Mathematical Issues Concerning the Boussinesq Approximation for Thermally Coupled Viscous Flows

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The flow of a viscous fluid driven by buoyancy forces is governed by balance equations for momentum, mass, and internal energy. Frequently, the Boussinesq approximation is employed to simplify the system, even in situations where dissipative heating cannot be neglected. The resulting equations violate the principle of conservation of total energy, which causes significant mathematical problems. We discuss these problems and possible remedies.

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The flow of a viscous fluid under gravity (or some other external force) is governed by balance equations for momentum, mass, and internal energy; these can be expressed in terms of the fluid velocity $\mathbf{v}$ and the thermodynamic variables p (pressure), ρ (density), and θ (temperature). The balance of momentum is given by

$$\rho\big(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \nabla\cdot\mathcal{S}(\mathbf{v},p) = \rho\mathbf{g}, \tag{1}$$

where $\mathcal{S}(\mathbf{v},p)$ is the stress tensor, $\mathbf{g}$ the acceleration due to the external force. For simplicity, we assume the fluid to be Newtonian, with a stress tensor of the form $\mathcal{S}(\mathbf{v},p) = \eta\big(\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{tr}}\big) - \big(\frac{2}{3}\eta\,\nabla\cdot\mathbf{v} + p\big)\,\mathcal{I}$, where η is the dynamic viscosity and $\mathcal{I}$ is the identity tensor, and suppose that $\mathbf{g} = -\nabla\phi$, for some potential ϕ. The balance of mass is given by

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{2}$$

the balance of internal energy by

$$\rho\big(e_t + (\mathbf{v}\cdot\nabla)e\big) - \nabla\cdot(\kappa\nabla\theta) = \mathcal{S}(\mathbf{v},p)\nabla\mathbf{v}, \tag{3}$$

where $e = C_v(\theta)$ is the internal energy density, with $C_v' = c_v$ the specific heat at constant volume and κ the thermal conductivity. Equations (1)–(3) must be supplemented by an equation of state, that is, a constitutive relation between the thermodynamic variables p, ρ, and θ.

Now suppose the fluid occupies a sufficiently regular bounded region of space Ω with fixed, mechanically impermeable, and thermally insulated walls. Then $\mathbf{v}$ and θ satisfy the boundary conditions $\mathbf{v} = 0$ and $\nabla\theta\cdot\mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ denotes the outward unit normal vector field. Integrating Equations (1)–(3) and using the boundary conditions, we obtain the following expressions for the rates of change of kinetic, potential, and internal energy:

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\rho\mathbf{v}^2 = -\int_\Omega \mathcal{S}(\mathbf{v},p)\nabla\mathbf{v} + \int_\Omega \rho\mathbf{g}\cdot\mathbf{v}, \qquad \frac{d}{dt}\int_\Omega \rho\phi = -\int_\Omega \rho\mathbf{g}\cdot\mathbf{v}, \qquad \frac{d}{dt}\int_\Omega \rho e = \int_\Omega \mathcal{S}(\mathbf{v},p)\nabla\mathbf{v}.$$

The three derivatives add up to zero, that is, total energy is conserved. Note that this is independent of the equation of state.

The simplest viable equation of state is $\rho = \rho_0 = \text{const}$. Assuming η to be constant as well, Eqs. (1)–(3) then simplify to

$$\rho_0\big(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \eta\Delta\mathbf{v} + \nabla p = \rho_0\mathbf{g}, \qquad \nabla\cdot\mathbf{v} = 0,$$

$$c_v\rho_0\big(\theta_t + (\mathbf{v}\cdot\nabla)\theta\big) - \nabla\cdot(\kappa\nabla\theta) = \tfrac{\eta}{2}|\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{tr}}|^2,$$

that is, the classical Navier-Stokes equations, along with a semilinear heat equation. Note that the first two equations are independent of θ: since the density is constant, there is no buoyancy, and temperature fluctuations do not affect the fluid motion. The quadratic source term on the right-hand side of the heat equation, which represents dissipative heating, causes major mathematical difficulties; but these have been addressed and overcome (see, for example, [12, Chapter 3.4]), even in the case of temperature-dependent viscosity (see [7, 15]).

To capture the effects of buoyancy, ρ must depend on θ. Provided that the temperature fluctuates in a fairly narrow range, it is reasonable to assume that ρ decreases linearly with θ; that is, $\rho = \rho_0(1-\alpha\bar\theta)$, where $\bar\theta = \theta - \theta_0$ is the temperature deviation from a reference temperature θ_0, $\rho_0 = \rho(\theta_0)$ is the reference density, $\alpha = -\rho'(\theta_0)/\rho(\theta_0)$ the thermal expansion coefficient.

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In the well-known Boussinesq approximation, this ansatz for ρ is used to compute the buoyancy force $\rho\mathbf{g}$ on the right-hand side of Eq. (1), but everywhere else in the equations, ρ is replaced by ρ_0. In other words, the fluid is considered "thermally compressible, yet mechanically incompressible" (see [13, 17] for rigorous justifications of this procedure). Assuming $\theta_0 = 0$, one obtains the equations

$$\rho_0\big(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \eta\Delta\mathbf{v} + \nabla p = \rho_0(1-\alpha\theta)\mathbf{g}, \qquad \nabla\cdot\mathbf{v} = 0,$$

$$c_v\rho_0\big(\theta_t + (\mathbf{v}\cdot\nabla)\theta\big) - \nabla\cdot(\kappa\nabla\theta) = \tfrac{\eta}{2}|\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{tr}}|^2.$$

This system, frequently used in the engineering literature, is energetically inconsistent; in fact, the production of kinetic energy due to buoyancy, $\alpha\rho_0\int_\Omega \theta\mathbf{g}\cdot\mathbf{v}$, is not balanced by a corresponding reduction in potential energy. Significant mathematical difficulties ensue, and the well-posedness of the associated initial-boundary value problems, specifically the global-in-time existence of any kind of weak solutions, is largely open. (The only result in this direction appears to be [10, Theorem 2.1], where a two-dimensional Bénard problem is treated.)

A possible (if not necessarily feasible) remedy is to neglect dissipative heating. This allows one to establish a-priori bounds for θ, independent of $\mathbf{v}$, and then the analysis parallels that of the classical Navier-Stokes equations without thermal coupling (see [8, 9, 14]). Another approach is to rewrite the balance of internal energy, Eq. (3), in terms of c_p (specific heat at constant pressure) instead of c_v, using the thermodynamic relation $c_p = c_v + \frac{\alpha\theta}{\rho}\frac{\partial p}{\partial\theta}$, and then replace p by the hydrostatic pressure $p_0 = -\rho_0\phi$ (see [1] for a justification); this leads to the system

$$\rho_0\big(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \eta\Delta\mathbf{v} + \nabla p = \rho_0(1-\alpha\theta)\mathbf{g}, \qquad \nabla\cdot\mathbf{v} = 0,$$

$$c_p\rho_0\big(\theta_t + (\mathbf{v}\cdot\nabla)\theta\big) - \nabla\cdot(\kappa\nabla\theta) = \tfrac{\eta}{2}|\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{tr}}|^2 + \alpha\rho_0\theta\mathbf{g}\cdot\mathbf{v}.$$

The additional source term on the right-hand side of the heat equation is referred to as "adiabatic heating" (see [11] and the references therein); it formally restores the conservation of total energy. Even so, global existence results have been obtained only for a non-Newtonian model [16].

A third possibility, motivated by earlier work by Díaz et al. [2, 6], is to use a Boussinesq-like ansatz for ρ not only in the buoyancy force, but also in the rate of change of internal energy. Assuming ρ to be a given continuous function of θ, positive and nonincreasing in a neighborhood (θ_1, θ_2) of the reference temperature θ_0, and letting $\rho_0 := \rho(\theta_0)$, we consider the system

$$\rho_0\big(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \eta\Delta\mathbf{v} + \nabla p = \rho(\theta)\mathbf{g}, \qquad \nabla\cdot\mathbf{v} = 0,$$

$$c_v\rho(\theta)\big(\theta_t + (\mathbf{v}\cdot\nabla)\theta\big) - \nabla\cdot(\kappa\nabla\theta) = \tfrac{\eta}{2}|\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{tr}}|^2,$$

subject to the constraint $\theta_1 \le \theta \le \theta_2$. First results indicate that this system, under homogeneous Dirichlet and Neumann boundary conditions for $\mathbf{v}$ and θ, respectively, admits global-in-time solutions in a weak sense.

In [3, 4], we study a much simplified model problem that still captures the characteristic difficulties: the parabolic system

$$v_t - \Delta v = \rho(\theta), \quad \rho(\theta)\theta_t - \Delta\theta = |\nabla v|^2, \tag{4}$$

for two scalar unknowns v and θ, satisfying homogeneous Dirichlet and Neumann boundary conditions, respectively, where $\rho \in C(\mathbb{R})$ is positive and nonincreasing on a bounded interval (a, b). Given initial data $v_0 \in H_0^1(\Omega)$, $\theta_0 \in H^1(\Omega)$ with $a \le \theta_0 \le b$, and given any $T \in (0, \infty)$, we obtain functions $v \in L^2((0,T), H^2(\Omega)) \cap C([0,T], H^1(\Omega))$ and $\theta \in L^2((0,T), H^1(\Omega)) \cap C([0,T], L^2(\Omega))$ with $a \le \theta \le b$, satisfying the boundary and initial conditions, and satisfying Eqs. (4) in the weak sense, with $|\nabla v|^2$ replaced by some function $G_v \in L^1((0,T), L^1(\Omega))$ with $|\nabla v|^2\chi_{\theta<1} \le G_v \le |\nabla v|^2$. Additional results concern the existence of *regular solutions* (with $\theta \in L^2((0,T), H^2(\Omega)) \cap C([0,T], H^1(\Omega))$), *exact solutions* (with $G_v = |\nabla v|^2$), and *strong solutions* (regular and exact). Uniqueness of strong solutions is proved in the spatially two-dimensional case, for sufficiently regular intial data.

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