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Finite extinction and null controllability via delayed feedback non-local actions

A.C. Casal [a,*], J.I. Díaz [b], J.M. Vegas [b]

[a] Dept. Matemática Aplicada, ETSAM, Univ. Politécnica de Madrid, E–28040, Spain
[b] Dept. Matemática Aplicada, Univ. Complutense de Madrid, E–28040, Spain

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ABSTRACT

We give sufficient conditions to have the finite extinction for all solutions of linear parabolic reaction-diffusion equations of the type

$$\frac{\partial u}{\partial t} - \Delta u = -M(t)u(t-\tau, x) \tag{1}$$

with a delay term $\tau > 0$, on Ω, an open set of $\mathbb{R}^N$, $M(t)$ is a bounded linear map on $L^p(\Omega)$, $u(t,x)$ satisfies a homogeneous Neumann or Dirichlet boundary condition. We apply this result to obtain distributed null controllability of the linear heat equation $u_t - \Delta u = v(t,x)$ by means of the delayed feedback term $v(t,x) = -M(t)u(t-\tau,x)$.

1. Introduction

In 1969 Winston and Yorke [1] (see also Hale [2], p. 70) showed that all the solutions of some delay-differential equation of the type

$$x'(t) = -\beta(t)x(t-1), \quad t \geq 0 \tag{2}$$

with

$$\beta(t) = \begin{cases} \sin^2 \pi t & \text{for } t \in [2n, 2n+1] \\ 0 & \text{for } t \in (2n-1, 2n) \end{cases} \tag{3}$$

vanish after some time $t \geq T$. The purpose of this paper is to analyze the possibility of extending this property to higher-dimensional delay equations, including functional partial differential equations, and to show that a suitable modification of this idea can be used to obtain exact null controllability in parabolic equations in a very simple way. This is only one of the aspects of a more general analysis on finite extinction phenomena started by the authors in [3].

More specifically, we consider the parabolic-delay linear problem with Dirichlet homogeneous boundary conditions:

$$(P) \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = -M(t)u(t-\tau, \mathbf{x}) & \text{in } (0,\infty) \times \Omega, \\ u(t,\mathbf{x}) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u(t,\mathbf{x}) = \xi(t,\mathbf{x}) & \text{on } (-\tau, 0) \times \Omega, \end{cases} \tag{4}$$

* Corresponding author. Tel.: +34 913366561; fax: +34 913366563.
E-mail addresses: alfonso.casal@upm.es (A.C. Casal), diaz.racefyn@insde.es (J.I. Díaz), jm_vegas@mat.ucm.es (J.M. Vegas).

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where $\xi : [-\tau, 0] \to \mathcal{F}(\Omega)$ is a given continuous "initial function" with values in some appropriate space $\mathcal{F}(\Omega)$ of functions defined on Ω and $M : [0, \infty) \to \mathcal{L}(\mathcal{F}(\Omega))$ is continuous, where $\mathcal{L}(\mathcal{F}(\Omega))$ is the space of bounded linear maps on $\mathcal{F}(\Omega)$.

Let us recall that if $M(t) \equiv 0$ the *finite extinction phenomenon* cannot arise because of such well-known properties of linear parabolic equations as the unique continuation property or the strong maximum principle. Also, in the case of zero delay $\tau = 0$, extinction in finite time is typical of *nonlinear*equations containing a strong absorption term (see [8]), like in the case of reaction-diffusion equations of the type

$$\frac{\partial u}{\partial t} - \Delta u = -\lambda |u|^{m-1} u \tag{5}$$

for $\lambda, \ m > 0$. It is well known (see e.g., Antontsev, Díaz and Shmarev [4] and the references therein) that the finite extinction phenomenon takes place if and only if $m < 1$, that is, in the nonlinear non-Lipschitz case.

Going back to the linear problem (P), we will show that if M vanishes outside the interval $[\tau, 2\tau]$, commutes with the semigroup $S(t)$ generated by the Laplacian operator with the given boundary conditions, and its integral has the value $S(\tau)$ in $\mathcal{L}(\mathcal{F}(\Omega))$, that is, if

$$\begin{cases} M(t) = 0 & \text{for } t \in [0, \tau] \cup [2\tau, \infty) \\ M(t)S(t) = S(t)M(t) & \text{for } t \geq 0 \\ \displaystyle\int_0^\infty M(t)\mathrm{d}t = S(\tau) \end{cases} \tag{6}$$

then all solutions vanish identically for $t \geq 2\tau$, so we have global *extinction in finite time*,thus generalizing the results obtained in Winston and Yorke [1] and in Casal, Diaz and Vegas [3].

A simple example of $M(t)$ with properties (6) is

$$M(t) = b(t)S(\tau) \tag{7}$$

where $b : [0, \infty) \to \mathbb{R}$ is continuous and vanishes on $[0, \tau] \cup [2\tau, \infty)$. This observation enables us to apply this idea to the problem of exact null controllability for the heat equation and prove that all the solutions $u(t, x)$ of

$$\frac{\partial u}{\partial t} - \Delta u = v(t, x) \tag{8}$$

(with Dirichlet boundary conditions) can be driven to zero at time 2τ by using the easily implementable delayed feedback controller

$$v(t) = -b(t)S(\tau)u(t - \tau) \tag{9}$$

which can also be expressed as

$$v(t) = -b(t)w(t) \tag{10}$$

where w satisfies the heat equation with the same initial condition $u(0, \cdot)$:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w = 0 & \text{on } (0, \infty) \times \Omega \\ w(0, x) = u(0, x) & \text{on } \Omega \end{cases} \tag{11}$$

and Dirichlet boundary conditions.

Let us finally mention that the boundary condition can be changed and that, in fact, the result also applies to the case of systems of equations and higher order parabolic equations, since the proof relies exclusively on semigroup theory and no application of the maximum principle is involved. For some considerations on the initial condition see Remark 1.

2. On the finite extinction phenomenon

Let A be the infinitesimal generator of a linear C^0-semigroup on a Banach space X, let $M : \mathbb{R}^+ \to L(X)$ be a continuous map and let $\tau > 0$ be given. The abstract linear delay-differential equation

$$\begin{cases} u'(t) = Au(t) - M(t)u(t - \tau) & \text{for } t \geq 0 \\ u(\theta) = \xi(\theta) & \text{for } -\tau \leq \theta \leq 0 \end{cases} \tag{12}$$

where $\xi : [-\tau, 0] \to X$ can be given a precise (integral) sense by the method of steps, because on each interval $[k\tau, (k+1)\tau]$ $(k = 1, 2, \ldots)$ Eq. (12) can be considered as a linear nonhomogeneous equation in which the forcing term is known from the previous step:

$$u(t) = \mathrm{e}^{At}\xi(0) + \int_0^t \mathrm{e}^{A(t-s)}M(s)\xi(s - \tau)\mathrm{d}s \quad \text{for } 0 \leq t \leq \tau$$

$$u(t) = \mathrm{e}^{A(t-\tau)}u(\tau) + \int_\tau^t \mathrm{e}^{A(t-s)}M(s)u(s - \tau)\mathrm{d}s \quad \text{for } \tau \leq t \leq 2\tau$$

and so on. This means that as long as the delay term appears, as above, in the equation we do not need the full machinery of the theory of abstract functional differential equations as developed, for instance, in Ha [5], and we can guarantee the *existence and uniqueness of mild solutions* for every continuous initial function ϕ. By "mild (or integral) solution" we understand a continuous function $u : [0, T) \to X$ which satisfies

$$u(t) = e^{At}\xi(0) + \int_0^t e^{A(t-s)} M(s)u(s-\tau)ds \quad \text{for } t \geq 0. \tag{13}$$

Regularity questions, of course, are more involved and depend on both the regularity of the semigroup and that of the forcing term. If A generates an *analytic semigroup*, it is well known that the hypotheses on the nonhomogeneous term are not very strong, and Hölder continuity is enough to ensure that the mild solution is also strong. In the general C^0 semigroup case (hyperbolic equations, for instance), more conditions are needed. For the purposes of this paper, continuity of the initial function $\xi : [-\tau, 0] \to X$ is enough, and no "fractional powers" A^α are needed.

Our main result is the following:

Theorem 1. *Let A be the infinitesimal generator of a linear C^0-semigroup $\{e^{At}\}_{t\geq 0}$ on a Banach space X and let $M : R^+ \to L(X)$ be a continuous map such that*

(i) $M(t) = 0$ *on* $[0, \tau] \cup [t^*, \infty)$, *where* $0 < \tau < t^* \leq 2\tau$.
(ii) *M commutes with the semigroup, that is:* $M(t)e^{At} = e^{At}M(t)$ *for every* $t \geq 0$.
(iii) $\int_0^\infty M(t)dt = \int_\tau^{t^*} M(t)dt = e^{A\tau}$.

Then:

(a) *All mild solutions of the abstract retarded functional differential equation* (12)

$$u'(t) = Au(t) - M(t)u(t-\tau) \tag{14}$$

vanish for $t \geq t^*$.

(b) *In particular, if $b : R^+ \to R$ is a continuous function which vanishes outside the interval* $[\tau, t^*]$ *and* $\int_0^\infty b(t)dt = 1$, *then*

$$M(t) = b(t)e^{A\tau} \tag{15}$$

satisfies conditions (i), (ii) *and* (iii).

Proof. Observe that $u(t) = e^{At}\xi(0)$ on the first time interval $[0, \tau]$ since $M(s)$ vanishes there. Let us rewrite the integral (13) that defines $u(t)$ on the second time interval $[\tau, 2\tau]$:

$$u(t) = e^{A(t-\tau)}u(\tau) - \int_\tau^t e^{A(t-s)} M(s)u(s-\tau)ds \quad \text{for } \tau \leq t \leq 2\tau \tag{16}$$

substitute $u(\tau) = e^{A\tau}\xi(0)$, $u(s-\tau) = e^{A(s-\tau)}\xi(0)$ and apply the assumed commutation property (ii) for M:

$$\begin{aligned} u(t) &= e^{At}\xi(0) - \int_\tau^t e^{A(t-s)} M(s) e^{A(s-\tau)}\xi(0)ds \\ &= e^{At}\xi(0) - \int_\tau^t e^{A(t-s)} e^{A(s-\tau)} M(s)\xi(0)ds \\ &= e^{At}\xi(0) - \int_\tau^t e^{A(t-\tau)} M(s)\xi(0)ds \\ &= \left[e^{At} - \left(\int_\tau^t M(s)ds \right) e^{A(t-\tau)} \right] \xi(0). \end{aligned}$$

We now evaluate at $t = t^*$ (which belongs to $[\tau, 2\tau]$ by assumption: this is very important), apply hypothesis (c) and obtain

$$\begin{aligned} u(t^*) &= \left[e^{At^*} - \left(\int_\tau^{t^*} M(s)ds \right) e^{A(t^*-\tau)} \right] u(0) \\ &= \left[e^{At^*} - e^{A\tau} e^{A(t^*-\tau)} \right] u(0) = \left[e^{At^*} - e^{At^*} \right] u(0) = 0 \end{aligned}$$

independently of $u(0)$. The proof is complete. □

Remark 1. Since the function $M(t)$ has the property that $M(t) \equiv 0$ for $t \in [0, \tau]$, the solution does not depend on the whole initial history ϕ, but only on $\phi(0)$. This is a very useful consequence of the fact that the delay element does not start acting until $t = \tau$. As the next section shows, this makes the practical implementation of the right-hand side of (12) as a controller much simpler.

3. Application to exact null-controllability

Observe now the following interesting fact: Since $u(\sigma) = e^{A\sigma}u(0)$ for $\sigma \in [0, \tau]$, we have

$$v(t) = -b(t)e^{A\tau}u(t-\tau) = -b(t)e^{A\tau}e^{A(t-\tau)}u(0) \tag{17}$$

$$= -b(t)e^{At}u(0) \quad \text{for } t \in [\tau, 2\tau]. \tag{18}$$

In other words, the right-hand side of (12) may be written in terms of $w(t) = e^{At}u(0)$, which is just the solution of the linear homogeneous problem $w' = Aw$ with the *same* initial condition $u(0)$. We thus have the following easy parallel implementation of the distributed null-controllability problem for the heat equation:

Theorem 2. *Let us consider the distributed null controllability problem*

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = v(t, x) & \text{on } (0, +\infty) \times \Omega \\ u(t, x) = 0 & \text{on } (0, +\infty) \times \partial\Omega \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) & \text{on } \Omega \end{cases} \tag{19}$$

and we want to find a control v such that $u(2\tau, x) = 0$ on Ω. This goal can be achieved by setting

$$v(t, \cdot) = -b(t)e^{A\tau}u(t-\tau, \cdot) \quad \text{for } t \in [\tau, 2\tau] \tag{20}$$

where b satisfies the hypotheses of Theorem 2. This controller can be implemented in parallel as follows:

$$(C)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = -b(t)w(t, \mathbf{x}) & \text{on } (0, +\infty) \times \Omega, \\ \dfrac{\partial w}{\partial t} - \Delta w = 0 & \text{on } (0, +\infty) \times \Omega, \\ u(t, \mathbf{x}) = 0, \ w(t, \mathbf{x}) = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) \quad \text{given} & \text{on } \Omega, \\ w(0, \mathbf{x}) = u(0, \mathbf{x}) = u_0(\mathbf{x}) & \text{on } \Omega. \end{cases} \tag{21}$$

4. Final remarks

For some special domains Ω it is possible to give an explicit description of some nonlocal operators $M(t)$ satisfying the required conditions. So, in the case of Dirichlet boundary conditions, if $\Omega = (0, +\infty)$, we can take

$$M(t)[\phi](x) = \frac{b(t)}{\sqrt{4\pi\tau}} \int_0^{+\infty} \left(e^{-\frac{(x-y)^2}{4\pi\tau}} - e^{-\frac{(x+y)^2}{4\pi\tau}}\right)\phi(y)dy \tag{22}$$

(see, e.g., [6], page 260), and if $\Omega = (0, l)$ with $l < +\infty$, we assume that

$$M(t)[\phi](x) = b(t) \int_0^l G(\tau, x, y)\phi(y)dy \tag{23}$$

with

$$G(t, x, y) = \frac{2}{l} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} \sin\frac{n\pi y}{l} e^{-\frac{n^2\pi^2 t}{l^2}}. \tag{24}$$

(see, e.g., [7], page 517). More generally, if Ω is bounded and $-A$ is self-adjoint with eigenvalues $\{\lambda_n\}_{n\geq 1}$ and normalized eigenfunctions $\{\phi_n\}_{n\geq 1}$, then

$$M(t)[\phi](x) = b(t) \int_\Omega G(\tau, x, y)\phi(y)dy \tag{25}$$

with

$$G(t, x, y) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y)e^{-\lambda_k t}. \tag{26}$$

In all cases, $b : [0, \infty) \to \mathbb{R}$ is continuous and vanishes on $[0, \tau] \cup [2\tau, \infty)$.

Many other explicit examples can be given (see the above mentioned references, and those in [9] and [10]).

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