Energy methods for free boundary problems: new results and some remarks on numerical algorithms

CANUM 2002
Anglet, May 28, 2002

1. Introduction

Different energy methods since the beginning of the eighties for the study of the free boundaries giving rise by the solutions of nonlinear partial differential equations

S.N. Antontsev, J.I. Díaz and S.I. Shmarev,

Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics,

Progress in Nonlinear Differential Equations and Their Applications, **48**, Birkhäuser, Boston, 2002.

In the notes (technical version): some new applications of such methods and some remarks on the persistence of the free boundaries for the associated discretized problems.

This presentation: more pedagogical version

Energy methods:

• of special interest in the study of free boundary problems when the traditional methods based on the comparison principles fail (typical examples: higher-order equations or a system of pde's)

$$\begin{cases} m(x)\frac{\partial s}{\partial t} = \operatorname{div}\left(K_0(x)a(x,s)\nabla s + K_1(x,s)\nabla p + \mathrm{f}_0\right) + q, \\ 0 = \operatorname{div}\left(K(x,s)\nabla p + \mathrm{f}(x,s)\right), \end{cases}$$

G.Gagneux and M.Madaune-Tort, *Analyse mathematique de modeles non lineaires de l'ingenierie petroliere*, Mathematiques & Applications, **22**, Springer-Verlag, Paris, 1995.

• even when the comparison principle holds, it may be extremely difficult to construct suitable sub or super-solutions (transport terms, variable coefficients, unbounded right-hand side term, etc.)

Very recent personal conclusion:

relevance of an ordinary differential equation in the study of many different techniques for free boundary problems

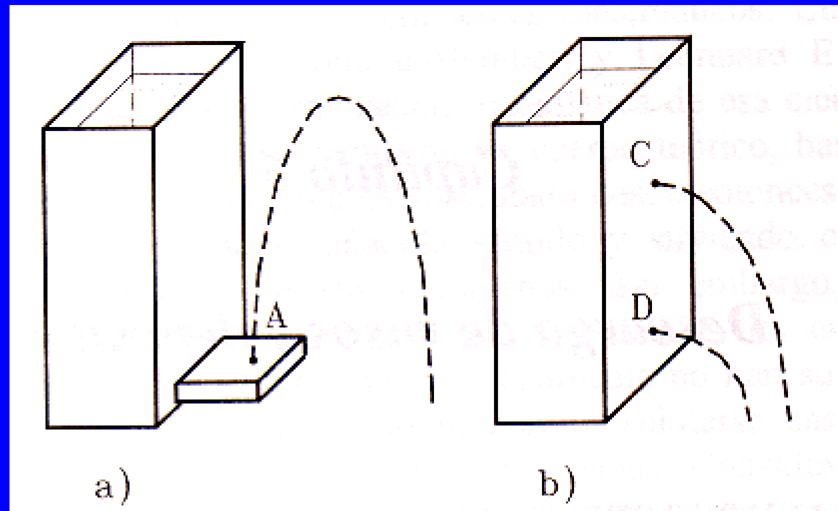
$$\begin{cases} \dfrac{dh}{dt}(t) + C\sqrt{h(t)} = 0 \\ h(0) = h_0 > 0 \end{cases}$$

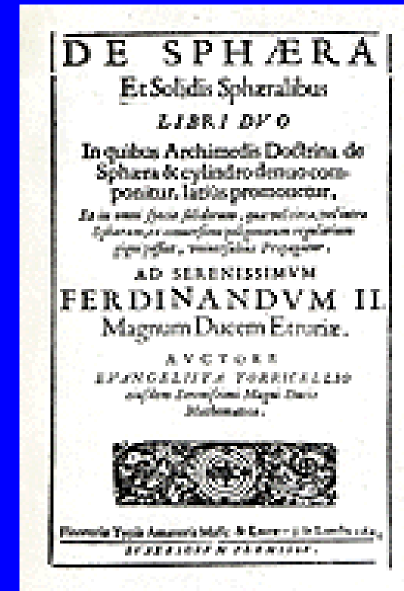
* Example for nonuniqueness of solutions

* Torricelli's law of flow $\quad v = \sqrt{2gh}$

The efflux of a liquid from a small orifice in the walls of a vessel

Evangelista Torricelli (Rome 1608-Florence 1647)




DE SPHÆRA
Et Solidis Sphæralibus
LIBRI DVO
In quibus Archimedis Doctrina de Sphæra & cylindro denuo componitur, latius promouetur,
AD SERENISSIMVM
FERDINANDVM II.
Magnum Ducem Etruriæ.
AVCTORE
EVANGELISTA TORRICELLIO

De motu gravium naturaliter accelerato, Firenze, 1643

Opera geometrica (1644)

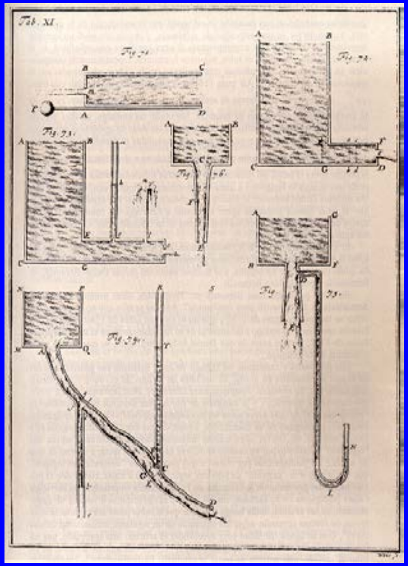
I. Newton: Propositio XXXVI. Problem VIII. Second Edition of *Principia* (1713) [First edition: 1687]

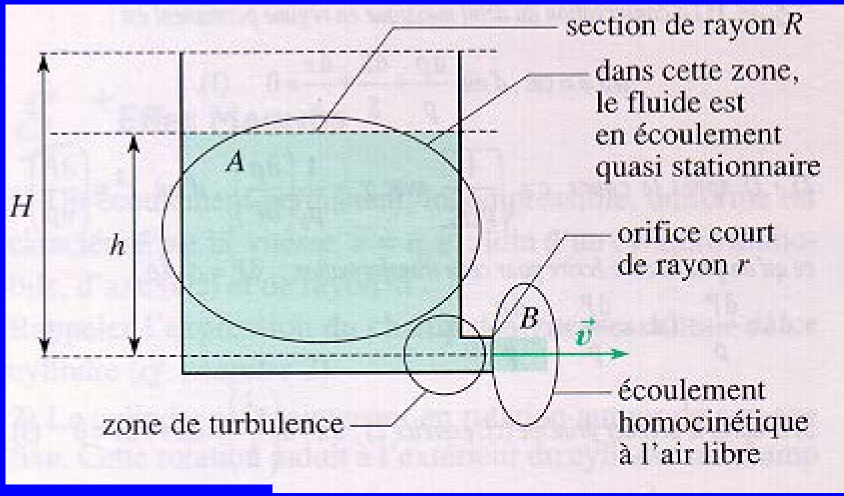
F. Di Trocchio, *Las mentiras de la ciencia*, Alianza, Madrid, 1998

P. Varignon, J. Lagrange, ...

Daniel Bernoulli (1700-1782)

Hydrodynamics (1738)

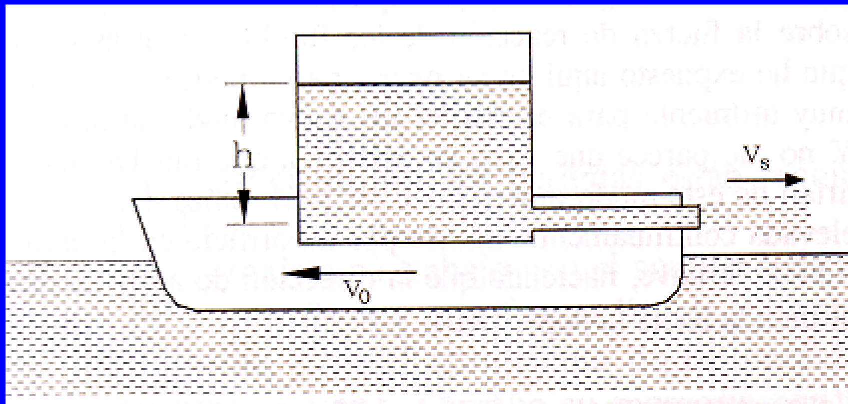




$$\frac{v_A^2}{2} + g z_A + \frac{p_A}{\rho} = \frac{v_B^2}{2} + g z_B + \frac{p_B}{\rho}$$

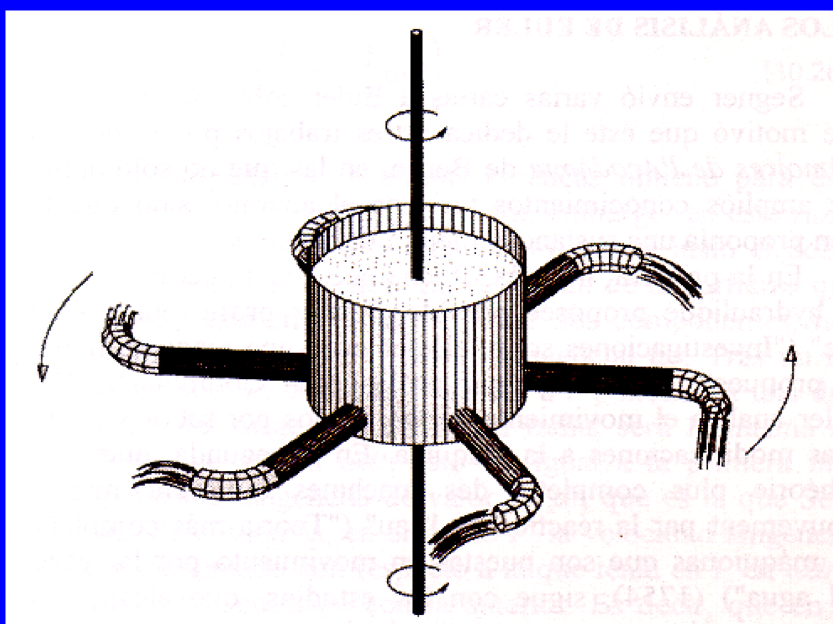
$$z_A - z_B = h,\ R^2 V = r^2 v \qquad V = -\frac{dh}{dt}(t)$$

$$-\frac{dh}{dt}(t) = \frac{r^2}{R^2}\sqrt{2gh} \qquad \int_0^H \frac{dh}{\sqrt{h}} = \frac{r^2}{R^2}\sqrt{2gT}$$



Reaction ship

(Daniell Bernoulli, Hydrodynamica, 1734)



J.A. Segner (1750)

L. Euler (1750)

Baptism (Anglet, May 28 , 2002)

The Torricelli-Bernoulli equation

$$\begin{cases} \dfrac{dh}{dt}(t) + C\sqrt{h(t)} = 0 \\ h(0) = h_0 > 0 \end{cases}$$

More in general:

Lemma.

Consider

$$\begin{cases} \dfrac{dh}{dt}(t) + a(t)\varphi(h(t)) = 0 \\ h(0) = h_0 > 0 \end{cases}$$

and assume

$$\Psi(\tau) = \int_0^\tau \frac{ds}{\varphi(s)} < +\infty, \forall \tau > 0, \varphi \geq 0,$$

$$a \geq 0,\ a \in L^1_{loc}(0,+\infty)$$

Then $$h(t) = \Psi^{-1}\left(\left[\Psi(h_0) - \int_0^t a(r)dr\right]_+\right)$$

In particular, if

$$\Psi(h_0) - \int_0^{T_0} a(r)dr = 0 \text{ for some } T_0 > 0$$

Then $\exists T_0 > 0$ such that $h(t) \equiv 0 \; \forall t \geq T_0.$

Corollary. The last conclusion holds if

$$\frac{dh}{dt}(t) + a(t)\varphi(h(t)) \leq 0$$

Remark. Backwards variable $h(t) = E(\rho), \rho = R - t$

Corollary. Let $$\begin{cases} \dfrac{dE}{d\rho}(\rho) - \widetilde{a}(\rho)\varphi(E(\rho)) \geq 0 \\ E(R) = E_0 > 0 \end{cases}$$

Assume that

$$\Psi(E_0) - \int_{R_0}^{R} a(r)dr = 0 \text{ for some } R_0 \in (0, R)$$

Then $$E(\rho) \equiv 0 \ \forall \rho \in [0, R_0]$$

The rest of the talk: some applications to energy methods.

Remark. For an application to the construction of super and subsolutions for

$$-div(|\nabla u|^{p-2} \nabla u) + g(u) = f(x)$$

with

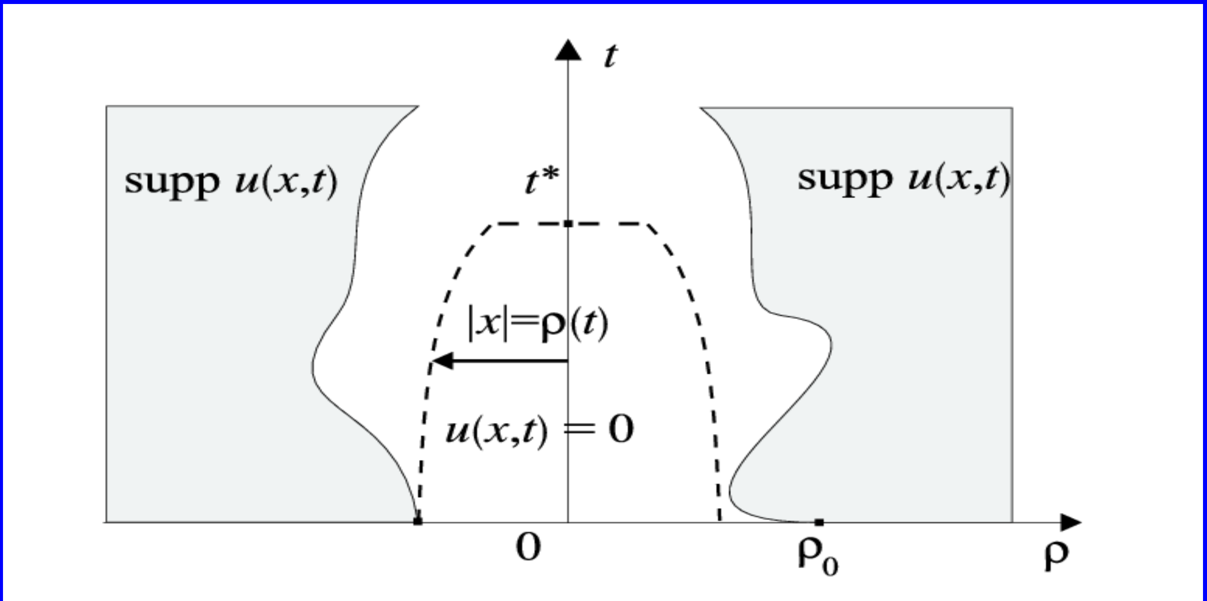
$$\int_0^{\tau} \frac{ds}{\sqrt[p]{G(s)}} < +\infty, \ G(s) = \int_0^s g(r)dr$$

see: J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Pitman, London. 1985.

2. A local energy method

$$v_t - (v^m)_{xx} = 0, \quad v(\cdot, 0) = v_0,$$

Finite speed of propagation: solutions corresponding to compactly supported initial data remain with compact support at least for some time, i.e. u_0 of compact supp in $\Omega \Rightarrow u(\cdot, t)$ of compact supp in Ω for $t < t^*$.



For $x_0 \in \mathbb{R}$ and $r > 0$ $\qquad B_r := \{x : |x - x_0| < r\}$

$$u := v^m,\ p := 1/m.$$

$$(u^p)_t - u_{xx} = 0, \quad u(\cdot,0) = u_0 := v_0^m.$$

Assume $\quad u_0(x) = 0 \quad \text{in } B_{\rho_0} \quad \text{for some } \rho_0 > 0.$

Multiplying by u

$$\tfrac{p}{p+1}\int_{B_\rho} u(t)^{p+1} + \int_0^t\int_{B_\rho} |u_x|^2 = \int_0^t\int_{\partial B_\rho} u u_x n.$$

Define the **energy functions**

$$b(\rho,t) := \sup_{0\le\tau\le t}\int_{B_\rho} u(\tau)^{p+1}, \quad E(\rho,t) := \int_0^t\int_{B_\rho} |u_x|^2.$$

Hölder's inequality in (2) and using

$$E_\rho(\rho,t) = \int_0^t\int_{\partial B_\rho} |u_x|^2,$$

gives

$$\tfrac{p}{p+1}b(\rho,t) + E(\rho,t) \le \left(\int_0^t\int_{\partial B_\rho} u^2\right)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}} =: I(E_\rho)^{\frac{1}{2}}.$$

To estimate I (a key step of the method) we apply the *interpolation-trace inequality* (Díaz-Veron (1985)).

Lemma

Let $D \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $\varphi \in W^{1,p}(D)$ with $1 < p < \infty$. Then

$$||\varphi||_{q,\partial D} \leq C \left(||\nabla\varphi||_{p,D} + ||\varphi||_{\theta,D} \right)^{\gamma} ||\varphi||_{r,D}^{1-\gamma},$$

with $C > 0$ and for

$$1 \leq \theta < \infty, \quad 1 \leq q, r \leq \infty, \quad \text{and} \quad \gamma = \frac{p(qN - r(N-1))}{(q(p(N+r) - Nr))}$$

Using this lemma together with Young and Hölder inequalities we get

$$I \leq ct^{\frac{1-\gamma}{2}}(E+b)^{\beta}, \qquad \beta = \gamma/2 + (1-\gamma)/(p+1)$$

$$\beta > 1/2 \Longleftrightarrow p < 1.$$

$$I(E_\rho)^{\frac{1}{2}} \leq \varepsilon(E+b) + c_\varepsilon t^{\frac{1-\gamma}{\kappa}}(E_\rho)$$

$\varepsilon > 0,\ \kappa := 2(1-\beta) < 1.$

$$E \leq c(E_\rho)^{\frac{1}{\kappa}},$$

$$E(\rho, t) = 0 \quad \text{for all } t \in (0, t^*) \quad \rho \leq r(t).$$

$$\rho_0 - \frac{1}{c} E^{1-\kappa}(\rho_0, t) =: r(t),$$

* Method applicable to more general equations

$$\frac{\partial \psi(u)}{\partial t} - \mathrm{div}\, \mathbf{A}(x,t,u,D\,u) + B(x,t,u,D\,u) + C(x,t,u) + \beta(u) \ni f(x,t),$$

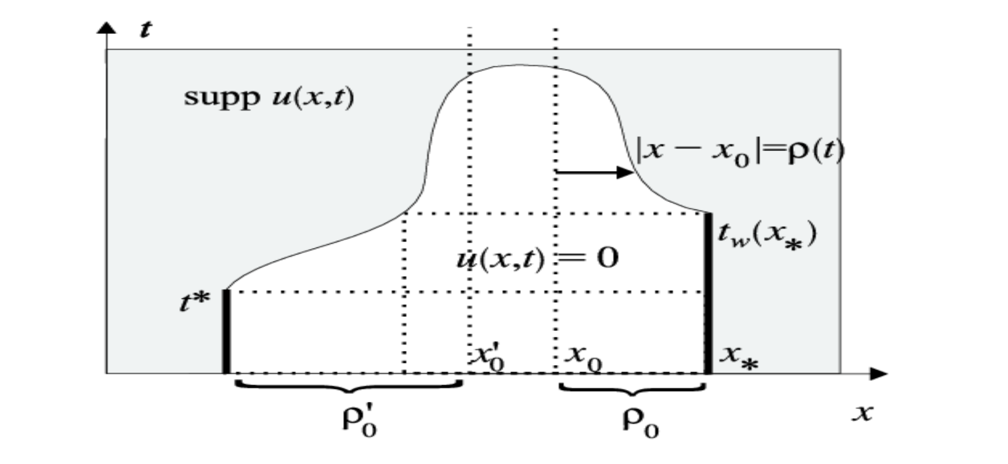
$$|\mathbf{A}(x,t,r,\mathbf{q})| \le C_1 |\mathbf{q}|^{p-1}, C_2\,|\mathbf{q}|^p \le \mathbf{A}(x,t,r,\mathbf{q}) \cdot \mathbf{q}, \qquad (2a)$$

$$|B(x,t,r,\mathbf{q})| \le C_3\,|r|^{\alpha}\,|\mathbf{q}|^{\beta},\ 0 \le C(x,t,r)\,r, \qquad (2b)$$

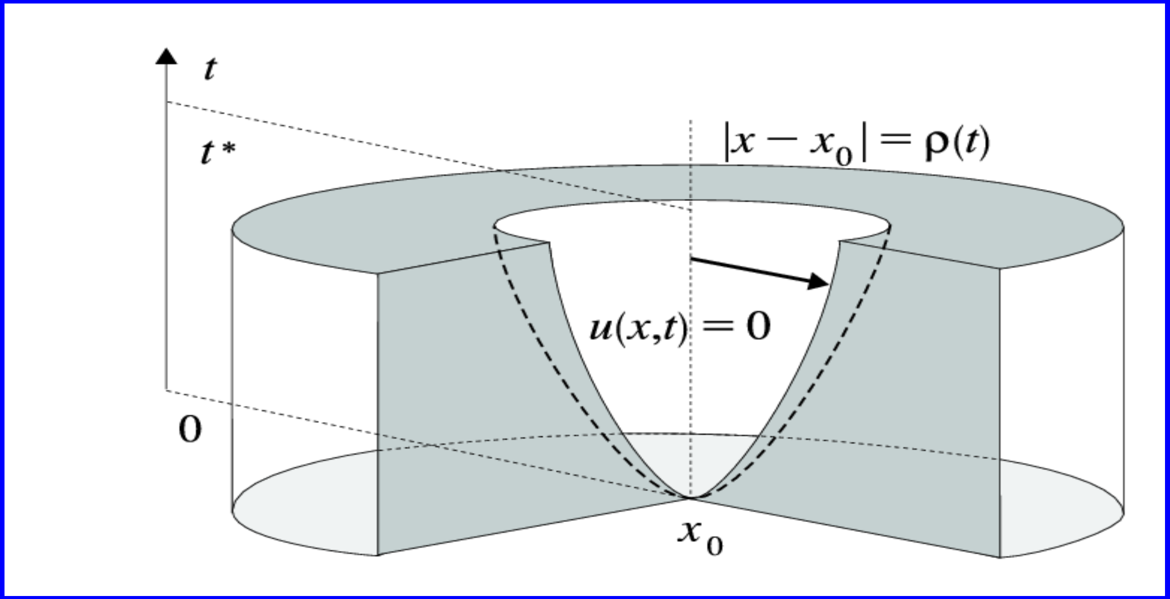
$$C_6\,|r|^{\gamma+1} \le G(r) \le C_5\,|r|^{\gamma+1},\ \text{where } G(r) = \psi(r)\,r - \int_0^r \psi(\tau)\,d\tau.$$

* Independent of boundary conditions

* Other qualitative properties: finite waiting time



* Formation of a “dead core” (positive initial datum)



* New: case of the obstacle problem and other multivalued equations

•Application to systems and higher order equations:

•New: **Stopping a viscous fluid by an external feedback field**

(S.N. Antontsev, J.I. Díaz and H.B. de Oliveira)

$$\Omega = (0,+\infty) \times (0,L),\ L > 0,$$

$$\begin{cases} -\nu\Delta\mathbf{u} = \mathbf{f} - \nabla p, \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}(0,y) = \mathbf{u}_*(y), y \in (0,L), & \mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0},\ x \in (0,+\infty), \\ |\mathbf{u}(x,y)| \to 0,\ \text{as } x \to +\infty \text{ and} & y \in (0,L). \end{cases}$$

Question : can we find an external localized force field f stopping the fluid at a finite distance: i.e. such that

$$\mathbf{u}(x,y) = \mathbf{0} \quad \text{for } x \geq x_{\mathbf{u}} \text{ and } y \in (0,L)$$ **?**

$$\mathbf{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{f}(\mathbf{x},\mathbf{u}) = (f_1(\mathbf{x},\mathbf{u}), f_2(\mathbf{x},\mathbf{u})),$$

$$-\mathbf{f}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u} \geq \delta\chi_{(0,x_{\mathbf{f}})}(\mathbf{x})|u(\mathbf{x})|^{1+\sigma} + \mathbf{g}(\mathbf{x}) \cdot \mathbf{u}$$

$$\begin{cases} \nu\Delta^2\psi + \frac{\partial f_1}{\partial y}(\mathbf{x}, \psi_y, -\psi_x) - \frac{\partial f_2}{\partial x}(\mathbf{x}, \psi_y, -\psi_x) = 0 & \text{in } \Omega, \\ \psi(x,0) = \psi(x,L) = \frac{\partial \psi}{\partial n}(x,0) = \frac{\partial \psi}{\partial n}(x,L) = 0 & \text{for } x \in [0,+\infty), \\ \psi(0,y) = \int_0^y u_*(s)ds, \ \frac{\partial \psi}{\partial n}(0,y) = v_*(y), \quad , & \text{for } y \in (0,L), \\ \psi(x,y), \ |\nabla \psi(x,y)| \to 0, & \text{as } x \to +\infty, \text{for } y \in (0 \end{cases}$$

$$\mathbf{u} = (\psi_y, -\psi_x)$$

half-planes technique introduced by F. Bernis (1984),....

* Global energy method: global extinction time

$$\begin{cases} \frac{\partial}{\partial t}\left(u|u|^{\gamma-1}\right) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{\sigma-1}u = f + \operatorname{div} \mathbf{g} & \text{in } Q := \Omega \times (0,+\infty), \\ u = 0 & \text{on } \sum_T = \Gamma \times (0,+\infty). \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Thanks for your attention