problem,

$$\int_{\mathbf{T}_{l,k}} (u_{l+1,2n+1})_k \phi_i dx = \int_{\mathbf{T}_{l,k}} (u_l)_k \phi_i dx - \frac{3dt}{2} \int_{\mathbf{T}_{l,k}} (u_{l+1,2n})_k^{\frac{1}{2}} ((u_{l+1,2n+1})_k)_x \phi_{ix} dx$$
$$+ dt \int_{\mathbf{T}_{l,k}} (u_{l+1,2n})_k^{\frac{1}{2}} (u_{l+1,2n+1})_k \phi_i dx - dt \int_{\mathbf{T}_{l,k}} \frac{M}{2} \delta(u_l) \phi_i dx,\ \forall\, \phi_i \in \mathbf{B_{l,k}}, \tag{2.5}$$

where, $(u_{l+1,2n})_k$ has been rescaled before being introduce in (2.5) so that $\displaystyle\int (u_{l+1,2n})_k^{\frac{3}{2}} = \frac{M}{2}$, according to ($P_{2n}$), i.e., $(u_{l+1,2n})_k = C_{l+1,2n}(u_{l+1,2n-1})_k$. The resulting system of equations for the nodal values at the $(2n+1)$th-step is solved with the Gauss Seidel method. In order to initiate the iterative scheme, one can take as $(u_{l+1,p=1})_k$ the values obtained in the previous time step, that is to say, $(u_{l+1,p=1})_k = u_l$. The scheme finishes assuming the values for the $(l+1)$-time level given by $u_{l+1} = (u_{l+1,p=N})_k$.

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