FEEDBACK DELAY AS A CONTROL TOOL: THE COMPLEX GINZBURG-LANDAU EQUATION WITH LOCAL AND NONLOCAL DELAYED PERTURBATIONS

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Abstract. This survey collects several works by the authors dealing with the role of suitable feedback delayed terms as a suitable control, to different purposes, concerning the behaviour of solutions to the complex Ginzburg-Landau

$$\frac{\partial u}{\partial t} - (1+i\epsilon)\,\Delta u + (1+i\beta)\,|u|^2\,u - (1-i\omega)\,u = F\,(u(x,t-\tau))$$

for $t > 0$, with

$$F\,(u(x,t-\tau)) = e^{i\chi_0}\left\{\frac{\mu}{|\Omega|}\int_\Omega u\,(x,t-\tau)\,dx + \nu u\,(x,t-\tau)\right\},$$

where μ, $\nu \geq 0$, $\tau > 0$ but the rest of real parameters ϵ, β, ω and χ_0 do not have a prescribed sign. Besides the existence and uniqueness of weak solutions of problem, we analyze the tole of delayed feedback perturbations to control the chemical turbulence. The Höpf bifurcation is also considered in presence of the delayed terms

Dedicated to Miguel. A.F. Sanjuán on occasion of his 60th Birthday

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1. Introduction

This survey collects several works by the authors dealing with the role of suitable feedback delayed terms as a suitable control, to different purposes, concerning the behaviour of solutions to the complex Ginzburg-Landau equation (CGLE)

$$\frac{\partial \mathbf{u}}{\partial t} - (1+i\epsilon)\,\Delta \mathbf{u} + (1+i\beta)\,|\mathbf{u}|^2\,\mathbf{u} - (1-i\omega)\,\mathbf{u} = \mathbf{F}\,(\mathbf{u}(x,t-\tau))\,.$$

Some introductory words on the complex Ginzburg-Landau equations: Before to deal with to the role of delayed terms, it is important to point out that the complex Ginzburg-Landau is as very relevant generalization of the nonlinear Schrödinger equations used for many modeling purposes after the pioneering work by Ginzburg and Landau [40] in 1950 in superconductivity. Of course, the *Ginzburg-Landau equation* has been systematically used to study different types of phenomena in superconductor theory. Moreover, a rich variety of mathematical models of PDEs have also been inspired by the original model of Ginzburg and Landau to study a large number of physical phenomena, (see for instance Kuramoto [48], Levy [50], Temam [65] and references therein). Notice that the presence of complex coefficients introduces important differences with the classical Ginzburg-Landau equations arising in superconductivity [24]. Let us see with some detail how this equation arises in the study of some apparently different problems as some reaction-diffusion systems. The evolution of a chemical system consisting of n species which are reacting with each other and allowed to diffuse in a spatially extended medium, is generally described by a n-component reaction-diffusion equation for the $n-$concentrations $\mathbf{c}(x,t)$

$$\frac{\partial \mathbf{c}}{\partial t} = \mathbf{F}(\mathbf{c};p) + \mathbf{D}\Delta\mathbf{c}, \tag{1}$$

where $\mathbf{F}$ denotes the typically nonlinear reaction term representing chemical kinetics, $\mathbf{D}\Delta\mathbf{c}$ the diffusion term (being $\mathbf{D}$ the diffusion matrix) and p a scalar control parameter. Let us assume that this system has a homogeneous, stationary solution $\mathbf{c_s}$ which undergoes a *Hopf bifurcation* at $p = p_0$: i.e., for $p \in (p_0, p_0+\varepsilon)$ the stationary solution $\mathbf{c_s}$ becomes a time periodic solution, at least for $\varepsilon > 0$ small enough. It has been shown by Kuramoto and others that the dynamics of any reaction-diffusion system (1) in the vicinity of a Hopf bifurcation is described, by means of suitable parametrizations, by a nonlinear parabolic equation with complex coefficients, the so-called *complex Ginzburg-Landau equation* (CGLE), see, e.g., [48], [36]. The relation between reaction-diffusion systems and the CGLE has been treated in many texts, here we will follow the presentation

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of [46]. After a convenient choice of variables $\mathbf{X} = \mathbf{c} - \mathbf{c_s}$ (the *concentration deviations*) and $\epsilon = p - p_0$, the system can be reformulated as

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{J}\mathbf{X} + \mathbf{f}(\mathbf{x}, \epsilon) + \mathbf{D}\Delta\mathbf{X},$$

where $\mathbf{J}$ is the Jacobian matrix for the homogeneous system evaluated at $\mathbf{X_s} = \mathbf{0}$, i.e. $\mathbf{F}(\mathbf{c}; p) - \mathbf{F}(\mathbf{c}_s; p_0) = \mathbf{J}\mathbf{X} + \mathbf{f}(\mathbf{x}, \epsilon)$. At the *bifurcation point*, $\mathbf{J}$ has two imaginary eigenvalues $\pm \mathrm{i}\omega_0$, being ω_0 the so-called *Hopf frequency*. The corresponding *right eigenvectors* $\mathbf{e}_1$ and $\mathbf{e}_2 = \bar{\mathbf{e}}_\mathbf{1}$ (normalized with left eigenvectors $\mathbf{e}_i^+$ according to $\mathbf{e}_i^+ \mathbf{e}_j = \delta_{ij}$) span the *center subspace* E^c of the homogeneous solution. The *center manifold* W^c is tangent to E^c at $\mathbf{X} = \mathbf{0}$, $\epsilon = 0$. The other $n-2$ eigenvalues are all assumed to be large and negative. This assures that a homogeneous solution converges fast toward W^c provided that $\mathbf{X}$ and ϵ are sufficiently small (for details and further references see [46]). This allows us to express the concentration deviations $\mathbf{X}$ in terms of *amplitude coordinates* $\mathbf{Y} \in E^c$ by

$$\mathbf{X} = \mathbf{Y} + \mathbf{h}(\mathbf{Y}, \epsilon).$$

This equation describes a mapping from coordinates in the center subspace E^c onto the center manifold W^c. The function $\mathbf{h}(\mathbf{Y}, \epsilon)$ is selected in such a way to successively eliminate as many nonlinear terms as possible from the kinetic equations starting from the lowest order [46]. Each kind of bifurcation is characterized by the specific terms which cannot be eliminated (the so-called *resonant terms*). In this way we obtain a general equation valid for all reaction-diffusion equations undergoing a given bifurcation. In the case of the Hopf bifurcation, neglecting the diffusion term, to third order we obtain the so-called *Stuart-Landau equation*

$$\frac{d\mathbf{Y}}{dt} = (i\omega_0 + \sigma_1\epsilon)\mathbf{Y} - g|\mathbf{Y}|^2\mathbf{Y},$$

where $\mathbf{Y}$ is a complex amplitude given by $\mathbf{Y} = Y\mathbf{e}_1 + \overline{Y}\mathbf{e}_2$. The parameters σ_1 and g are complex and given by solutions of lengthy equations given in [46]. The Stuart-Landau equation represents the *normal form* of a homogeneous system close to a Hopf bifurcation. Performing a similar derivation, but including diffusion, we arrive at

$$\frac{\partial \mathbf{Y}}{\partial t} = (i\omega_0 + \sigma_1\epsilon)\mathbf{Y} - g|\mathbf{Y}|^2\mathbf{Y} + d\Delta\mathbf{Y},$$

with $d = \mathbf{e}_1^+ \cdot \mathbf{D}\mathbf{e}_1$. After rescaling of space, time, and introducing $\mathbf{A}$ for $\mathbf{Y}$, we finally arrive at the rescaled complex Ginzburg-Landau equation

$$\frac{\partial \mathbf{A}}{\partial t} = (1 - i\omega)\mathbf{A} - (1 + \mathrm{i}\alpha)|\mathbf{A}|^2\mathbf{A} + (1 + i\beta)\Delta\mathbf{A}, \tag{2}$$

where $\mathbf{A}$ is the *complex oscillation amplitude*, ω the *linear frequency parameter*, α the *nonlinear frequency parameter*, and β the *linear dispersion coefficient*. All reaction-diffusion systems sufficiently close to a Hopf bifurcation are described by the complex Ginzburg-Landau equation. The specific details of the original system are incorporated in the parameter values. If one wishes to express the solution of the CGLE in the original variables, to first order the concentrations of the chemical species are expressed by

$$\mathbf{c} = \mathbf{c_s} + \sqrt{\epsilon}(Y(x,t)\mathbf{e}_1 + \overline{Y}(x,t)\mathbf{e}_2).$$

Different scalings of the CGLE are considered in the literature [6]. Here, we assume that the Hopf frequency is not scaled out, and hence contributes to ω in Eq. (2). We also refer the reader to Appendix B of [48] for the detailed derivation of the CGLE associated to the Brusselator model. This bifurcation is omnipresent in reaction-diffusion systems like the Belousov-Zhabotinsky reaction or the oxidation of CO on platinum. The basic solution of CGL, uniform oscillations, are unstable and lead to *space-time chaos* if $1 + \alpha\beta < 0$. Different space-time chaos control schemes have been explored over the last years, both for the CGLE [11], [12], [22], and for the CO reaction model [23], also validated in the corresponding experiment [47].

Some introductory words on the delay as control. It is well-known that feedback delayed term can be introduced to control very complex phenomena (see, e.g. the expositions made in [9], [39] and [67]). In recent years, many studies have investigated the mechanisms capable of controlling the behavior of dynamic systems, in particular, of reaction-diffusion systems in a regime of space-time chaos. A method widely studied for its efficiency is that of *Time-delay autosynchronization* (TDAS), based on a work by Pyragas [58]. The idea of this method is to apply a signal to the system (*feedback* $\mathbf{F}$) that is proportional to the difference between the current state of the system (measured by an appropriate variable $\mathbf{A}$) at a certain time and the state of the system at a previous time $t - \tau$, where τ represents an adjustable delay: $\mathbf{F}(t, \tau) \propto \mathbf{A}(t - \tau) - \mathbf{A}(t)$. In the case of some finite-dimensional dynamical systems, it was proved that using this method it is possible to manipulate the dynamics of a system and create and stabilize a multitude of spatial-temporal patterns, not present or unstable without TDAS. We will collect here several works dealing with the application of this general philosophy to the case of the complex Ginzburg-Landau equation. Sometimes the feedback consists of a global variable or a spatial average of a local variable. This type of feedback is easy to implement in a real system and therefore has been applied frequently. In the next Section we extend the concept of feedback by admitting local components.

It is remarkable, that even if the Hopf bifurcation is supercritical, and hence the limit cycle a stable solution of the Stuart-Landau equation, the oscillations in the spatially-extended system may be unstable. The resulting states of spatiotemporal chaos appear if the Benjamin-Feir-Newell criterion $1+\alpha\beta < 0$ is fulfilled, a phenomenon that is induced by the diffusive coupling and that is therefore genuine to a system with spatial degrees of freedom. Considerable efforts have been made to understand this type of chaotic behavior and to apply methods to suppress this kind of turbulence and replace it by regular dynamics. In the context of the reaction-diffusion systems, the introduction of forcing terms or global feedback terms have been shown to be efficient ways to control turbulence [54], [47]. Still, control of chaotic states in nonlinear systems is a wide field of research that we cannot review here [60]. Global feedback methods, where a spatially independent quantity (or, e.g., a spatial average of a space-dependent quantity) is coupled back to the system dynamics, have attracted much attention since in many cases the models are simpler and easier to be carried out experimentally. Nevertheless, local methods have gained interest in recent years since they allow to access to other solutions of the systems and may also be implemented, such as in the light-sensitive BZ reaction or in neurophysiological experiments [54]. Feedback methods with an explicit time delay amplify the range of possibilities of control that can be applied to the system and provide the researcher with an additional adjustable parameter. On the level of the mathematical description, the model equations become delay differential equations, [9] [39] Obviously, time delay feedback can be applied to any solution of the dynamics, not necessarily to a chaotic one.

In 1996, Battogtokh and Mikhailov [12], introduced a nonlocal delayed term in the generalized equation in order to control the system and suppress turbulence (see also Battogtokh, A. Preusser and Mikhailov [11]). The equation appears in the study of some chemical reactions and models the concentration of various reacting species. D. Battogtokh and A. Mikhailov analyze numerically this model and control the turbulence thanks to the delayed term. The idea is to adjust two real parameters: the feedback intensity μ and the delay time τ. The results were made rigorous later in a series of articles, which we indicate below. This work is a natural companion of those rigorous studies. For instance, a first rigorous approach was presented in Casal and Díaz [30], where the control of turbulence in oscillatory reaction-diffusion systems is made through a combination of global and local feedback by means of a pseudo-linearization technique (see also Casal and Díaz [30], [29], Casal, Diaz, Padial, Tello[28],Casal, Díaz and Stich [31], [32] and Casal, Díaz, Stich and Vegas [33]). In Diaz, Padial, J.I. Tello and L. Tello [38], we consider weaker assumptions on the initial data and parameters than in the above mentioned papers and others results in the literature (see, e.g. [4]).

On some collected researches: Different parts of this exposition. In Section 2 we will analyze the basic questions of the existence and uniqueness of solutions to the associate initial-boundary value problem such as developed in the paper [38]. Let us mention than different types of boundary condition can be imposed to the solutions of the complex nonlinear PDE. The mathematical treatment is different in each case but there are many common points in the diverse approaches. In Section 2 of this survey we will consider the case of a global delayed problem in which two real parameters play a fundamental role: the feedback intensity, μ, and the delay time, τ. The problem is reduced to find a complex valued field u in $Q := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain for $N \leq 3$ with regular boundary $\partial\Omega$ and $t > 0$.

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - (1+i\epsilon)\,\Delta\mathbf{u} + (1+i\beta)\,|\mathbf{u}|^2\,\mathbf{u} - (1-i\omega)\,\mathbf{u} = \mathbf{F}\left(\mathbf{u}(x,t-\tau)\right), & \text{in } Q, \\ \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{0} & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & \text{on } \Omega, \\ \mathbf{F}(u(s)) = \mathbf{F}_0(s) & s\in(-\tau,0), \end{cases} \tag{3}$$

where the global delayed feedback term is given by

$$\mathbf{F}\left(\mathbf{u}(x,t-\tau)\right) = F_1\left(u(x,t-\tau)\right) + iF_2\left(u(x,t-\tau)\right) := \mu e^{i\chi_0}\left\{\frac{1}{|\Omega|}\int_\Omega \mathbf{u}(x,t-\tau)\,dx\right\}, \tag{4}$$

here $\omega, \beta, \epsilon,\ \tau,\ \mu$ and χ_0 are given real numbers without prescribed sign, $\mathbf{u}_0(x)$ and $\mathbf{F}_0(s)$ are given complex functions and $\mathbf{n}$ is the outward normal vector to $\partial\Omega$. We point out that, in contrast with most of the delayed problems, here the *initial past history* is composed of a pointwise information at $t = 0$ (the usual initial condition $\mathbf{u}(x,0) = \mathbf{u}_0(x)$ on Ω) and only a partial information on the function $\mathbf{u}(s)$ when $s \in (-\tau, 0)$: only the integral of the unknown is prescribed for $s \in (-\tau, 0)$. Under suitable conditions on $\mathbf{u}_0(x)$ and $\mathbf{F}_0(s)$ we prove (in Theorems

1 and 2) that there exists a unique solution of (3). A second model concerns the case, already used in [12], [11] and [30], in which the delayed feedback term involves the own unknown

$$\mathbf{F}\left(\mathbf{u}(x,t-\tau)\right)=e^{i\chi_0}\left\{\frac{\mu}{|\Omega|}\int_\Omega \mathbf{u}\left(x,t-\tau\right)dx+\nu\mathbf{u}\left(x,t-\tau\right)\right\}. \tag{5}$$

In this case it is clear that the required *initial past history* must be more complete and so the new formulation is the usual one for delayed problems. As a matter of fact, as we will mention later, the nonlinear perturbation can be easily treated under a more general growth condition of the type $(1+i\beta)\left|\mathbf{u}\right|^{m-1}\mathbf{u}$, for any $m>0$. In particular, when $m\in(0,1)$ we comment how to apply the techniques introduced in a series of works concerning the pure Schrödinger equation with a non-Lipschitz perturbation to our case (see [13], [16], [14] and [17]). See also the study made in [4], for complex Ginzburg-Landau equations without any delayed term. Thus our second problem can be formulated in the terms

$$\begin{cases} \dfrac{\partial\mathbf{u}}{\partial t}-(1+i\epsilon)\,\Delta\mathbf{u}+(1+i\beta)\left|\mathbf{u}\right|^{m-1}\mathbf{u}-(1-i\omega)\,\mathbf{u}=\mathbf{F}\left(\mathbf{u}(x,t-\tau)\right), & \text{in } Q,\\ \dfrac{\partial\mathbf{u}}{\partial\mathbf{n}}=\mathbf{0}, & \text{on } \partial\Omega\times(0,T),\\ \mathbf{u}\left(x,s\right)=\mathbf{u}_0\left(x,s\right), & s\in[-\tau,0],\ x\in\Omega. \end{cases} \tag{6}$$

In the special case of $m\in(0,1)$ and $\mathbf{F}$ given by (5) with $\mu=0$ (i.e., with only local delayed feedback terms) we prove that several qualitative properties as the finite speed of propagation or the finite extinction time property obtained previously in the literature for complex formulations problems without delayed term (see [13], [16], [4], [14] and [17]) can be easily extended to the mentioned delayed formulation.

The Sections are organized as follows: the existence and uniqueness of solutions for problems (3) and (6) is obtained in Section 2. The proof of the existence of solutions use an iterative argument as well as a Galerkin method when $t\in[0,\tau)$ jointly with suitable a priori estimates which allow to justify the passing to the limit. The uniqueness of solutions is given for $N\leq 3$. Finally the study of some qualitative properties, for $m\in(0,1)$ and $\mathbf{F}$ given by (5) with $\mu=0$, will be collected,where some energy methods will be applied.

In Section 3 we will study the stabilization of the uniform oscillations for the *complex Ginzburg-Landau equation* by means of some global delayed feedback according the papers [30], [28], [34]. Our main interest in this Section concerns the control of chemical turbulence in oscillatory reaction-diffusion systems made through a combination of global and local delayed feedback. Now we will assume the domain be given by $\Omega=(0,L_1)\times(0,L_2)$. We define the faces of the boundary.

$$\Gamma_j=\partial\Omega\cap\{x_j=0\}\,,\Gamma_{j+2}=\partial\Omega\cap\{x_j=L_j\}\,,\ j=1,2,$$

on which we assume periodic boundary conditions and, so, the problem under study can be formulated as

$$(P_1)\begin{cases} \frac{\partial\mathbf{u}}{\partial t}-(1+i\epsilon)\Delta\mathbf{u}=(1-i\omega)\mathbf{u}-(1+i\beta)\left|\mathbf{u}\right|^{\mathbf{2}}\mathbf{u}+\mu e^{i\chi_0}\mathbf{F}(\mathbf{u},t,\tau) & \Omega\times(0,+\infty),\\ \mathbf{u}|_{\Gamma_j}=\mathbf{u}|_{\Gamma_{j+2}},\left(-\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\big|_{\Gamma_j}=\right)\frac{\partial\mathbf{u}}{\partial x_j}\Big|_{\Gamma_j}=\frac{\partial\mathbf{u}}{\partial x_j}\Big|_{\Gamma_{j+2}}\left(=\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\big|_{\Gamma_{j+2}}\right), & \partial\Omega\times(0,+\infty),\\ \mathbf{u}(x,s)=\mathbf{u}_0(x,s) & \Omega\times[-\tau,0], \end{cases}$$

where $\mathbf{n}$ is the outpointing normal unit vector,and

$$\mathbf{F}(\mathbf{u},t,\tau)=[m_1\mathbf{u}(t)+m_2\overline{\mathbf{u}}(t)+m_3\mathbf{u}(t-\tau,x)+m_4\overline{\mathbf{u}}(t-\tau)]$$

with

$$\overline{\mathbf{u}}(s)=\frac{1}{|\Omega|}\int_\Omega\mathbf{u}(s,x)dx.$$

Here the parameters $\epsilon,\beta,\omega,\mu,\chi_0,m_i$ and τ are real numbers, in contrast with the solution $\mathbf{u}(x,t)=u_1(x,t)+iu_2(x,t)$. We point out that most of our results remain true for N-dimensional domains (with $N>2$) as well as for Neumann boundary conditions (a previous study dealing with the one-dimensional case was carried out in [28]).

With the basis of a sound experimental work, many recent studies of a more descriptive nature, but of a great originality and interest have been written. In those studies the delay term $\mathbf{F}(\mathbf{u},t,\tau)$ has been taken corresponding to $m_4=1,m_i=0$ for $i=1,2,3$ and introduced as a control mechanism (see, e.g., [11], [12], [53], [61]). Our main goal is to carry out a rigorous analysis of those studies We also want to investigate the possibility of controlling the turbulence by using other terms (see Remark 4). In particular our treatment does not use the Fourier transform, apparently hard to be rigorously justified in this setting.

We focus our attention on the so called *slowly varying complex amplitudes* defined by $\mathbf{u}(x,t) = \mathbf{v}(x,t)e^{-i\omega t}$. Thus, $\mathbf{v}$ satisfy (P_2) :

$$(P_2)\begin{cases} \frac{\partial \mathbf{v}}{\partial t} - (1+i\epsilon)\Delta \mathbf{v} = \mathbf{v} - (1+i\beta)|\mathbf{v}|^{\mathbf{2}}\,\mathbf{v} + \\ +\mu e^{i\chi_0}\left[m_1\mathbf{v} + m_2\overline{\mathbf{v}} + e^{i\omega\tau}(m_3\mathbf{v}(t-\tau,x) + m_4\overline{\mathbf{v}}(t-\tau))\right] & \text{in } \Omega\times(0,+\infty), \\ \mathbf{v}|_{\Gamma_j} = \mathbf{v}|_{\Gamma_{j+2}}, \left(-\frac{\partial \mathbf{v}}{\partial \mathbf{n}}|_{\Gamma_j} =\right) \frac{\partial \mathbf{v}}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial \mathbf{v}}{\partial x_j}\Big|_{\Gamma_{j+2}} \left(= \frac{\partial \mathbf{v}}{\partial \mathbf{n}}|_{\Gamma_{j+2}}\right), & \text{on } \partial\Omega\times(0,+\infty), \\ \mathbf{v}(x,s) = \mathbf{u}_0(x,s)e^{i\omega s} & \text{on } \Omega\times[-\tau,0]. \end{cases} \tag{7}$$

We study in this Section the stability of *uniform oscillations*, i.e., special solutions of (P_2) of the form $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ which determines completely ρ_0 and θ. As we shall see, the only effect of the delay τ is that it controls the effective phase shift $\chi(\tau)$.

In absence of delay ($\tau = 0$), and for $|\Omega| = +\infty$ and $\mu = 0$, it is known (see [48] and [53]) that the *Benjamin-Feir condition* $\beta < -\frac{1}{\epsilon}$ implies the instability of such uniform oscillations. Here we shall assume merely that

$$\beta \leq 0 \text{ and } \varepsilon \geq 0 \tag{8}$$

and we shall prove that this instability holds, in absence of delay, for $L < +\infty$ once $\chi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\mu > \frac{1}{|\cos\chi_0|}$. Moreover, we shall also prove that when $\tau > 0$ is suitably chosen then the uniform oscillation becomes linearly stable. We point out that the above stabilization phenomenon requires a non zero complex component perturbation (notice that χ_0 can not be zero) and that it applies to the case of $\mu > 0$ and $\epsilon = \beta = \omega = 0$. In this approach we will use the *pseudo-linearization principle* introduced by the authors in [29].

Finally, in Section 4 we will analyze several bifurcation effects produced by the delay time in the behavior of solutions of the complex Ginzburg-Landau equation with this type of feedback as shown in [33]. We prove a Hopf bifurcation result for the equation without diffusion (the Stuart-Landau equation) when the amplitude of the delayed term is suitably chosen. This simplified formulation has the advantage that closed analytical solutions are possible and the necessary eigenvalue computations can be carried out in full. The diffusion case is considered firstly in the case of the whole space and later on a bounded domain with periodicity conditions. In the case in which the space is the whole $\mathbb{R}$ (we consider here the one-dimensional case) we performed a linear stability analysis of uniform oscillations with respect to spatiotemporal perturbations following the treatment made in [63]: we express the complex oscillation amplitude $\mathbf{A}$ as the superposition of a homogeneous mode $\mathbf{H}$ (corresponding to uniform oscillations) with spatially inhomogeneous perturbations,

$$\mathbf{A}(x,t) = \mathbf{H}(t) + \mathbf{A}_+(t)e^{i\kappa x} + \mathbf{A}_-(t)e^{-i\kappa x}\,.$$

With the help of computational arguments we get several bifurcation diagrams where, besides the delay time it is possible to use the feedback magnitude term. Among many other detailed informations, we obtain numerical evidence of the fulfillment of the delicate transversality condition. The Section ends by analyzing the case in which the bifurcation takes place starting from an uniform oscillation and originating a path over a torus. This time the study is carried out in two spatial dimensions over a rectangle in which we impose periodic boundary conditions. We show the applicability of an abstract result ([67]) to our formulation thanks to a suitable choice of the involved functional spaces. In this way, the spatial perturbations can be considered in their greatest generality.

2. Existence and Uniqueness of solutions: problems (3) and (6)

In this Section we will use the following notations**:** $W^{s,p}(D)$ and $H^s(D)$ denotes the standard Sobolev spaces which consist of real scalar (or vector) valued functions defined on D (an open subset of $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$). Sobolev spaces of complex valued functions are denoted by $\mathcal{W}^{s,p}(D)$ and $\mathcal{H}^s(D)$ with calligraphic letters, as well, as continuous functions $\mathcal{C}(D)$ defined over a domain D. We use $\|\cdot\|$ and $(\cdot,\cdot)$ for the usual norm and the inner product of $L^2(D)$ or $\mathcal{L}^2(D)$, respectively. Given a general Banach space B, $\|\cdot\|_B$ denotes the norm of Banach space B. Its topological dual space will be denoted by B'. By $\langle\cdot,\cdot\rangle_{B',B}$ we denote the duality product between B' and B.

We first introduce the notion of weak solution of problem (3).

Definition 1. *Let $T \leq \infty$, and assume $\mathbf{u}_0 \in \mathcal{L}^4(\Omega)\cap\mathcal{H}^1(\Omega)$ and $\mathbf{F}_0 \in \mathcal{L}^2(-\tau,0)$. A function $\mathbf{u} : \Omega\times(-\tau,T) \to \mathbb{C}$ is called a* weak solution *of problem (3) if*

$$\mathbf{u} \in C([0,T] : \mathcal{L}^2(\Omega)) \cap \mathcal{L}^2(0,T : \mathcal{H}^1(\Omega)) \cap \mathcal{L}^4(0,T : \mathcal{L}^4(\Omega)) \cap \mathcal{L}^2\left(-\tau, 0 : \mathcal{L}^1(\Omega)\right),$$

$$\mathbf{u}_t \in \mathcal{L}^2(0,T : (\mathcal{H}^1(\Omega))'),$$

for every $t \in (0,T)$

$$\langle \frac{\partial}{\partial t}\mathbf{u},\varphi\rangle_{(\mathcal{H}^1(\Omega))'\times\mathcal{H}^1(\Omega)} = (1-i\omega)\int_\Omega \mathbf{u}\bar{\varphi}dx - (1+i\beta)\int_\Omega |\mathbf{u}|^2\,\mathbf{u}\bar{\varphi}dx \tag{9}$$

$$-(1+i\epsilon)\int_\Omega \nabla\mathbf{u}\cdot\nabla\bar{\varphi}dx + \mathbf{F}\left(\mathbf{u}(t-\tau)\right)\int_\Omega \bar{\varphi}dx, \quad \forall\varphi\in\mathcal{H}^1(\Omega),$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \ \ in\ \mathcal{L}^2(\Omega)$$

and

$$\mathbf{F}(\mathbf{u}(\cdot)) = \mathbf{F}_0(\cdot) \ \ in\ \mathcal{L}^2(-\tau,0),$$

where $\mathbf{F}\left(\mathbf{u}(t-\tau)\right)$ is given by (4).

In the case of problem (6) a stronger notion of weak solution must be introduced:

Definition 2. *Let $T\le\infty$, and assume that $\mathbf{u}_0 \in\mathcal{C}\left([-\tau,0]:\mathcal{L}^2(\Omega)\right)$, $\mathbf{u}_0(\cdot,0)\in\mathcal{L}^{m+1}(\Omega)\cap\mathcal{H}^1(\Omega)$ $(m>0)$. A function $\mathbf{u}:\Omega\times(-\tau,T)\to\mathbb{C}$ is called a* weak solution *of problem (6) if*

$$\mathbf{u}\in\mathcal{L}^2(0,T:\mathcal{H}^1(\Omega))\cap\mathcal{L}^{m+1}(0,T:\mathcal{L}^{m+1}(\Omega))\cap\mathcal{L}^2\left(-\tau,0:\mathcal{L}^2(\Omega)\right),$$

$$\mathbf{u}_t\in\mathcal{L}^2(0,T:(\mathcal{H}^1(\Omega))'),$$

for every $t\in(0,T)$

$$\langle \frac{\partial}{\partial t}\mathbf{u},\varphi\rangle_{\mathcal{H}^{-1}(\Omega)\times\mathcal{H}^1(\Omega)} = (1-i\omega)\int_\Omega \mathbf{u}\bar{\varphi}dx - (1+i\beta)\int_\Omega |\mathbf{u}|^{m-1}\,\mathbf{u}\bar{\varphi}dx \tag{10}$$

$$-(1+i\epsilon)\int_\Omega \nabla\mathbf{u}\cdot\nabla\bar{\varphi}dx + \int_\Omega \mathbf{F}\left(\mathbf{u}(x,t-\tau)\right)\bar{\varphi}dx, \quad \forall\varphi\in\mathcal{H}^1(\Omega)$$

and

$$\mathbf{u}=\mathbf{u}_0 \ \ in\ \mathcal{C}\left([-\tau,0]:\mathcal{L}^2(\Omega)\right),$$

where $\mathbf{F}\left(\mathbf{u}(x,t-\tau)\right)$ is given by (5).

It is useful to rewrite the complex Gingzburg-Landau problem (3) in terms of the real components $(\mathbf{u}_1,u_2)$ of the solution $\mathbf{u}$, i.e. $\mathbf{u}=u_1+iu_2$. The associated real system in Q is the following:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 - \epsilon\Delta u_2 + \left(u_1^2+u_2^2\right)(-u_1+\beta u_2) + u_1 + \omega u_2 + F_1(u(x,t-\tau)), & \text{in } Q,\\ \dfrac{\partial}{\partial t}u_2 = \epsilon\Delta u_1 + \Delta u_2 - \left(u_1^2+u_2^2\right)(\beta u_1+u_2) + u_2 - \omega u_1 + F_2(u(x,t-\tau)), & \text{in } Q,\\ u_1(x,t) = \mathcal{R}eal(U_0(x,t)) \text{ and } u_1(x,t) = \mathcal{I}m(U_0(x,t)), & \text{in } (-\tau,0)\times\Omega,\\ \dfrac{\partial u_1}{\partial\mathbf{n}} = \dfrac{\partial u_2}{\partial\mathbf{n}} = 0, & \text{on } \partial\Omega\times(0,T). \end{cases}$$

for F_1 and F_2 defined in (4) as the real and imaginary part of F respectively.

The main results of this section is enclosed in the following theorem.

Theorem 1. *i) Assume $\mathbf{F}_0\in\mathcal{L}^2(-\tau,0)$ and let $\mathbf{u}_0$ be such that*

$$\mathbf{u}_0\in\mathcal{L}^4(\Omega)\cap\mathcal{H}^1(\Omega).$$

Then, there exists at least a weak solution to (3) in $(0,\infty)$.

ii) Assume $\mathbf{u}_0 \in\mathcal{C}\left([-\tau,0]:\mathcal{L}^2(\Omega)\right)$, $\mathbf{u}_0(x,0)\in\mathcal{L}^{m+1}(\Omega)\cap\mathcal{H}^1(\Omega)$. Then, there exists at least a weak solution to (6) in $(0,\infty)$.

Remark 1. *Although there are some works in the literature dealing with a partial information on the initial history (see, e.g.* [3] *and its references) we point out that the initial information required in problem (3) is weaker than in those series of works.*

To prove the existence of a weak solution of (3) we first obtain some a priori estimates in the following lemma.

Lemma 1. *Let $T < \infty$ and assume $\mathbf{F}_0 \in \mathcal{L}^2(-\tau, 0)$ and let $\mathbf{u}_0$ be such that*

$$\mathbf{u}_0 \in \mathcal{L}^4(\Omega) \cap \mathcal{H}^1(\Omega).$$

Let $\mathbf{u} \in \mathcal{L}^2\left(0, T : \mathcal{L}^4(\Omega)\right)$ be a weak solution of (3). Then

$$\mathbf{u} \in \mathcal{L}^\infty\left(0, T : \mathcal{L}^2(\Omega)\right). \tag{11}$$

Moreover the norm of $\mathbf{u}$ in this space, as well as in the spaces $\mathcal{L}^2\left(0, T : \mathcal{H}^1(\Omega)\right)$ and $\mathcal{L}^2\left(0, T : \mathcal{L}^4(\Omega)\right)$ has a bound only depending of $\mathbf{F}_0, \mathbf{u}_0$, μ, τ, β and T.

Lemma 2. *Let $T < \infty$ and assume*

$$\mathbf{u}_0 \in \mathcal{L}^4(\Omega) \cap \mathcal{H}^1(\Omega).$$

and

$$\mathbf{F}_0 \in \mathcal{L}^2(-\tau, 0).$$

Let $\mathbf{u}$ be a "strong" solution of (3). Then

$$\mathbf{u} \in \mathcal{H}^1\left(0, T : \mathcal{L}^2(\Omega)\right) \cap \mathcal{L}^\infty\left(0, T : \mathcal{H}^1(\Omega)\right) \cap \mathcal{L}^\infty(0, T : \mathcal{L}^4(\Omega)).$$

Moreover, there exists $K > 0$ such that

$$\int_0^t \int_\Omega |\mathbf{u}_t|^2 dx + \frac{1}{4}\int_\Omega |\mathbf{u}(t)|^4 dx + \frac{1}{4}\int_\Omega |\nabla \mathbf{u}(t)|^2 dx \leq \tag{12}$$

$$K \int_{-\tau}^0 |\mathbf{F}_0(s)|^2 ds + \frac{1}{4}\int_\Omega |\mathbf{u}_0(x)|^4 dx + \frac{1}{4}\int_\Omega |\nabla \mathbf{u}_0(x)|^2 dx, \ \textit{a.e. } t \in (0, T).$$

The proof of Theorem 1 uses a Galerkin approximation expansion, the above lemmas suply some uniform estimates and we pass to the limits by using the compactness Aubin-Lions Lemma (see [38]) for details.

The proof of the uniqueness of solutions follows a contradiction argument using the previous estimates. We recall that in this Section we are assuming that $N \leq 3$.

Theorem 2. *Assume the conditions on $\mathbf{F}_0$, $\mathbf{u}_0$ and $\mathbf{u}_0$ given in parts i) and ii) of Theorem 1. Then, problems (3) and (6) have at most one weak solution for the following cases:*

- $m \in [1, \infty)$, *if* $N = 1, 2$
- $m \in [1, 5)$, *if* $N = 3$,
- $m \in (0, 1)$, *if* $N = 1, 2, 3$ *provided* β *satisfies*

$$|\beta| \leq \frac{1 - m}{2m^{\frac{1}{2}}}. \tag{13}$$

Remark 2. *The above two theorems also holds, with minor changes, for other type of boundary conditions such as, Dirichlet boundary conditions or periodic boundary conditions (as considered in* [12], [11] *and* [30]*). We also point out that the assumption $(1 + i\epsilon)$ on the coefficient of the complex diffusion operator is absolutely crucial since when the real part of such a coefficient vanishes the equation becomes a nonlinear Schrödinger delayed equation and some additional conditions on the coefficient of the nonlinear part (in this paper assumed of the form $(1 + i\beta)$) are required (see, e.g. the existence and uniqueness results for the case $m \in (0, 1)$ given in* [18]*).*

It is possible to get some qualitative properties of solutions of the delayed problem when $m \in (0, 1)$ by adapting to the case of delayed problems the energy methods presented in the monograph [8] and, more concretely, their adaptation to complex Ginzburg-Landau equations with absorption made in [4]. Due to the presence of the "bad term" $-(1 - i\omega)\mathbf{u}$ in the equation in all this section we shall need an extra information on the solutions: we will always assume that the solution is bounded. This condition could be avoided in absence of such a term in the equation.

A first qualitative result concerns the so called *finite extinction time.* This property is of interest in many different contexts. For instance in Control Theory it usually associated to the "zero exact controllability property".

Theorem 3. *Let $m \in (0,1)$ and $\alpha \in (0,1)$ such that*

$$\|\mathbf{u}\|_{L^\infty(Q)} \leq |1-\alpha|^{\frac{1}{1-m}}. \tag{14}$$

i) Assume

$$\|\mathbf{u}_0\|^2_{\mathcal{L}^2(\Omega)} \text{ small enough.} \tag{15}$$

Assume also that there exists $t^* \in (0,\tau)$ such that

$$|\mathbf{F}_0(s)|^{\frac{m+1}{m}} \leq c\,[(t^*-\tau)-s]_+^{\frac{\delta}{1-\delta}} \text{ for a.e. } s \in (-\tau,0), \tag{16}$$

for some $c>0$ and some $\delta \in (0,1)$. Then any bounded solution of the nonlocal problem (6) (i.e. with $\nu=0$) satisfies

$$\|\mathbf{u}(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)} \leq c\kappa\,[t^*-t]_+^{\frac{1}{1-\delta}} \text{ for any } t \in [0,\tau)$$

for some $c>0$. In particular $\mathbf{u}(\cdot,t) \equiv \mathbf{0}$ in Ω for any $t \in [t^*,\tau)$. In addition, $\mathbf{u}(\cdot,t) \equiv 0$ in Ω for any $t \in [nt^*, n\tau)$ for any $n \in \mathbb{N}$.

ii) Assume

$$\|\mathbf{u}_0(\cdot,s)\|^{\frac{2(m+1)}{m}}_{\mathcal{L}^2(\Omega)} \leq \kappa\,[(t^*-\tau)-s]_+^{\frac{\delta}{1-\delta}} \text{ for a.e. } s \in (-\tau,0), \tag{17}$$

for some $\kappa>0$ and some $\delta \in (0,1)$. Then any bounded solution of the (6), with $\nu>0$, satisfies that

$$\|\mathbf{u}(\cdot,t)\|^2_{\mathcal{L}^2(\Omega)} \leq c\kappa\,[t^*-t]_+^{\frac{1}{1-\delta}} \text{ for any } t \in [0,\tau)$$

for some $c>0$. In particular $\mathbf{u}(\cdot,t) \equiv \mathbf{0}$ in Ω for any $t \in [t^*,\tau)$. In addition $\mathbf{u}(\cdot,t) \equiv \mathbf{0}$ in Ω for any $t \in [nt^*, n\tau)$, for any $n \in \mathbb{N}$.

Roughly speaking, for the proof of this results we follow the energy method presented in Section 6.2 of the monograph [8] (see the applications to complex equations made in [4] and [18]). In fact, we will use the following improvement of a suitable energy inequality:

Lemma 3. [18] *Let $y \in W^{1,1}_{\mathrm{loc}}\big([0,\infty);\mathbb{R}\big)$ with $y \geq 0$ over $[0,\infty)$, $\delta \in (0,1)$, $\alpha, T_0 > 0$ and,*

$$y_\star = \left(\kappa\,\delta^\delta(1-\delta)\right)^{\frac{1}{1-\delta}}, \tag{18}$$

$$x_\star = \left(\kappa\,\delta\,(1-\delta)\,T_0\right)^{\frac{1}{1-\delta}}. \tag{19}$$

If,

$$y(0) \leq x_\star,$$

and if for almost every $t>0$,

$$y'(t) + \kappa\, y(t)^\delta \leq y_\star\,(T_0-t)_+^{\frac{\delta}{1-\delta}},$$

then, there exists $k^*>0$ such that

$$y(t) \leq k^*(T_0-t)_+^{\frac{1}{1-\delta}} \text{ for any } t>0. \tag{20}$$

Remark 3. *If the initial history $\mathbf{F}_0(s)$ and $\mathbf{u}_0$ (respectively $\mathbf{u}_0(\cdot,s)$) doesn't vanishes also for $s \in [-\tau,\underline{t}) \cup \{0\}$, for some $\underline{t}<0$ then Theorem 3 proves that the solution of the nonlocal (6), i.e. with $\nu=0$ (respectively the local case, with $\nu>0$) is discontinuous at the times $t=n\tau$ for any $n \in \mathbb{N}$. The technique of proof of Theorem 3 could also be used to prove that the solutions may vanishes too on intervals of the form $[n\tau, n\tau+\varepsilon]$ avoiding the above mentioned discontinuity.*

Remark 4. *The detailed analysis made in* [4] *shows that, in fact,*

$$\delta = \frac{m+1}{\theta(m+1)+2(1-\theta)} \text{ with } \theta = \frac{N(1-m)}{N(1-m)+2(1+m)}.$$

Remark 5. *The above assumptions are, in some sense, necessary. Indeed, if for instance $\|\mathbf{u}_0\|^2_{\mathcal{L}^2(\Omega)}$ is big enough then it is possible to take the parameters such that any function $y(t)$ satisfying the ordinary differential inequality with zero in the right hand side satisfy that $y(\tau)>0$ and thus the contribution of the global initial memory cannot be of any help in the rest of values of time $t \in [\tau,+\infty)$. Analogously, the decay condition indicated in the assumption (16) is the optimal decay which is compatible with the decay of any function $y(t)$ satisfying the ordinary differential inequality with zero in the right hand side and with an exponent $\delta \in (0,1)$. On the other hand, it is well known that if $\delta \geq 1$ then $y(t)>0$ for any $t \in [0,+\infty)$.*

Remark 6. *Assumption (14) is used to obtain the finite time extinction of the solution. If assumption (14) is not satisfied, for some initial data, the solution reaches a non-zeroconstant value in finite time. Such a constant is, in fact, the average (in space) of the solution and its behavior is determined by an ordinary differential equation. See for instance* [8]*, Chapter 2, section 7.2 and references therein where finite time convergence to the average of the solution is studied for a porous-media type equation.*

Theorem 4. *Let $m \in (0,1)$ and $\alpha \in (0,1)$ such that*

$$\|\mathbf{u}\|_{L^\infty(Q)} \leq |1-\alpha|^{\frac{1}{1-m}}. \tag{21}$$

i) Assume that there exists $\rho_0 > 0$ such that

$$\mathbf{u}_0 = 0 \text{ in } B_{\rho_0}. \tag{22}$$

Assume also that there exists $s_{\mathbf{F}_0} \in (0,\tau)$ such that

$$\mathbf{F}_0(s) = \mathbf{0} \text{ a.e. } s \in (-\tau, -s_{\mathbf{F}_0}). \tag{23}$$

Let $\mathbf{u}$ be a bounded solution of the nonlocal (6) (i.e. with $\nu = 0$). Then there exists $\rho_1 \in (0,\rho_0)$ and $t_1 \in (0,\tau)$ (both depending of the energies associated to $\mathbf{u}$) such that

$$\mathbf{u} = \mathbf{0} \text{ in } Q_{\rho_1,t_1}.$$

ii) Assume that there exists $s_{\mathbf{F}_0} \in (0,\tau)$ and $\rho_0 > 0$ such that

$$\mathbf{u}_0(\cdot,s) = \mathbf{0} \text{ on } B_{\rho_0} \text{ for } s = 0 \text{ and for a.e. } s \in (-\tau,-s_{\mathbf{F}_0}). \tag{24}$$

Let $\mathbf{u}$ be a bounded solution of the local (6) (i.e. with $\nu > 0$). Then there exists $\rho_1 \in (0,\rho_0)$ and $t_1 \in (0,\tau)$ (both depending of the energies associated to $\mathbf{u}$) such that

$$\mathbf{u} = \mathbf{0} \text{ in } Q_{\rho_1,t_1}.$$

Remark 7. *As in Theorem 5.3 of* [4]*, it is possible to show a "waiting time property" (showing that in fact $\rho_1 = \rho_0$ for any $t \in (0,t_0)$, for some $t_0 \in (0,\tau)$) for the solution $\mathbf{u}$ of the nonlocal (6) (i.e. with $\nu = 0$), if we make the decay stronger assumption*

$$\int_{B_\rho} |\mathbf{u}_0(x)|^2\,dx \leq \delta(\rho-\rho_0)_+^{\frac{1}{1-\xi}} \text{ for a.e. } \rho \in (0,\rho_0+\varepsilon) \tag{25}$$

for some $\delta,\varepsilon > 0$ and with $\xi \in (0,1)$ the exponent arising in (25). in the case of the local (6) (i.e. with $\nu > 0$) it must be required a similar decay stronger assumption now on $\mathbf{u}_0(\cdot,s)$: to be more precise, we must assume that there exists $s_{\mathbf{F}_0} \in (0,\tau]$, $\rho_0 > 0$ and $\delta,\varepsilon > 0$ such that

$$\int_{B_\rho} |\mathbf{u}_0(x,0)|^2\,dx + \int_{-\tau}^{-s_{\mathbf{F}_0}} \int_{B_\rho} |\mathbf{u}_0(x,s)|^{\frac{m+1}{m}}\,dxds \leq \delta(\rho-\rho_0)_+^{\frac{1}{1-\xi}} \text{ for a.e. } \rho \in (0,\rho_0+\varepsilon). \tag{26}$$

Remark 8. *As in the case of the equation without delay terms, it remains an open question to know if the above finite speed of propagation also holds for the pure Schrödinger equation with the same absorption perturbation term. A partial answer was given in* [17].

3. Turbulence control by feedback delay perturbations

We start by pointing out that the existence and uniqueness of a solution of (P_1) can be proven, as in the precedent Section, once we assume that $\mathbf{u}_0 \in \mathbf{C}([-\tau,0] : \mathbf{L}^2(\Omega))$. As mentioned in the Introduction, we are now interested in the stability analysis of the time-periodical function $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$. In order to avoid the application of techniques for the study of the stability of periodic solutions we can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t) = \mathbf{v}(x,t)e^{i\theta t}$ where $\mathbf{v}(x,t)$ is a solution of (P_2). Thus $\mathbf{z}(x,t)$ satisfies

$$(P_3)\begin{cases} \frac{\partial \mathbf{z}}{\partial t} - (1+i\epsilon)\Delta\mathbf{z} = (1+i\theta)\mathbf{z} - (1+i\beta)|\mathbf{z}|^2\mathbf{z} + \\ +\mu e^{i\chi_0}\left[m_1\mathbf{z} + m_2\overline{\mathbf{z}} + e^{i(\omega+\theta)\tau}(m_3\mathbf{z}(t-\tau,x) + m_4\overline{\mathbf{z}}(t-\tau))\right] & \text{in } \Omega\times(0,+\infty), \\ \mathbf{z}|_{\Gamma_j} = \mathbf{z}|_{\Gamma_{j+2}}, \left(-\frac{\partial \mathbf{z}}{\partial \mathbf{n}}\big|_{\Gamma_j} =\right) \frac{\partial \mathbf{z}}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial \mathbf{z}}{\partial x_j}\Big|_{\Gamma_{j+2}} \left(= \frac{\partial \mathbf{z}}{\partial \mathbf{n}}\big|_{\Gamma_{j+2}}\right), & \text{on } \partial\Omega\times(0,+\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x,s)e^{i(\omega-\theta)s} & \text{on } \Omega\times[-\tau,0]. \end{cases} \tag{27}$$

Now, $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_{uosc}(x,t)e^{i\theta t} = \mathbf{z}_\infty = \rho_0$ is an stationary solution of (P_3): i.e.

$$\mathbf{0} = (1+i\theta)\mathbf{z}_\infty - (1+i\beta)|\mathbf{z}_\infty|^2\mathbf{z}_\infty + \mu e^{i\chi_0}\left[m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3+m_4)\right]\mathbf{z}_\infty. \tag{28}$$

In order to keep some resemblance with [11] and [12] we shall assume that

$$m_1+m_2=0 \text{ and } m_3+m_4=1 \tag{29}$$

Then we get the expressions $\rho_0(\tau)=(1+\mu\cos\chi(\tau))^{1/2}$, where $\chi(\tau)=\chi_0+(\omega+\theta(\tau))\tau$ and with $\theta(\tau)$ given as the solution of the implicit equation

$$\theta=\beta-\mu(\sin(\chi_0+(\omega+\theta)\,\tau)-\beta\cos(\chi_0+(\omega+\theta)\,\tau)). \tag{30}$$

Notice that if $\mu=0$ we deduce that $\rho_0(\tau)=1$ and that $\theta(\tau)=\beta$ for any τ and that $\rho_0(0)=(1+\mu\cos\chi_0)^{1/2},\theta(0)=\beta-\mu(\sin\chi_0-\beta\cos\chi_0)$. It is not difficult to prove (see subsection 3.3 below) the existence and uniqueness of such a function $\theta(\tau)$ and that $\theta\in C^1$.

Our main stabilization result is the following

Theorem 5. Assume (8), (29), $\chi_0\in(\pi,\frac{3\pi}{2})$,

$$3-m_1-2m_3\geq 0,\ m_1+m_3\geq 0,\ 3+2m_3>0, \tag{31}$$

$$\mu>\max\{\frac{1}{|\cos\chi_0|},\frac{3\beta-\omega+3(\omega+\beta)\sin\chi_0+\cos\chi_0}{5(-\beta)\sin\chi_0\cos\chi_0+1},$$
$$\frac{m_3(3\beta-\omega-\varepsilon\frac{\pi^2}{L^2})+3(\omega+\beta)\sin\chi_0+(m_1+m_3)\cos\chi_0}{(3-m_1-2m_3)\sin^2\chi_0+(m_1+m_3)\cos^2\chi_0+(-\beta)(3+2m_3)\sin\chi_0\cos\chi_0}\}.$$

Then there exists some $\tau_0\in(0,1)$ such that if we assume $\tau\in(\tau_0,1)$ we get that

$$|\mathbf{v}(x,t)-\rho_0|\leq Me^{-\alpha t}\left\|\mathbf{u}_0(\cdot,\cdot)e^{i\omega\cdot}-\rho_0\right\|.$$

For the proof we shall first introduce a new and quite general *pseudo-linearization principle.* Then, we shall show the applicability of it to the delayed problem and, at the end, we shall study the eigenvalues of the linear part to find the range of parameters for the stability of the linear part.

3.1. **The pseudo-Linearization Principle.** It is useful to analyze to study the stabilization, as $t\to\infty$, of the solutions in a more general framework: the nonlinear abstract functional differential equation

$$\begin{cases}\frac{du}{dt}(t)+Au(t)+Bu(t)\ni F(u_t(.)) & in\ X,\\ u(s)=u_0(s) & s\in[-\tau,0],\end{cases} \tag{32}$$

on a Banach space X, where

$$u_t(\theta)=u(t+\theta),\theta\in[-\tau,0],$$

to the associated equilibria: $w\in D(A)\subset D(B)\subset X$ such that

$$Aw+Bw\ni F(\widehat{w}(.)),$$

where $\widehat{w}\in C:=C([-\tau,0]:X)$ is the function which takes constant values equal to w. Our main goal in this subsection is to extend, to a broad class of nonlinear operators A, the usual linearized stability principle saying, roughly speaking, that for the special case of A linear (single valued) and B and F are differentiable, the asymptotic stability of the zero solution of the linearized equation,

$$\begin{cases}\frac{dv}{dt}(t)+Av(t)+\mathrm{D}B(w)v(t)=\mathrm{D}F(\widehat{w})v_t(.) & in\ X,\\ v(s)=u_0(s) & s\in[-\tau,0].\end{cases} \tag{33}$$

implies that $u(t:u_0)\to w$ as $t\to\infty$, at least if $u_0(.)$ is close enough to $\widehat{w}$. We point out that some relevant examples of nonlinear functional equations arise in the most different contexts (see, for instance, Díaz and Hetzer [37] for one example in Climatology, Chukwu [35] for a family of examples dealing with the wealth of nations and the general exposition made in Hale [44]).

The motivation to keep A nonlinear after the process of linearization (reason why we used the term of *pseudo-linearization principle*) comes from the fact that if we use the representation for the unknown of the delayed nonlinear equation (P_3) as $\mathbf{z}(x,t)=\rho(x,t)e^{i\phi(x,t)}$ then we arrive to a coupled nonlinear system of delayed equations for ρ and ϕ which can be described in terms of the representation operator given by $\mathbf{P}:\mathbb{R}^2\to\mathbb{C}$, $\mathbf{P}(\rho,\phi)=\rho e^{i\phi}$. Indeed, notice that $\mathbf{P}$ is nonlinear and that if $\mathbf{q}=(\rho,\phi)$ then $\mathbf{z}(x,t)=\mathbf{P}(\mathbf{q}(x,t))$ and the (P_3) can be formulated as $\frac{d\mathbf{P}(\mathbf{q}(\cdot,t))}{dt}+A\mathbf{P}(\mathbf{q}(\cdot,t))+B\mathbf{P}(\mathbf{q}(\cdot,t))=F(\mathbf{P}(\mathbf{q}(\cdot))_t)$. By using that the matrix $\mathbf{C}(\mathbf{q}(\cdot,t))=grad\mathbf{P}(\mathbf{q}(\cdot,t))$ is not singular, we can arrive to the simpler formulation

$$\frac{d\mathbf{q}}{dt}(\cdot,t)+\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}[A\mathbf{P}(\mathbf{q}(\cdot,t))+B\mathbf{P}(q(\cdot,t))]=\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}F(\mathbf{P}(\mathbf{q}(\cdot))_t). \tag{34}$$

Notice that, although this delayed system can be also (formally) linearized (this is the procedure followed in Battogtokh and Mikhailov [12] and Mertens *et al.* [53] the above diffusion operator $\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot,t))$ becomes now quasilinear on $\mathbf{q}$ and thus the mathematical justification is much more delicate.

There are some others linearization principles in the literature. Their motivation is usually a particular problem, but its applicability is wider. Close to ours we can mention that of W. M. Ruess [59], although the formulation, scope and proof are different. Besides its applicability to the problem in this work, ours can also be applied to the case in which A is nondifferentiable and nonlinear, among many others (see [34]).

Coming back to the abstract formulation, the structural assumptions we shall assume in this paper are the following

(H1): $A \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, with

$$\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathcal{P}(X) \text{ such that } A + \omega I \text{ is a m-accretive operator}\},$$

(see Brezis [26] for the case of $X = H$ a Hilbert space and the works by Benilan, Crandall, Pazy and others for the case of a general Banach space: see the monographs [20] and [66]),

(H2): the operators semigroup $T(t) : \overline{D_x(A)}^X \to X$, $t \geq 0$, generated by A, is compact

(see Vrabie [66]),

(H3): $B \in \mathcal{A}(0 : X)$, B is single valued, Fréchet differentiable, and B is dominated by A; i.e.

$$\begin{array}{c} D_X(A) \subset D_X(B) \text{ and } |Bu| \leq k\left|A^0u\right| + \sigma(|u|) \\ \text{for any } u \in D_X(A) \text{ and for some } k < 1 \text{ and some continuos function } \sigma : \mathbb{R} \to \mathbb{R}, \end{array} \tag{35}$$

where, here and in what follows, $|.|$ denotes the norm in the space X (in contrast with the norm in space C which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $\left|A^0u\right| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$,

(H4): $F : C \to X$ satisfies a local Lipschitz condition, i.e.,

$$\left\{ \begin{array}{l} \text{for any } R > 0 \text{ there exists } L\,(R) > 0 \text{ such that} \\ |F(\phi) - F(\psi)| \leq L\,(R)\,\|\phi - \psi\| \text{ for any } \phi, \psi \in C \text{ and } \|\phi\|\,,\|\psi\| \leq R. \end{array} \right. \tag{36}$$

(H5): there exists $\delta^F > 0$ such that $F : B^X_{\delta^F}(\widehat{w}) \to X$ is Fréchet differentiable with the Fréchet derivative $\mathrm{D}F(\widehat{w})$ given by $\mathrm{D}(F(\widehat{w}))\phi = \int_{-\tau}^0 d\eta(\theta)\phi(\theta), \phi \in C$, for $\eta : [-\tau, 0] \to B(X, X)$ of bounded variation and the Fréchet derivative is locally Lipschitz continuous, where $B^X_{\delta^F}(\widehat{w}) = \left\{\phi \in C; \|\phi - \widehat{x}\| < \delta^F\right\}$,

We further assume the main condition of our arguments:

(H6): the operator $y \to Ay + By - \mathrm{D}F(\widehat{w})\,(e^{\omega \cdot}y)$ belongs to $\mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ with $\operatorname{Re}\omega = \gamma < 0$ where $e^{\omega \cdot}v \in C$ is defined by

$$(e^{\omega \cdot}v)(s) = e^{\omega s}\widehat{v}(s),\ with\ \widehat{v}(s) = v,\ for\ any\ s \in [-\tau, 0],\ for\ v \in X. \tag{37}$$

In order to treat the case in which B is differentiable we introduce the conditions

(H7): there exists a Banach space Y and there exists $\delta^B > 0$ such that B is Fréchet differentiable as function from $B_{\delta^B}(w) = \left\{z \in D(B);\ |w - z| < \delta^B\right\}$ into Y, with the Fréchet derivative $\mathrm{D}B(w)$ locally Lipschitz continuous,

and

(H8): the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega^* \cdot}y\right)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\operatorname{Re}\omega^* = \gamma^* < 0$.

A concrete statement of the *pseudo-linearization principle is the following:*

Theorem 6. *Assume (H1)-(H6). Then there exists $\alpha > 0$, $\epsilon > 0$ and $M \geq 1$ such that if $u_0 \in B^X_\epsilon(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ then the solution $u(\cdot : u_0)$ of (58) exists on $[-\tau, +\infty)$ and*

$$|u(t : u_0) - w| \leq Me^{-\alpha t}\,\|u_0 - \widehat{w}\|\,,\ for\ any\ t > 0. \tag{38}$$

Moreover, if we also assume (H7), that (H1)-(H5) holds on the space Y and (H8) then there exists $\alpha^* > 0$, $\epsilon^* \in (0, \epsilon]$ and $M^* \geq 1$ such that if $u_0 \in B^{X \cap Y}_{\epsilon^*}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ then

$$|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \leq M^*e^{-\alpha^* t}(\|u_0 - \widehat{w}\|_X + \|u_0 - \widehat{w}\|_Y),\ for\ any\ t > 0. \tag{39}$$

Remark 9. *It is not difficult to show that the assumption (H8) is implied (when A is linear) by the condition: "if $\lambda \in \mathbb{C}$ is given so that there exists $y \in D(B)\setminus\{0\}$ such that $Ay + \mathrm{D}B(w)y - \lambda y \ni \mathrm{D}F(\widehat{w})\left(e^{\lambda \cdot}y\right)$ then $\operatorname{Re}\lambda > 0$". This allow to see Theorem 4.1 of Wu* [67] *(see also Parrot* [57] *and its references) as an special case of our abstract result with $B = 0$. In that case the "variation of the constants formula" can be used to get a different proof of the theorem since A is linear. Notice that if $B \neq 0$ and $D(B) \subsetneq X$ then the arguments of the proof of Wu* [67] *do not work (in spite of what is claimed in the Example 4.8 given there).*

Remark 10. *When A is linear, as in the case without delay, assumption (H7) implies that the zero solution of the linearized problem $\frac{dU}{dt}(t) + AU(t) + \mathrm{D}B(w)U(t)$-$\mathrm{D}F(\widehat{w})U_t(.) = 0$ in X, is locally asymptotically stable (Wu* [67]*).*

Remark 11. *It is possible to prove the existence of global solutions for a general class of initial data (not necessarily near $\widehat{w}$) by using that $A + B \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, some truncation of the nonlocal term $F(u_t)$ and passing to the limit by the compactness of the semigroup generated by A (see Vrabie* [66] *for some related results).*

An easy adaptation of the above proof leads to the following linearization result (now on a possibly smaller neighborhood of w) when A is differentiable

Theorem 7. *The conclusion of the above result remains true if we assume, additionally, that condition (H7) also holds for A and we replace condition (H8) by*

> **(H9):** *the operator $y \to \mathrm{D}A(w)y + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega \cdot}y\right)$ belongs to $\mathcal{A}(\omega)$, for some $\omega \in \mathbb{C}$ with $\operatorname{Re}\omega = \gamma < 0$* ■

Remark 12. *We claim that our arguments keeping A nonlinear after linearizing the rest of the terms (and in particular the way in which we apply Gronwall inequality) allow to extend, to the case of quasilinear equations, the so called "method of quasilinearization" which, introduced by Bellman and Kalaba* [19]*, we used to find solutions of a parabolic semilinear problem through the iteration of solutions of the linearized equation when starting in a super and a subsolution of the original semilinear problem (see, e.g., Lakshmikantham and Vatsala* [49]*, Carl and Lakshmikantham* [27] *and their references). This will be the subject of a future work by the authors.*

3.2. **Applications of the abstract result to the complex Ginzburg-Landau equation.** Motivated by the special form of the nonlinear term of the equation in (P_3) we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$ (notice that, in contrast with the case of scalar equations (see Parrot [57]) the space $\mathbf{L}^\infty(\Omega)$ is not a suitable space to check assumption (H1): see [10]. A detailed analysis of the associated diffusion operator is consequence of some previous results in the literature: see, for instance, Amann [5]. Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} \Delta & -\epsilon\Delta \\ \epsilon\Delta & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, if $\epsilon \neq 0$ the diffusion matrix has a non zero antisymmetric part. In particular, A is the generator of a semigroup of contractions $\{T(t)\}_{t\geq 0}$ on X and the compactness of the semigroup is a consequence of the compactness of the inclusion $D(A) \subset X$ (notice that, since $N = 2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems.

Concerning the rest of the terms of the equation in (P_3), we define $B\mathbf{u} = (1+i\beta)\left|\mathbf{u}\right|^2\mathbf{u}$ with $D(B) = \mathbf{L}^{12}(\Omega)$. By using the characterizarion of the semi inner-braket $[,]$ for the spaces $L^p(\Omega)$ (see, for instance Benilan, Crandall and Pazy [20]) it is easy to see that $\mathbf{B}$ verifies (H3). Moreover, by the results on the Frechet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi [7] we get that (H7) holds, with $\mathrm{D}B(\mathbf{y})\mathbf{v} = 3(1+i\beta)\left|\mathbf{y}\right|^2\mathbf{v}$, if we take $Y = \mathbf{L}^{4/3}(\Omega)$. It can be found in the above mentioned reference that assumption (H7) does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

The nonlocal term is defined by

$$\mathbf{F}(\mathbf{u}_t) = (1+i\theta)\mathbf{u}(t) + \mu e^{i\chi_0}\left[m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + e^{i(\omega+\theta)\tau}(m_3\mathbf{u}(t-\tau) + m_4\overline{\mathbf{u}}(t-\tau))\right],$$

is locally Lipschitz continuous and its Frechet derivative is given by

$$\mathrm{D}\mathbf{F}(\widehat{\mathbf{y}})\mathbf{v}(t) = -(1+i\theta)\mathbf{v}(t) - \mu e^{i\chi_0}\left[m_1\mathbf{v}(t) + m_2\overline{\mathbf{v}}(t) - e^{i(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau) - m_4\overline{\mathbf{v}}(t-\tau))\right] \tag{40}$$

since for any $\phi \in C$, the non-local operator $\phi \to \frac{1}{|\Omega|}\int_\Omega \phi(s)dx$ is linear and we can write $\mathrm{D}\mathbf{F}(\widehat{\mathbf{y}})\phi = \int_{-\tau}^0 d\eta(s)\phi(s)$, with

$$d\eta(s)\mathbf{v}(s) = \delta_0(s)(1+i\theta)\mathbf{v}(s) + \mu e^{i\chi_0}\left[\delta_0(s)(m_1\mathbf{v}(s) + m_2\overline{\mathbf{v}}(s)) + e^{i(\omega+\theta)\tau}\delta_{-\tau}(s)(m_3\mathbf{v}(s) + m_4\overline{\mathbf{v}}(s))\right] \tag{41}$$

for any $\mathbf{v} \in C([-\tau,\infty)\colon \mathbf{L^4}(\Omega))$ and any $s \in [-\tau,\infty)$, where $\delta_0(s), \delta_{-\tau}(s)$ denote the Dirac delta at the points $s=0$ and $s=-\tau$ respectively. By well-known results, we have that $\eta : [-\tau,0] \to B(X,X)$ has a bounded variation and so, conditions (H4) and (H5) hold (and analogously replacing X by Y).

Finally, assumption (H6) can be read as a condition on the stationary state $\mathbf{y}$ (a study of the eigenvalue of operator A can be found, for instance, in Temam [65]).

Remark 13. *By introducing the representation operator* $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$ *it is clear that the quasilinear operator* $A\mathbf{P}(\mathbf{q})$ *obtained from the operator* $A\mathbf{u}$=-$(1+i\epsilon)\Delta\mathbf{u}$ *satisfies also condition* $A \in \mathcal{A}(\omega)$ *(since* $\mathbf{P}$ *is merely a change of variables). We point out that,*

$$A\mathbf{P}(\mathbf{q}) = -(1+i\epsilon)[\Delta\rho - \rho\left|\nabla\phi\right|^2 + i(2\nabla\rho\cdot\nabla\phi + \rho\Delta\phi)]e^{i\phi}.$$

Then, the "formal linearization" of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ *at* $\mathbf{q}^*(x,y) := \mathbf{y} \equiv \rho_0$ *becomes*

$$D\mathbf{E}(\mathbf{q}^*)(\rho e^{i\phi}) = -(1+i\epsilon)[\Delta\rho + i\rho_0\Delta\phi]e^{i\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ *needs a slight modification of the above linear expression.*■

3.3. Study of the eigenvalues of the linearized problem.

In this subsection we shall study the eigenvalues $\lambda \in \mathcal{C}$, $\lambda = a+ib$ of the linearized problem and, which is crucial, we look for

$$\begin{cases} \quad \text{any } \lambda \in \mathcal{C} \text{ such that } \exists\, v \in D(A),\ v \neq 0,\ \text{such that} \\ 0 = \lambda v + Av + \mathrm{D}B(w)v - \mathrm{D}F(\widehat{w})(e^{\lambda\cdot}v),\ \text{and } Re\lambda < 0, \end{cases} \tag{42}$$

where $e^{\lambda\cdot}v \in C$ is defined by

$$(e^{\lambda\cdot}v)(s) = e^{\lambda s}\widehat{v}(s),\ \text{with } \widehat{v}(s) = v,\ \text{for any } s \in [-\tau,0]. \tag{43}$$

As in the case without delay, (42) implies that the zero solution of the linearized problem $\frac{dU}{dt}(t) + AU(t) + \mathrm{D}B(w)U(t)$-$\mathrm{D}F(\widehat{w})U_t(.) = 0$ *in* X, is locally asymptotically stable ([67]).

We go back now to the problems (7) and (27), and recall the expressions (28), (29) and (30)

$$\theta = \beta - \mu(\sin(\chi_0 + (\omega+\theta)\tau) - \beta\cos(\chi_0 + (\omega+\theta)\tau)). \tag{44}$$

Notice that if $\mu = 0$ we deduce that $\rho_0(\tau) = 1$ and that $\theta(\tau) = \beta$ for any τ and that $\rho_0(0) = (1 + \mu\cos\chi_0)^{1/2}, \theta(0) = \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)$. It is not difficult to prove (see the following Proposition) the existence and uniqueness of such a function $\theta(\tau)$ and that $\theta \in C^1$.

Proposition 1. There exists a unique function $\theta(\tau)$ such that

$$\theta(\tau) - \beta + \mu(\sin(\chi_0 + (\omega+\theta(\tau))\tau) - \beta\cos(\chi_0 + (\omega+\theta(\tau))\tau)) = 0$$

for any $\tau \in [0,1]$. Moreover $\theta \in C^1$.

Proof. It is enough to see, by the implicit function theorem, that $\theta(\tau)$ is characterized as the (unique) solution of the Cauchy problem associated to the ODE

$$\frac{d\theta}{d\tau}(\tau) = \frac{-[\mu(\cos(\chi_0 + (\omega+\theta(\tau))\tau)(\omega+\theta) + \beta\sin(\chi_0 + (\omega+\theta(\tau))\tau))](\omega+\theta(\tau))}{1 + \mu(\cos(\chi_0 + (\omega+\theta(\tau))\tau)\tau + \beta\sin(\chi_0 + (\omega+\theta(\tau))\tau))\tau}.$$

We recall that in our case, $\mathbf{z}_\infty = \rho_0$ and so we can arrive to the linear problem

$$(P_4)\begin{cases} -(1+i\epsilon)\Delta\mathbf{z} &= -(a+ib)\mathbf{z} + [(1+i\theta) - 3(1+i\beta)\rho_0^2)]\mathbf{z} \\ & \quad +\mu e^{i\chi_0}\left[m_1\mathbf{z} + m_2\overline{\mathbf{z}} + e^{-a\tau + i(\omega+\theta-b)\tau}(m_3\mathbf{z} + m_4\overline{\mathbf{z}})\right] \qquad in\ \Omega, \\ \dfrac{\partial\mathbf{z}}{\partial\overrightarrow{n}} &= \mathbf{0} \qquad on\ \partial\Omega. \end{cases}$$

As usual, the linear structure of the equation leads to the search of nontrivial solutions $\mathbf{z}(x)$ of the form $\mathbf{A_k}w^j_{\mathbf{k}}(x)$, with $j = 1,2$, where $w^j_{\mathbf{k}}(x)$ are the eigenfunctions for the usual Laplacian operator Δ with periodic boundary conditions on $\Omega = (0,L_1)\times(0,L_2)$ We recall that the eigenvalues of this problem are given by

$$\lambda_0^0 = 0,\ \lambda_{\mathbf{k}}^0 = 4\pi\left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2}\right);\quad k_1,k_2 \in \mathbb{N}$$

with the associate eigenfunctions

$$w_0 = \frac{1}{\sqrt{|\Omega|}}, w^1_{\mathbf{k}} = \sqrt{\frac{2}{|\Omega|}}\cos 2\pi\mathbf{kx}, w^2_{\mathbf{k}} = \sqrt{\frac{2}{|\Omega|}}\sin 2\pi\mathbf{kx},\ \text{with}\ |\Omega| = L_1L_2,$$

where we have written $\mathbf{kx} := \left(\frac{k_1}{L_1}x_1 + \frac{k_2}{L_2}x_2\right)$ (see, e.g., Temam [65]).

The following general Lemma will be used in the study of $\mathbf{z}(x)$

Lemma 4. Let A be a selfadjoint operator on $\mathbf{L}^2(\Omega)$ and let $\{\varphi_n\}$ be a family of eigenfunctions associated to the different eigenvalues $\{\lambda_n^0\}$. Assume that $\lambda_0^0 = 0$ is an eigenvalue and that $\varphi_0 = 1$ is an eigenfunction associated to λ_0. Then

$$\int_\Omega \varphi_n = 0 \; for \; any \; n \neq 0.$$

Proof. It is enough to recall that $\int_\Omega \varphi_n\varphi_m = 0 \; for \; any \; n \neq m$ since $\lambda_n^0 \neq \lambda_m^0$ □

$$\lambda_n^0 \int_\Omega \varphi_n\varphi_m = \int_\Omega A\varphi_n\varphi_m = \int_\Omega \varphi_n A\varphi_m = \lambda_m^0 \int_\Omega \varphi_n\varphi_m.$$

Then taking $m = 0$ we get the conclusion.■

In order to keep a coherent notation with the one used in [11] and[12] we introduce the notation $\lambda_{\mathbf{k}} = a_{\mathbf{k}} + ib_{\mathbf{k}}$ for the real and imaginary parts of the eigenvalues of the problem stated in (H8). Notice that, by the previous Lemma, $\int_\Omega w_{\mathbf{k}}^j = 0$ for any $\mathbf{k} \neq 0$ and $j = 1, 2$. Then we get that

$$\begin{aligned}(a_{\mathbf{k}} + ib_{\mathbf{k}}) - (1+i\epsilon)(-\lambda_{\mathbf{k}}) = (1+i\theta) - 3(1+i\beta)\rho_0^2 \\ +\mu e^{i\chi_0}\left[m_1 + m_2\delta_{0\mathbf{k}} + e^{-a\tau + i(\omega+\theta-b)\tau}(m_3 + m_4\delta_{0\mathbf{k}})\right]\end{aligned}$$

where $\delta_{0\mathbf{k}}$ denotes the Kronecker delta function. We arrive to

$$\left\{\begin{array}{lcl} a_{\mathbf{k}} & = & -\lambda_{\mathbf{k}}^0 - 2 - 3\mu\cos\chi(\tau) + \mu(m_1 + m_2\delta_{0k})\cos\chi_0 + \\ & & +\mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0k})\cos(\chi_0 + (\omega+\theta-b_{\mathbf{k}})\tau), \\ b_{\mathbf{k}} & = & \theta - \epsilon\lambda_{\mathbf{k}}^0 - 3\beta(1+\mu\cos\chi) + \mu(m_1 + m_2\delta_{0k})\sin\chi_0 + \\ & & +\mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0k})\sin(\chi_0 + (\omega+\theta-b_{\mathbf{k}})\tau). \end{array}\right. \tag{45}$$

The previous equations are transcendent and we cannot get an explicit expression for the real and imaginary part of the eigenvalues (for some similar transcendent equations arising in delayed ODEs see [44]).

Now, we focus our attention in the dependence of $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ with respect to τ. So, by the regularity of the involved functions we can assume

$$a_{\mathbf{k}} = a_{\mathbf{k}0} + a_{\mathbf{k}1}\tau + o(\tau), \; b_{\mathbf{k}} = b_{\mathbf{k}0} + b_{\mathbf{k}1}\tau + o(\tau),$$

as we get, for instance, by a "formal" series development in powers of τ argument. Here we used the Landau notation ($f(\tau) = o(\tau)$ means that $\frac{f(\tau)}{\tau} \to 0$ when $\tau \to 0$).

The terms of order zero in τ are obtained by making $\tau = 0$ in (45)

$$\left\{\begin{array}{c} a_{\mathbf{k}0} = -\left(2 + \lambda_{\mathbf{k}}^0\right) + \mu\cos\chi_0(m_1 + m_2\delta_{0\mathbf{k}} + m_3 + m_4\delta_{0\mathbf{k}}) \\ b_{\mathbf{k}0} = 4\beta - \epsilon\lambda_{\mathbf{k}}^0 + 3\mu\beta\cos\chi_0 + \mu\sin\chi_0(m_1 + m_2\delta_{0\mathbf{k}} + m_3 + m_4\delta_{0\mathbf{k}}). \end{array}\right. \tag{46}$$

So, we can state a first result concerning the case without any delay

Proposition 2. Assume $\tau = 0$, $\chi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, and $\mu > \frac{1}{|\cos\chi_0|}$. Then the uniform oscillation $v_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ is linearly unstable.

Proof. From (46) we see that $a_{00} > 0$ and since $\tau = 0$ we get the existence of at least one eigenvalue λ of the linearized problem with $Re(\lambda) > 0$ which implies the result.

The first order terms in τ are calculated below

Lemma 5. We have

$$\begin{array}{c} a_{\mathbf{k}1} = \left[\frac{da_{\mathbf{k}}}{d\tau}\right]_{\tau=0} = \left(2 + \lambda_{\mathbf{k}}^0\right) + \mu\left[3(\omega+\beta)\sin\chi_0 + (m_3 + m_4\delta_{0\mathbf{k}})\left(3\beta - \epsilon\lambda_{\mathbf{k}}^0 - \omega\right)\right] \\ +\mu^2\{-3\sin^2\chi_0 + 3\beta\sin\chi_0\cos\chi_0 + \\ +(m_3 + m_4\delta_{0\mathbf{k}})\left[\sin^2\chi_0 + 2\beta\sin\chi_0\cos\chi_0 + \right. \\ \left. +(m_1 + m_2\delta_{0\mathbf{k}} + m_3 + m_4\delta_{0\mathbf{k}})\right](\sin^2\chi_0 - \cos^2\chi_0)\}. \end{array} \tag{47}$$

Differentiating in (45) we get that

$$\begin{array}{c} a_{\mathbf{k}1} = \left[\frac{da_{\mathbf{k}}}{d\tau}\right]_{\tau=0} = \left[3\mu\sin\chi(\tau)\frac{d\chi}{d\tau}\right]_{\tau=0} + \left[(-a_{\mathbf{k}})\,\mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0\mathbf{k}})\cos(\chi_0 + (\omega+\theta-b_{\mathbf{k}})\tau)\right]_{\tau=0} \\ -\left[\mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0\mathbf{k}})\sin(\chi_0 + (\omega+\theta-b_{\mathbf{k}})\tau)\right]_{\tau=0}\left[\frac{d(\omega+\theta-b_{\mathbf{k}})\tau}{d\tau}\right]_{\tau=0} = \\ = (3\mu\sin\chi_0)\left(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)\right) - \\ -\left(-\left(2 + \lambda_{\mathbf{k}}^0\right) + \mu\cos\chi_0(m_1 + m_2\delta_{0\mathbf{k}} + m_3 + m_4\delta_{0\mathbf{k}})\right)\mu(m_3 + m_4\delta_{0\mathbf{k}})\cos\chi_0 - \\ -\mu(m_3 + m_4\delta_{0\mathbf{k}})(\omega + \beta - \mu(\sin\chi_0 - \beta\cos\chi_0) - b_{\mathbf{k}})\sin\chi_0. \end{array}$$

Thus, by using the expression for $b_{\mathbf{k}}$ (see (45)) we obtain that

$$
\begin{gathered}
a_{\mathbf{k}1} = (3\mu\sin\chi_0)\left(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0)\right) - \\
-\left(-\left(2+\lambda^0_{\mathbf{k}}\right)+\mu\cos\chi_0(m_1{+}m_2\delta_{0\mathbf{k}}{+}m_3{+}m_4\delta_{0\mathbf{k}})\right)\mu(m_3{+}m_4\delta_{0\mathbf{k}})\cos\chi_0 - \\
-\mu(m_3{+}m_4\delta_{0\mathbf{k}})(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0))\sin\chi_0 \\
(3\mu\sin\chi_0)\left(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0)\right) - \\
-\left(-\left(2+k^2\right)+\mu\cos\chi_0(m_1{+}m_2\delta_{0\mathbf{k}}{+}m_3{+}m_4\delta_{0\mathbf{k}})\right)\mu(m_3{+}m_4\delta_{0\mathbf{k}})\cos\chi_0 - \\
-\mu(m_3{+}m_4\delta_{0\mathbf{k}})(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0))\sin\chi_0 \\
+\mu(m_3{+}m_4\delta_{0\mathbf{k}})(4\beta-\epsilon\lambda^0_{\mathbf{k}}+3\mu\beta\cos\chi_0+\mu\sin\chi_0(m_1{+}m_2\delta_{0\mathbf{k}}+m_3{+}m_4\delta_{0\mathbf{k}}))\sin\chi_0.
\end{gathered}
$$

In consequence

$$
\begin{gathered}
a_{\mathbf{k}1} = \left(2+\lambda^0_{\mathbf{k}}\right)+\mu\left(3\left(\omega+\beta\right)\sin\chi_0-\left(m_3{+}m_4\delta_{0\mathbf{k}}\right)\left(\omega+\beta\right)+\left(4\beta-\epsilon\lambda^0_{\mathbf{k}}\right)\left(m_3{+}m_4\delta_{0\mathbf{k}}\right)\right) \\
-\mu^2\left(3\sin\chi_0(\sin\chi_0-\beta\cos\chi_0)+\cos^2\chi_0(m_1{+}m_2\delta_{0\mathbf{k}}{+}m_3{+}m_4\delta_{0\mathbf{k}})(m_3{+}m_4\delta_{0\mathbf{k}})\right) - \\
+\mu^2(m_3{+}m_4\delta_{0\mathbf{k}})\left[(\sin\chi_0-\beta\cos\chi_0)\sin\chi_0+(3\beta\cos\chi_0+\sin\chi_0(m_1{+}m_2\delta_{0\mathbf{k}}+m_3{+}m_4\delta_{0\mathbf{k}}))\sin\chi_0\right]
\end{gathered}
$$

which proves the result.

Proposition 3. Assume (8), $\chi_0 \in (\pi, \frac{3\pi}{2})$, (29) and

$$\mu > \max\{0, \frac{3\beta-\omega+3(\omega+\beta)\sin\chi_0+\cos\chi_0}{5(-\beta)\sin\chi_0\cos\chi_0+1}\}.$$

Then $a_{00}+a_{01}<0$.

Proof. By using (46), (47), and (29) we get

$$a_{00}+a_{01} = \mu[(3\beta-\omega+3(\omega+\beta)\sin\chi_0+\cos\chi_0)-\mu(5(-\beta)\sin\chi_0\cos\chi_0+1)].$$

Then, the assumptions imply the positivity of the coefficient of μ^2 and the result holds. ☐

Proposition 4. Assume (8), $\chi_0 \in (\pi, \frac{3\pi}{2})$, (29) *and*

$$\mu > \max\{0, \frac{m_3(3\beta-\omega-\varepsilon 4\pi\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right))+3(\omega+\beta)\sin\chi_0+(m_1+m_3)\cos\chi_0}{(3-m_1-2m_3)\sin^2\chi_0+(m_1+m_3)\cos^2\chi_0+(-\beta)(3+2m_3)\sin\chi_0\cos\chi_0}\}.$$

Then, for any $\mathbf{k}$, $a_{\mathbf{k}0}+a_{\mathbf{k}1}<0$. Moreover, for any $\mathbf{k}\neq 0$ and any $\tau\in(0,1]$,

$$a_{k(n)0}+a_{k(n)1}\tau < a_{k(1)0}+a_{k(1)1}\tau.$$

By using (46), (47) and that $0<\lambda^0_{(1,1)}<\lambda^0_{\mathbf{k}}$ for any $\mathbf{k}\in\mathbb{N}^2$, $\mathbf{k}\neq(1,1)$, we obtain that

$$
\begin{gathered}
a_{\mathbf{k}0}+a_{\mathbf{k}1} = \mu[(m_3(3\beta-\omega-\varepsilon 4\pi\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right))+3(\omega+\beta)\sin\chi_0+(m_1+m_3)\cos\chi_0) \\
-\mu((3-m_1-2m_3)\sin^2\chi_0+(m_1+m_3)\cos^2\chi_0+(-\beta)(3+2m_3)\sin\chi_0\cos\chi_0)].
\end{gathered}
$$

Again, the assumptions made on the parameters imply the positivity of the coefficient of μ^2 and the result holds. Moreover

$$a_{k(n)0}-a_{k(1)0}+(a_{k(n)1}-a_{k(1)1})\tau = -k(n)^2+k(1)^2-(m_3\epsilon k(n)^2-m_3\epsilon k(1)^2)\tau<0.$$

The proof of Theorem 5 is now complete since from Propositions 3 and 4 we deduce the existence of some $\tau_0\in(0,1)$ (independent of $\mathbf{k}\in\mathbb{N}^2$) such that for any $|\mathbf{k}|\geq 0$ we have $a_{\mathbf{k}0}+a_{\mathbf{k}1}\tau<0$ for any $\tau\in(\tau_0,1)$. This implies the hypothesis of the abstract result and the conclusion follows.

Remark 14. *Notice that Theorem 5 applies to the case $m_1=m_2=m_3=0$ which corresponds to a formulation similar to the one of* [11]*. Moreover, it also applies to the choice $m_1=\kappa, m_2=-1-\kappa$, $m_3=0$ and $m_4=1$, for any $\kappa\in(0,1)$ which corresponds to a formulation quite close to the pioneering paper* [58] *(concerning chaotic ODEs).*

Remark 15. *Since the eigenvalue $\lambda^0_0=0$, using Lemma it is possible to obtain the same result for Neumann conditions.*

Remark 16. *Numerical simulations show that while a purely local control is unsuitable to produce uniform oscillations, a mixed local and global control can be efficient and also able to create other patterns such as standing waves, amplitude death, or traveling waves (see* [62] *and its references).*

4. Hopf bifurcation and delay terms

Before to consider the case of the complex Ginzburg-Landau equation (CGLE), for the purposes of clarity and ease of understanding, we start by considering a very simplified version of the general model to be given later which has the advantage that closed analytical solutions are possible and the necessary eigenvalue computations can be carried out in full. Unfortunately, such precise calculations are not available for the general model and a fairly complete graphical-numerical study will be given in exchange.

4.1. **Hopf bifurcation for the Stuart-Landau equation with a time delay feedback.** In the Stuart-Landau equation, the diffusion term is absent, which amounts to restricting our study to the *spatially homogeneous* solutions (which always satisfy periodic boundary conditions). On the other hand, we assume that a *delayed linear feedback term* is added, so the equation under study in this section will be

$$\frac{\partial \mathbf{A}}{\partial t} = (1 - i\omega)\mathbf{A} - (1 + i\alpha)\left|\mathbf{A}\right|^2 \mathbf{A} + m_1 \mathbf{A} + m_3 \mathbf{A}(t - \tau). \tag{48}$$

More general control terms will be considered later. The change of variables $\mathbf{w}(t) = e^{-i\phi t}\mathbf{A}(t)$ gives

$$\frac{\partial \mathbf{w}}{\partial t} = (1 - i\omega - i\phi)\mathbf{w} - (1 + i\alpha)\left|\mathbf{w}\right|^2 \mathbf{w} + m_1 \mathbf{w} + m_3^- e^{i\phi\tau}\mathbf{w}(t - \tau). \tag{49}$$

We now choose $\phi = -\alpha - \omega$ and $m_3 = -ei^{i\phi\tau} m_1$ and denote the stationary solution of

$$\frac{\partial \mathbf{w}}{\partial t} = (1 + i\alpha)(\mathbf{w} - \left|\mathbf{w}\right|^2 \mathbf{w}) + m_1 \left[\mathbf{w} - \mathbf{w}(t - \tau)\right]. \tag{50}$$

by $\mathbf{w}_0$. In order to check if at some critical value of the delay $\tau = \tau^*$ a Hopf bifurcation takes place, we linearize the equation around $\mathbf{w}_0 = 1$ and check whether a pair of complex eigenvalues $\lambda(\tau) = a(\tau) \pm ib(\tau)$ of the linearization cross transversally the imaginary axis away from the origin, i.e., they satisfy $a(\tau^*) = 0,\ b(\tau^*) \neq 0$ and $a'(\tau^*) \neq 0$ (see, e.g., [67]).

Observe now that the complex term $\left|v\right|^2 v$, although perfectly differentiable from the real point of view (in fact, the complex map $z \longmapsto \left|z\right|^2 z = z^2\bar{z}$ is *real-analytic*), is *not an analytic* (or holomorphic) function from the complex viewpoint. Therefore it becomes convenient at this point to abandon the complex notation and write the system in real form ($\mathbf{w} = u + iv$) as follows

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} (1 - (u^2 + v^2)) \begin{pmatrix} u \\ v \end{pmatrix} + m_1 \begin{pmatrix} u - u(t - \tau) \\ v - v(t - \tau) \end{pmatrix}.$$

Let us fix our attention to the stationary solution $\mathbf{w}_0 = (u_0, v_0) = (1, 0)$. The linearization around $\mathbf{w}_0$ is given by

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + m_1 \begin{pmatrix} U - U(t - \tau) \\ V - V(t - \tau) \end{pmatrix} \tag{51}$$

and the eigenvalue-eigenvector pairs associated to this vector equation are the solutions of (51) of the special form $U(t) = e^{\lambda t}U_0,\ V(t) = e^{\lambda t}V_0$ where $\lambda \in \mathbb{C}$ and U_0, V_0 are (possibly complex) constant (nonzero) 2-vectors. One thus easily finds

$$\lambda \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} -2 + m_1 & 0 \\ -2\alpha & m_1 \end{pmatrix} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} - m_1 e^{-\lambda\tau} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix},$$

thus arriving to the *characteristic equation*

$$\begin{vmatrix} \lambda + 2 - m_1 + m_1 e^{-\lambda\tau} & 0 \\ 2\alpha & \lambda - m_1 + m_1 e^{-\lambda\tau} \end{vmatrix} = 0.$$

This means that we have a double collection of eigenvalues: those satisfying $\lambda - m_1 + m_1 e^{-\lambda\tau} = 0$ and those satisfying $\lambda + 2 - m_1 + m_1 e^{-\lambda\tau}$. Denoting $\lambda = a + ib$, we identify two classes of eigenvalues:

$$\lambda - m_1 + m_1 e^{-\lambda\tau} = 0 \Longleftrightarrow \left\{ \begin{array}{c} a - m_1 + m_1 e^{-a\tau}\cos b\tau = 0 \\ b - m_1 e^{-a\tau}\sin b\tau \end{array} \right. \quad \text{(Class 1)}$$

$$\lambda + 2 - m_1 + m_1 e^{-\lambda\tau} = 0 \Longleftrightarrow \left\{ \begin{array}{c} a + 2 - m_1 + m_1 e^{-a\tau}\cos b\tau = 0 \\ b - m_1 e^{-a\tau}\sin b\tau \end{array} \right. \quad \text{(Class 2)}$$

We now look for values $\tau = \tau^*$ for which $a = 0$ and $b \neq 0$. We find no eigenvalues of this kind for Class 1, since $-1 + \cos b\tau = 0$ implies $\sin b\tau = 0$, and hence $b = 0$ from the second equation.

However, Class 2 does give us some useful values:

$$2 - m_1 + m_1 \cos b\tau = 0 \Longrightarrow \cos b\tau = \frac{m_1 - 2}{m_1},$$

$$b - m_1 \sin b\tau = 0 \Longrightarrow \sin b\tau = \frac{b}{m_1}.$$

Thus,

$$1 = \cos^2 b\tau + \sin^2 b\tau = \left(\frac{m_1 - 2}{m_1}\right)^2 + \frac{b^2}{m_1^2} \Longrightarrow b^2 = m_1^2 - (m_1 - 2)^2 = 4(m_1 - 1).$$

Hence, if $m_1 > 1$, we have

$$\cos b\tau = \frac{m_1 - 2}{m_1} \Longrightarrow b\tau = \arccos\left(\frac{m_1-2}{m_1}\right)$$

which is well defined for every $m_1 > 1$.

Summarizing, the set of values

$$b^* = 2\sqrt{m_1 - 1},\ \tau^* = \frac{1}{b^*}\left[\arccos\left(\frac{m_1-2}{m_1}\right) + 2k\pi\right]$$

corresponds to a (possible) bifurcation point of Hopf type. For instance, for $m_1 = 2$ we have $b^* = 2$ and $\tau^* = k\pi + \pi/4$.

We now need to compute the derivative $a'(\tau^*)$. It is easier now to go back to the complex formulation of Class 2 eigenvalues

$$\lambda + 2 - m_1 + m_1 e^{-\lambda\tau} = 0,$$

and find $d\lambda/d\tau$ by implicit differentiation:

$$\frac{d\lambda}{d\tau} + m_1 e^{-\lambda\tau}\left(-\frac{d\lambda}{d\tau}\tau - \lambda\right) = 0 \Longrightarrow \frac{d\lambda}{d\tau} = \frac{\lambda e^{-\lambda\tau}}{1 - m_1 e^{-\lambda\tau}\tau} = \frac{\lambda}{1 - m_1 e^{\lambda\tau}\tau}.$$

Concentrating on the specific values $b^* = 2$ and $\tau^* = \pi/4$ we find, at the bifurcation values τ^*, $\lambda^* = ib^*$, that

$$\left.\frac{d\lambda}{d\tau}\right|_{(\tau^*,\lambda^*)} = \frac{ib^*}{1 - m_1 e^{ib^*\tau^*}\tau^*} = -\frac{4\pi}{\pi^2+4} + \frac{8}{\pi^2+4}i.$$

Hence

$$\frac{da}{d\tau}(\tau^*) = -\frac{4\pi}{\pi^2+4} \neq 0$$

and the transversality condition is satisfied. Therefore, a Hopf bifurcation occurs, and a periodic orbit of approximate period

$$T \simeq \frac{2\pi}{b(\tau^*)} = \pi$$

exists for delay values τ near τ^*.

Remark 17. *To decide the sub- or supercritical character of the bifurcation a much longer analysis is necessary. On the other hand, for $\tau > 1/2$ there are always positive* real *eigenvalues coming from the first class, which means that the stationary point has become already unstable* before *the delay reaches $\tau^* = \pi/4$ value. Hence the periodic orbit cannot capture the stability lost by the stationary point, since that stability was already lost.*

4.2. **Hopf bifurcation for the complex Ginzburg-Landau equation on the whole space and with delayed time feedback.** We come back to the consideration of the complex Ginzburg-Landau equation subjected to a time-delay feedback with local and global terms but now for the case of a spatial domain given by the whole space:

$$\begin{aligned} \partial_t \mathbf{A} &= (1 - \mathrm{i}\omega)\mathbf{A} - (1+\mathrm{i}\alpha)|\mathbf{A}|^2\mathbf{A} + (1+\mathrm{i}\beta)\partial_{xx}\mathbf{A} + \mathbf{F}, \\ \mathbf{F} &= \mu e^{\mathrm{i}\xi}\left[m_1\mathbf{A} + m_2\langle\mathbf{A}\rangle + m_3\mathbf{A}(t-\tau) + m_4\langle\mathbf{A}(t-\tau)\rangle\right], \end{aligned} \tag{52}$$

where

$$\langle\mathbf{A}\rangle = \frac{1}{L}\int_0^L \mathbf{A}(x,t)dx$$

denotes the spatial average of $\mathbf{A}$ over a one-dimensional medium of length L. There are many previous works in the literature dealing with such type of formulations: [30], [31], [62],.[63].

Extensive simulations [31] and an analytical stability analysis [63] for a special case representing a Pyragas-type feedback [58] ($m_3 = -m_1 = m_l$, $m_4 = -m_2 = m_g$) showed the range of patterns that can be stabilized as function of the local and global feedback terms. If the feedback is global, uniform oscillations can be stabilized for a large range of feedback parameters, while as the contribution of the local feedback term becomes larger, the parameter regions increase where the homogeneous fixed point solution, standing waves and traveling waves are found.

Uniform oscillations $\mathbf{A}(t) = \rho_0 \exp(-\mathrm{i}\theta t)$ are a solution of Eqs. (52) with amplitude and frequency given by

$$\begin{aligned} \rho_0 &= \sqrt{1 + \mu(m_g + m_l)(\cos(\xi + \theta\tau) - \cos\xi)}, \\ \theta &= \omega + \alpha + \mu(m_g + m_l)\left[\alpha(\cos(\xi+\theta\tau) - \cos\xi) - (\sin(\xi+\theta\tau) - \sin\xi)\right]. \end{aligned} \tag{53}$$

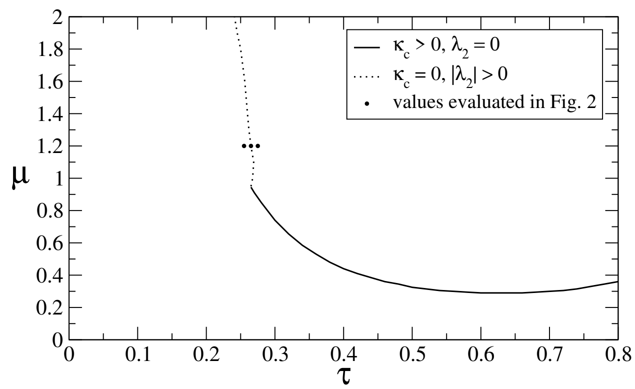

FIGURE 1. Control diagram in (μ, τ)-space for $m_l = 0.4$, $m_g = 0.6$. The other parameters are $\alpha = -1.4$, $\beta = 2$, $\omega = 2\pi - \alpha$, $\xi = \pi/2$. At the solid curve, uniform oscillations become unstable with respect to perturbations with $\kappa_c > 0$ and $\lambda_2(\kappa_c) = 0$, at the dotted curve, with $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$. The dots indicate parameter values further studied in Fig. **??**.

In [63], it is performed a linear stability analysis of uniform oscillations with respect to spatiotemporal perturbations. There, we expressed the complex oscillation amplitude $\mathbf{A}$ as the superposition of a homogeneous mode $\mathbf{H}$ (corresponding to uniform oscillations) with spatially inhomogeneous perturbations,

$$\mathbf{A}(x,t) = \mathbf{H}(t) + \mathbf{A}_+(t)e^{icx} + \mathbf{A}_-(t)e^{-icx}\,. \tag{54}$$

Notice that here we are using the fact that the equation takes place on the whole space, which allows the justification of the spatially inhomogeneous perturbations of the form $\mathbf{A}_+(t)e^{icx} + \mathbf{A}_-(t)e^{-icx}$. Inserting (54) into (52), and assuming that the amplitudes $\mathbf{A}_\pm$ are small, we obtain a set of equations for $\mathbf{H}$, $\mathbf{A}_+$, and $\mathbf{A}_-^*$ (see [63] for details of this derivation). To investigate linear stability of uniform oscillations with respect to spatiotemporal perturbations, we make the ansatz

$$\begin{aligned} \mathbf{A}_+ &= A_+^0 \exp(-i\theta t)\exp(\lambda t), \\ \mathbf{A}_-^* &= A_-^{*0} \exp(i)\exp(\lambda t), \end{aligned} \tag{55}$$

where $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ is a complex eigenvalue. Using ansatz (55), we arrive at the following eigenvalue equation:

$$\mathbf{F} = (A + iB - i\lambda_2 + D_1 + iD_2)(A - iB - i\lambda_2 + C_1 + iC_2), \tag{56}$$

where we have defined

$$\begin{aligned} F &= (1+\alpha^2)\rho_0^4, \\ A &= 1 - \lambda_1 - 2\rho_0^2 - \kappa^2, \\ B &= \theta - \omega - 2\alpha\rho_0^2 - \beta\kappa^2, \\ C_1 &= \mu m_l e^{-\lambda_1\tau}\cos(\xi + \theta\tau + \lambda_2\tau) - \mu m_l \cos\xi, \\ C_2 &= -\mu m_l e^{-\lambda_1\tau}\sin(\xi + \theta\tau + \lambda_2\tau) + \mu m_l \sin\xi, \\ D_1 &= \mu m_l e^{-\lambda_1\tau}\cos(\xi + \theta\tau - \lambda_2\tau) - \mu m_l \cos\xi, \\ D_2 &= \mu m_l e^{-\lambda_1\tau}\sin(\xi + \theta\tau - \lambda_2\tau) - \mu m_l \sin\xi. \end{aligned}$$

We point out that the above eigenvalue equation can be obtained also by a formal linearization argument involving the Fréchet derivatives as in the next section. There is no general analytic solution to Eq. (56) for $\lambda_{1,2}$. Thus, Eq. (56) must be solved numerically for a given set of parameters. We keep the CGLE parameters α, β, ω and the feedback parameters m_l, m_g, and ξ constant and solve Eq. (56) with the FindRoot routine of the Mathematica package [52]. We then find, for each point in the (τ, μ)-space, the functional dependence of λ_1 and λ_2 on κ. Notice that if we assume $\kappa = 0$ the study can be applied to the case of the Stuart-Landau equation, as before.

In general, Eq. (56) has multiple solutions, reflected by multiple branches in the dispersion relation. Stability is determined by the sign of λ_1. The curves $\lambda_1(\kappa)$ either lie below $\lambda_1 = 0$, so that uniform oscillations are stable, or they display an interval of κ-values, where $\lambda_1 > 0$, so that uniform oscillations are unstable. At criticality, we have $\lambda_1 = 0$, $\partial_\epsilon \lambda_1 \neq 0$, where ϵ stands for either μ or τ. For the critical wavenumber κ_c, there are two

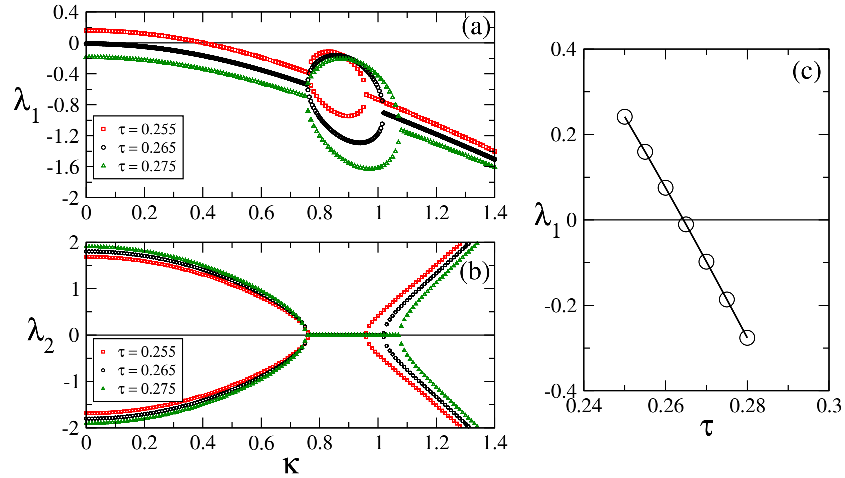

FIGURE 2. Dispersion relations for three parameter sets close to criticality: $\tau = 0.255$ (red squares), $\tau = 0.265$ (black circles), $\tau = 0.275$ (green triangles). (a) Real part of the eigenvalue as function of the wavenumber κ. (b) Imaginary part of the eigenvalue. The instability is characterized by $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$ and occurs for $\mu = 1.2$ at $\tau = 0.264399$. (c) Real part of the eigenvalue as function of τ, demonstrating transversality.

possibilities: $\kappa_c = 0$ or $\kappa_c \neq 0$ ($\pm\kappa_c$ are solutions, although below, we consider only $\kappa_c > 0$ without loss of generality).

Two instabilities are particularly important in our system: the first one is associated with $\kappa_c > 0$ and $\lambda_2(\kappa_c) = 0$, and the second one with $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$. In Fig.1, we show as an example the control diagram in (μ, τ)-space for $m_l = 0.4$, $m_g = 0.6$. Stable uniform oscillations are observed above the solid curve and to the right of the dotted curve. At the solid curve, uniform oscillations become unstable with respect to perturbations with $\kappa_c > 0$ and $\lambda_2(\kappa_c) = 0$, at the dotted curve, with $\kappa_c = 0$ and $\lambda_2(\kappa_c) \neq 0$. In Fig.2 (a,b), the dispersion relations $\lambda_{1,2} = \lambda_{1,2}(\kappa)$ are shown for three τ values close to criticality, demonstrating clearly the nature of the underlying instability. In Fig. 2(c), we show that λ_1 crosses $\lambda_1 = 0$ as τ is varied, hence demonstrating transversality. As the uniform oscillations become unstable with respect to a mode with complex conjugated eigenvalues and since ρ_0 remains finite, we infer the presence of a secondary Hopf bifurcation.

4.3. **Hopf bifurcation for the delayed CGLE in a bounded domain.** We consider now the case of two spatial dimensions varying on the domain $\Omega = (0, L_1) \times (0, L_2)$. Our goal is to show a bifurcation phenomenon near uniform oscillations for the CGLE in terms of the delay term as parameter. We define the faces of the boundary

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}\,, \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}\,,\ j = 1, 2,$$

on which we assume periodic boundary conditions and, hence, the problem under study can be formulated as

$$(P_1) \left\{ \begin{array}{ll} \begin{array}{c} \partial_t \mathbf{u} - (1 + i\beta)\Delta\mathbf{u} = (1 - i\omega)\mathbf{u} - (1 + i\alpha)|\mathbf{u}|^2\mathbf{u} \\ +\mu e^{i\xi}\mathbf{F}(\mathbf{u}, t, \tau) \end{array} & \Omega \times (0, \infty), \\ \left. \begin{array}{l} \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+2}}\,, \\ \left(-\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\Big|_{\Gamma_j} =\right) \frac{\partial\mathbf{u}}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial\mathbf{u}}{\partial x_j}\Big|_{\Gamma_{j+2}} \left(= \frac{\partial\mathbf{u}}{\partial\mathbf{n}}\Big|_{\Gamma_{j+2}}\right) \end{array} \right\} & \partial\Omega \times (0, \infty), \\ \mathbf{u}(x, s) = \mathbf{u}_0(x, s) & \Omega \times [-\tau, 0], \end{array} \right.$$

where $\mathbf{n}$ is the outpointing normal unit vector, and

$$\mathbf{F}(\mathbf{u}, t, \tau) = [m_1\mathbf{u}(x, t) + m_2\langle\mathbf{u}(t)\rangle + m_3\mathbf{u}(x, t - \tau) + m_4\langle\mathbf{u}(t - \tau)\rangle]$$

with

$$\langle\mathbf{u}(s)\rangle = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}(x, s)dx.$$

Again, the parameters $\alpha, \beta, \omega, \mu, \xi, m_i$ and τ are real, while $\mathbf{u}(x, t) = \mathbf{u}_1(x, t) + i\mathbf{u}_2(x, t)$ is complex.

We study the stability of *uniform oscillations*, i.e., solutions of (P_1) of the form $\mathbf{v}_{\mathrm{uo}}(t) = \rho_0 e^{-i\theta t}$ which determines completely ρ_0 and θ. We are interested in the Hopf bifurcation close to $\mathbf{v}_{\mathrm{uo}}(t)$ which gives rise to some paths on a suitable torus (for a different study dealing with invariant tori see [64]).

In order to avoid the application of very sophisticated techniques (dealing with periodic solutions), we can reduce the study to the Hopf bifurcation near a stationary solution of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t) = \mathbf{v}(x,t)e^{i\theta t}$ where $\mathbf{v}(x,t)$ is a solution of (P_1). Thus, $\mathbf{z}(x,t)$ satisfies

$$(P_2)\begin{cases} \partial_t \mathbf{z} - (1+i\beta)\Delta\mathbf{z} = (1+i\theta)\mathbf{z} - (1+i\alpha)|\mathbf{z}|^2\mathbf{z} + \mu e^{i\xi}\times \\ \times[m_1\mathbf{z} + m_2\langle\mathbf{z}\rangle + e^{i(\omega+\theta)\tau}(m_3\mathbf{z}(t-\tau) + m_4\langle\mathbf{z}(t-\tau)\rangle)] & \Omega\times(0,\infty), \\ \left.\begin{array}{l}\mathbf{z}|_{\Gamma_j} = \mathbf{z}|_{\Gamma_{j+2}}, \\ \left(-\left.\frac{\partial\mathbf{z}}{\partial\mathbf{n}}\right|_{\Gamma_j}=\right)\left.\frac{\partial\mathbf{z}}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial\mathbf{z}}{\partial x_j}\right|_{\Gamma_{j+2}}\left(=\left.\frac{\partial\mathbf{z}}{\partial\mathbf{n}}\right|_{\Gamma_{j+2}}\right)\end{array}\right\} & \partial\Omega\times(0,\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x,s)e^{i(\omega-\theta)s} & \Omega\times[-\tau,0]. \end{cases}$$

Now, $\mathbf{v}_{\mathrm{uo}}(t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_\infty(t)e^{i\theta t} = \mathbf{z}_\infty = \rho_0$ is an stationary solution of (P_2), i.e.,

$$\mathbf{0} = (1+i\theta)\mathbf{z}_\infty - (1+i\alpha)\left|\mathbf{z}_\infty\right|^2\mathbf{z}_\infty + \mu e^{i\xi}\left[m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3+m_4)\right]\mathbf{z}_\infty.$$

4.3.1. *An abstract Hopf bifurcation theorem for semilinear functional equations.* We shall apply to our setting an abstract result due to J. Wu (see [67], Theorem 2.1) stated for problems of the type

$$\begin{cases} \frac{du}{dt}(t) + Au(t) = L(\mu, u_t(.)) + g(u_t(.)) & in\ X, \\ u(s) = u_0(s) & s\in[-\tau,0]. \end{cases}$$

on a Banach space X, where $u_t : [-\tau,0]\to X$, under the following list of conditions:

(H_1) A generates an analytic compact semigroup $\{T(t)\}_{t\geq 0}$;

(H_2) The point spectrum of A consists of a sequence of real number $\{\mu_k\}_{k\geq 1}$ with the corresponding eigenspace M_k and the projection $P_k : X\to M_k$. Moreover, if $\sum_{k=1}^\infty x_k = 0$ for $x_k\in M_k$ then each x_k must be zero;

(H_3) Every $x\in D(A)$ has a unique expression $x = \sum_{k=1}^\infty P_k x$ and $Ax = \sum_{k=1}^\infty \mu_k P_k x$;

(H_4) The mapping $L : \mathbb{R}\times C\to X$ (with $C := C([-\tau,0] : X)$) is C^k-smooth ($k\geq 4$) and is given by

$$L(\mu,\phi) = \int_{-\tau}^0 \phi(\theta)d\eta(\mu,\theta)$$

for any $(\mu,\phi)\in\mathbb{R}\times C$, for a function $\eta(\mu,.)$: $[-\tau,0]\to B(X,X)$ of bounded variation. Moreover, $L(\mu,P_k\phi)\in M_k$, $k\geq 1$, $\phi\in C$ and $L(\mu,\sum_{k=1}^\infty P_k\phi) = \sum_{k=1}^\infty L(\mu,P_k\phi)$ for any $\phi\in C$ such that $\sum_{k=1}^\infty P_k\phi\in C$, where $P_k\phi$ is defined by $(P_k\phi)(\theta) = P_k\phi(\theta)$ for $\theta\in[-\tau,0]$;

(H_5) $g : \mathbb{R}\times C\to X$ has k-th-continuous Fréchet derivatives with $g(\mu,0) = 0$ and $Dg(\mu,0) = 0$ for $\mu\in\mathbb{R}$;

(H_6) There exists $\mu_0\in\mathbb{R}$ and $\omega_0 > 0$ such that $\pm i\omega_0$ are simple characteristic values of the linear equation

$$\dot{u}(t) + Au(t) = L(\mu_0, u_t(.)) \tag{58}$$

and all other characteristic values have negative real parts;

(H_7) *Transversality condition.* If μ is near μ_0 the eigenvalues of the corresponding problem (58) are given by $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, $\lambda(\mu_0) = i\omega_0$, $\lambda(\mu)$ is C^k-smooth in μ and

$$\alpha'(\mu_0)\neq 0.$$

Remark 18. *A careful reading of the proof of Theorem 2.1 of* [67] *allows to see that the use of the same notation u_t in the terms $L(\mu,u_t(.))$ and $g(u_t(.))$ does not needs that the kernels involved in each of the possible nonlocal terms be exactly the same. So, in particular, the conclusion remains valid in the special case in which $g(u_t(.)) = g(u(.))$, i.e., without delay or neutral term.*

4.3.2. *Application to the delayed CGLE on a bounded domain.* Motivated by the special form of the nonlinear term of the equation in (P_2) we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$. A detailed analysis of the associated diffusion operator is consequence of some previous results in the literature: see, e.g., Amann [5]. Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} \Delta & -\beta\Delta \\ \beta\Delta & \Delta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, if $\beta\neq 0$ the diffusion matrix has a nonzero antisymmetric part. In particular, A is the generator of a semigroup of contractions $\{T(t)\}_{t\geq 0}$ on X and the compactness of the semigroup is consequence of the compactness of the inclusion $D(A)\subset X$ (notice that, since $N = 2$, $\mathbf{W}^{1,4}(\Omega)\subset\mathbf{W}^{1,4/3}(\Omega)\subset\mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems. A study of the eigenvalues of A can be found, e.g., in Temam [65]

Concerning the rest of the terms of the equation in (P_2), we define $g(\mathbf{u}) = -(1+i\alpha)\left|\mathbf{u}\right|^2 \mathbf{u}$ with $D(g) = \mathbf{L}^{12}(\Omega)$. By using the characterization of the semi inner-braket $[,]$ for the spaces $L^p(\Omega)$ (see, e.g., Benilan, Crandall and Pazy [20]) it is easy to see that $\mathbf{B} = -\mathbf{g}$ is an accretive operator on X, which is dominated by A; i.e.,

$$D_X(A) \subset D_X(B) \text{ and } |Bu| \leq k\left|A^0 u\right| + \sigma(|u|)$$

for any $u \in D_X(A)$, some $k < 1$ and some continuous function $\sigma : \mathbb{R} \to \mathbb{R}$.

Here and in what follows, $|.|$ denotes the norm in the space X (in contrast to the norm in space C which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $\left|A^0 u\right| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$. In particular, the operator $A + B$ is also an accretive operator on X.

In order to calculate the Fréchet differential of Nemitsky operator $g(\mathbf{u})$, it is useful to start analyzing the Gateaux derivative of the complex function $\mathbf{h}(\mathbf{z}) := \|\mathbf{z}\|^2 \mathbf{z}$ in the direction of an arbitrary vector $\mathbf{v}$ of $\mathbb{C}$

$$\lim_{\substack{\beta\in\mathbb{R} \\ |\beta|\to 0}} \frac{\mathbf{h}(\mathbf{z}_0 + \beta\mathbf{v}) - \mathbf{h}(\mathbf{z}_0)}{|\beta|} = \mathbf{z}_0^2\overline{\mathbf{v}} + 2\left\|\mathbf{z}_0\right\|^2 \mathbf{v}.$$

Then, we identify the Fréchet differential of operator $g(\mathbf{u})$ as

$$D\mathbf{B}(\mathbf{y})\mathbf{v} = (1 + i\alpha)[\mathbf{y}^2\overline{\mathbf{v}} + 2\left\|\mathbf{y}\right\|^2 \mathbf{v}]. \tag{59}$$

Since we have $\|D\mathbf{B}(\mathbf{y})\| \leq c\left\|\mathbf{y}\right\|^2$, by the results on the Fréchet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi [7]) we get that, if we take $Y = \mathbf{L}^{4/3}(\Omega)$, then exists $\delta^B > 0$ such that $\mathbf{B}$ is Fréchet differentiable as function from $B_{\delta^B}(w) = \left\{z \in D(\mathbf{B});\ |w - z| < \delta^B\right\}$ into Y, and that the Fréchet derivative is locally Lipschitz continuous.

The nonlocal term is defined by

$$\begin{aligned} F(\mathbf{u}_t) &= (1 + i\theta)\mathbf{u}(t) \\ &\quad + \mu e^{i\xi}\left[m_1\mathbf{u}(t) + m_2\langle\mathbf{u}(t)\rangle + e^{i(\omega+\theta)\tau}(m_3\mathbf{u}(t-\tau) + m_4\langle\mathbf{u}(t-\tau)\rangle)\right], \end{aligned}$$

is locally Lipschitz continuous and its Fréchet derivative is given by

$$\begin{aligned} \mathrm{D}F(\widehat{\mathbf{y}})\mathbf{v}(t) &= -(1 + i\theta)\mathbf{v}(t) \\ &\quad - \mu e^{i\xi}\left[m_1\mathbf{v}(t) + m_2\langle\mathbf{v}(t)\rangle - e^{i(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau) - m_4\langle\mathbf{v}(t-\tau)\rangle)\right]. \end{aligned}$$

In consequence, the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega^* \cdot}y\right)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\mathrm{Re}\omega^* = \gamma^* < 0$. This means that the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega^* \cdot}y\right) + \omega^* y$ is accretive in $Y = \mathbf{L}^{4/3}(\Omega)$. We recall (see Ambrosetti and Prodi [7]) that this differentiability of B does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

We also recall that in [30] the existence (and uniqueness) of a mild solution of problem (P_2) was obtained through a pseudolinearization argument near a stationary solution $\widehat{w}$:

Theorem 8.

Theorem 9. [30]*Assume $(H_1) - (H_7)$. Then there exists $\alpha > 0$, $\beta > 0$ and $M \geq 1$ such that if $u_0 \in B_\beta^X(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ then the solution $u(\cdot : u_0)$ of (58) exists on $[-\tau, +\infty)$ and*

$$|u(t : u_0) - w| \leq Me^{-\alpha t}\left\|u_0 - \widehat{w}\right\|, \text{ for any } t > 0.$$

Moreover, there exists $\alpha^* > 0$, $\beta^* \in (0, \beta]$ and $M^* \geq 1$ such that if $u_0 \in B_{\beta^*}^{X\cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ then, for any $t > 0$,

$$|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \leq M^* e^{-\alpha^* t}(\left\|u_0 - \widehat{w}\right\|_X + \left\|u_0 - \widehat{w}\right\|_Y).$$

We can get better a priori estimates on the sup norm of the solution $\mathbf{u}$ if we assume more regular initial data in such a way that $u_0 \in B_{\beta^*}^{X\cap Y}(\widehat{w})$, $u_0(s) \in D(A) \cap D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$. Indeed, the solution can be found (after technical arguments) as a fixed point for the application $f \to Q_1(Q_2(f))$, with $w = Q_2 f$ (for $f \in W^{1,1}(0, T : X)$, for any arbitrary $T > 0$) being the solution of the problem

$$\begin{cases} \frac{dw}{dt}(t) + Aw(t) + B(w(t)) = f(t) & \text{in } X, \\ w(0) = w_0, \end{cases}$$

and Q_1 a suitable operator (see [66], Theorem 5.3.1). Since X is a reflexive Banach space, we know (see, e.g., [20], Lemma 7.8) that $w_0 \in D(A) \cap D_X(B)$ implies that $w(t) \in D(A) \cap D_X(B)$ for a.e. $t \in (0, T)$ and that

$$\left\|Aw(t)\right\|_X \leq C(\left\|Aw_0\right\|_X + \left\|B(w_0)\right\|_X, \left\|f\right\|_{W^{1,1}(0,T:X)}).$$

Thus, by the Sobolev imbedding theorems we know that

$$\|w(t)\|_{\mathbf{C}(\overline{\Omega})} \leq M$$

for a.e. $t \in (0,T)$ with $M = M(\|Aw_0\|_X + \|B(w_0)\|_X, \|f\|_{W^{1,1}(0,T:X)})$. In particular, this property remains true for the fixed point of $Q_1(Q_2(f))$ (see [66], Theorem 5.3.1) and thus

$$\|\mathbf{u}(t)\|_{\mathbf{C}(\overline{\Omega})} \leq M^*$$

for a suitable $M^* = M * (\|Au_0\|_{C([-\tau,0];X)} + \|B(w_0)\|_{C([-\tau,0];X)}, F)$. In consequence, without any loss of generality we can replace function $\mathbf{g}$ by the truncated one $\mathbf{g}_{M^*}(\mathbf{u})$:

$$\mathbf{g}_{M^*}(\mathbf{u}) = \begin{cases} -(1+i\alpha)\left|\mathbf{u}\right|^2 \mathbf{u} & \text{if } |\mathbf{u}| \leq M^*, \\ -2(1+i\alpha)\left(2M^*\right)^2 \mathbf{u} & \text{if } |\mathbf{u}| \geq M^*, \end{cases}$$

and with $\mathbf{g}_{M^*}(\mathbf{u})$ a C^k-smooth function generating an accretive operator $\mathbf{B}_{M^*} = -\mathbf{g}_{M^*}$ on X dominated by A as before. This proves that, at least for regular initial data, $\mathbf{u}$ coincides with the solution of

$$\begin{cases} \frac{du}{dt}(t) + Au(t) = L(\mu, u_t(.)) + g_{M^*}(u_t(.)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0]. \end{cases}$$

Thanks to this argument we can verify now the assumption (H_5) since by the results of Ambrosetti and Prodi (see [7], Sect. 3, Chap. 1) we know that the Nemitsky operator associated to g_{M^*} has k-th-continuous Fréchet derivatives on any $\mathbf{L}^p(\Omega)$, $p > 1$.

Remark 19. *By introducing the representation operator* $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$ *it is clear that the quasilinear operator* $A\mathbf{P}(\mathbf{q})$ *obtained from the operator* $A\mathbf{u}$=-$(1 + i\beta)\Delta\mathbf{u}$ *satisfies also condition* $A \in \mathcal{A}(\omega)$ *(since* $\mathbf{P}$ *is merely a change of variables). We point out that*

$$A\mathbf{P}(\mathbf{q}) = -(1+i\beta)[\Delta\rho - \rho\left|\nabla\phi\right|^2 + i(2\nabla\rho\cdot\nabla\phi + \rho\Delta\phi)]e^{i\phi}.$$

Then, the formal linearization of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ *at* $\mathbf{q}^*(x,y) := \mathbf{y} \equiv \rho_0$ *becomes*

$$D\mathbf{E}(\mathbf{q}^*)(\rho e^{i\phi}) = -(1+i\beta)[\Delta\rho + i\rho_0\Delta\phi]e^{i\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ *needs a slight modification of the above linear expression. Nevertheless by applying the representation operator* $\mathbf{P}$*, after the linearization used in the abstract theorem, we get a curious result relating two nonlinear problems which are closed (in some sense) in the same spirit as the* pseudo-linearization principle *obtained in* [30].

4.3.3. *Some comments on the associated transversality assumption.* Concerning problem (P_2), we give an outline of the study of eigenvalues and its implications on the associated transversality condition. The eigenvalue equation can be obtained by a linearization argument involving the Fréchet derivative of the nonlinear part, as in the preceding section.

As usual, the linear structure of the equation leads to the search of nontrivial solutions $\mathbf{z}(x)$ of the form $\mathbf{A_k} w^j_{\mathbf{k}}(x)$, with $j = 1, 2$, where $w^j_{\mathbf{k}}(x)$ are the eigenfunctions for the usual Laplacian operator Δ with periodic boundary conditions on $\Omega = (0, L_1) \times (0, L_2)$. The eigenvalues of this problem are given by $\lambda^0_0 = 0$, $\lambda^0_{\mathbf{k}} = 4\pi\left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2}\right)$; $k_1, k_2 \in \mathbb{N}$ with the associate eigenfunctions

$$w_0 = \frac{1}{\sqrt{|\Omega|}}, w^1_{\mathbf{k}} = \sqrt{\frac{2}{|\Omega|}}\cos 2\pi\mathbf{kx}, w^2_{\mathbf{k}} = \sqrt{\frac{2}{|\Omega|}}\sin 2\pi\mathbf{kx}, \text{ with } |\Omega| = L_1L_2,$$

where we have written $\mathbf{kx} := \left(\frac{k_1}{L_1}x_1 + \frac{k_2}{L_2}x_2\right)$. This study can be found in Temam [65]. We introduce the notation $\lambda_{\mathbf{k}} = a_{\mathbf{k}} + ib_{\mathbf{k}}$ for the real and imaginary parts of the eigenvalues of the problem, and taking into account Fréchet derivative of the nonlinear part (59), the eigenvalue equations for the problem (P_2) are

$$\begin{cases} (a_{\mathbf{k}} + ib_{\mathbf{k}})[v_r + iv_i] - (1+i\beta)(-\lambda_{\mathbf{k}})[v_r + iv_i] = \\ (1+i\theta)[v_r + iv_i] - (1+i\alpha)[3\rho_0^2 v_r + i\rho_0^2 v_i] + \\ \mu e^{i\xi}\left[m_1 + m_2\delta_{0\mathbf{k}} + e^{-a\tau + i(\omega+\theta-b)\tau}(m_3 + m_4\delta_{0\mathbf{k}})\right][v_r + iv_i], \end{cases}$$

where v_r and v_i are the real and imaginary parts of the linearization $\mathbf{v}$, and $\delta_{0\mathbf{k}}$ denotes the Kronecker delta function. We arrive at

$$\begin{cases} a_{\mathbf{k}} v_r - b_{\mathbf{k}} v_i = & -\lambda^0_{\mathbf{k}} v_r + \beta\lambda^0_{\mathbf{k}} v_i + \left(\left[1-3\rho_0^2\right] v_r + \left[\alpha\rho_0^2 - \theta\right] v_i\right) + \\ & \mu(m_1 + m_2\delta_{0k})\left[v_r\cos\xi - v_i \sin\xi\right] + \{\mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0k}) \\ & [\cos(\xi + (\omega+\theta-b_{\mathbf{k}})\tau)v_r - \sin(\xi + (\omega+\theta-b_{\mathbf{k}})\tau)v_i]\}, \\ b_{\mathbf{k}} v_r + a_{\mathbf{k}} v_i = & -\beta\lambda^0_{\mathbf{k}} v_r + \lambda^0_{\mathbf{k}} v_i + (v_i + \theta v_r) - \left[\rho_0^2 v_i - 3\alpha\rho_0^2 v_r\right] + \\ & \mu(m_1 + m_2\delta_{0k})\left[v_r\sin\xi + v_i \cos\xi\right] + \{\mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0k}) \\ & [\sin(\xi + (\omega+\theta-b_{\mathbf{k}})\tau)v_r + \cos(\xi + (\omega+\theta-b_{\mathbf{k}})\tau)v_i]\} \end{cases}$$

To show the procedure, without loss of generality, we consider the case

$$m_3 + m_4\delta_{0\mathbf{k}} = 0. \tag{60}$$

This represents a special, and important, choice of the combination of instantaneous and delayed terms in the global feedback, none of them necessarily zero. The equations for the eigenvalues become

$$\begin{cases} a_{\mathbf{k}} v_r - b_{\mathbf{k}} v_i = & -\lambda^0_{\mathbf{k}} v_r + \beta\lambda^0_{\mathbf{k}} v_i + \left(\left[1-3\rho_0^2\right] v_r + \left[\alpha\rho_0^2 - \theta\right] v_i\right) + \\ & \mu(m_1{+}m_2\delta_{0k})\cos\xi v_r - \mu(m_1{+}m_2\delta_{0k})\sin\xi v_i \\ b_{\mathbf{k}} v_r + a_{\mathbf{k}} v_i = & -\beta\lambda^0_{\mathbf{k}} v_r + \lambda^0_{\mathbf{k}} v_i + (v_i + \theta v_r) - \left[\rho_0^2 v_i - 3\alpha\rho_0^2 v_r\right] + \\ & \mu(m_1{+}m_2\delta_{0k})\sin\xi v_r + \mu(m_1{+}m_2\delta_{0k})\cos\xi v_i \end{cases}$$

If we call

$$\begin{aligned} C_1\left(\mu, m_1, m_2, \xi, \lambda^0_{\mathbf{k}}\right) &= 1 - \lambda^0_{\mathbf{k}} - \mu(m_1 + m_2\delta_{0k})\cos\xi, \\ C_2\left(\mu, m_1, m_2, \xi, \lambda^0_{\mathbf{k}}\right) &= 1 + \lambda^0_{\mathbf{k}} + \mu(m_1 + m_2\delta_{0k})\cos\xi, \\ D\left(\beta, \mu, m_1, m_2, \xi, \lambda^0_{\mathbf{k}}\right) &= -\beta\lambda^0_{\mathbf{k}} + \mu(m_1 + m_2\delta_{0k})\sin\xi, \end{aligned}$$

we obtain

$$\begin{cases} \left(a_{\mathbf{k}} - \left[C_1 - 3\rho_0^2\right]\right) v_r - \left(b_{\mathbf{k}} + \left[\alpha\rho_0^2 - \theta - D\right]\right) v_i = 0 \\ \left(b_{\mathbf{k}} - \left[-3\alpha\rho_0^2 + \theta + D\right]\right) v_r + \left(a_{\mathbf{k}} - \left[C_2 - \rho_0^2\right]\right) v_i = 0 \end{cases}$$

The compatibility of this system implies

$$\det\begin{pmatrix} a_{\mathbf{k}} - \left[C_1 - 3\rho_0^2\right] & -b_{\mathbf{k}} - \left[\alpha\rho_0^2 - \theta - D\right] \\ b_{\mathbf{k}} - \left[-3\alpha\rho_0^2 + \theta + D\right] & a_{\mathbf{k}} - \left[C_2 - \rho_0^2\right] \end{pmatrix} = 0,$$

that is

$$\begin{cases} \left(a_{\mathbf{k}} - \left[C_1 - 3\rho_0^2\right]\right)\left(a_{\mathbf{k}} - \left[C_2 - \rho_0^2\right]\right) = \\ \left(b_{\mathbf{k}} - \left[-3\alpha\rho_0^2 + \theta + D\right]\right)\left(b_{\mathbf{k}} + \left[\alpha\rho_0^2 - \theta - D\right]\right). \end{cases} \tag{61}$$

This expression is of the same type as (56) and, similarly, there is no general analytic solution for $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$. Thus, Eq. (61) must also be solved numerically for a given set of parameters, to find the numerical values of the eigenvalues as in the equation (56). One of the relevant parameter spaces of the representation is the one of (τ, μ) because they are the parameters of the perturbation.

Although the explicit analytical representation of the functions $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ is not possible, we can still say something analytic in the study of the transversality, already proved by the above numerical computation. From the equation (61), it is possible to find the implicit derivative

$$\left[\frac{d}{d\tau} a_{\mathbf{k}}\right]_{a_{\mathbf{k}}=0}.$$

The analytic computation are rather involved. We show how to proceed in a simpler, and still very important example

$$m_1{+}m_2\delta_{0\mathbf{k}} = 0, \tag{62}$$

where a remark similar as the one made for the expression (60) remains valid, in this case for the local part of the perturbation. For the case (62), we have

$$\begin{aligned} C_1\left(\mu, m_1, m_2, \xi, \lambda^0_{\mathbf{k}}\right) &= 1 - \lambda^0_{\mathbf{k}}, \\ C_2\left(\mu, m_1, m_2, \xi, \lambda^0_{\mathbf{k}}\right) &= 1 + \lambda^0_{\mathbf{k}}, \\ D\left(\beta, \mu, m_1, m_2, \xi, \lambda^0_{\mathbf{k}}\right) &= -\beta\lambda^0_{\mathbf{k}}. \end{aligned}$$

If we expand Eq. (61) for this case,

$$\left\{\begin{array}{l} \quad a_{\mathbf{k}}^2 - 2\left[1-2\rho_0^2\right] a_{\mathbf{k}} + \left(\left[1-\lambda_{\mathbf{k}}^0 - 3\rho_0^2\right]\left[1+\lambda_{\mathbf{k}}^0-\rho_0^2\right]\right) = \\ -b_{\mathbf{k}}^2 + 2\left[-\beta\lambda_{\mathbf{k}}^0 + \alpha\rho_0^2+\theta\right] b_{\mathbf{k}} + \left(\left[-\beta\lambda_{\mathbf{k}}^0 + 3\alpha\rho_0^2+\theta\right]\left[+\beta\lambda_{\mathbf{k}}^0+\alpha\rho_0^2-\theta\right]\right), \end{array}\right.$$

and differentiate implicitly

$$\left\{\begin{array}{l} \quad 2a_{\mathbf{k}}\frac{d}{d\tau}a_{\mathbf{k}} - 2\left[1-2\rho_0^2\right]\frac{d}{d\tau}a_{\mathbf{k}} - a_{\mathbf{k}}\frac{d}{d\tau}\left(2\left[1-2\rho_0^2\right]\right) + \\ \frac{d}{d\tau}\left(1-\left(\lambda_{\mathbf{k}}^0\right)^2 - 2\left[2+\lambda_{\mathbf{k}}^0\right]\rho_0^2 + 3\rho_0^4\right) = \\ \quad -2b_{\mathbf{k}}\frac{d}{d\tau}b_{\mathbf{k}} + 2\left[-\beta\lambda_{\mathbf{k}}^0+\alpha\rho_0^2+\theta\right]\frac{d}{d\tau}b_{\mathbf{k}} - b_{\mathbf{k}}\frac{d}{d\tau}\left(2\left[-\beta\lambda_{\mathbf{k}}^0+\alpha\rho_0^2+\theta\right]\right) + \\ \frac{d}{d\tau}\left(\left[-\beta\lambda_{\mathbf{k}}^0+3\alpha\rho_0^2+\theta\right]\left[+\beta\lambda_{\mathbf{k}}^0+\alpha\rho_0^2-\theta\right]\right). \end{array}\right.$$

The derivative of the real part $a_{\mathbf{k}}$ in the value $a_{\mathbf{k}}=0$ can be written as

$$\left\{\begin{array}{l} \left[-2(1-2\rho_0^2)\frac{d}{d\tau}a_{\mathbf{k}}\right]_{a_{\mathbf{k}}=0} = \\ \left[-\frac{d}{d\tau}\left(1-\left(\lambda_{\mathbf{k}}^0\right)^2 - 2\left[2+\lambda_{\mathbf{k}}^0\right]\rho_0^2+3\rho_0^4\right)\right]_{a_{\mathbf{k}}=0} \\ +2\left[-b_{\mathbf{k}}\frac{d}{d\tau}b_{\mathbf{k}} + \left[-\beta\lambda_{\mathbf{k}}^0+\alpha\rho_0^2+\theta\right]\frac{d}{d\tau}b_{\mathbf{k}} - b_{\mathbf{k}}\frac{d}{d\tau}\left(\left[-\beta\lambda_{\mathbf{k}}^0+\alpha\rho_0^2+\theta\right]\right)\right]_{a_{\mathbf{k}}=0} \\ +\left[\frac{d}{d\tau}\left(\left[-\beta\lambda_{\mathbf{k}}^0+3\alpha\rho_0^2+\theta\right]\left[+\beta\lambda_{\mathbf{k}}^0+\alpha\rho_0^2-\theta\right]\right)\right]_{a_{\mathbf{k}}=0}. \end{array}\right.$$

The coefficient of the derivative of $a_{\mathbf{k}}$,

$$-2(1-2\rho_0^2) = -2\left[1-2(1+\mu\cos\xi)\right] = 2(1+2\mu\cos\xi)$$

does not vanish either for stability reasons as can be seen, e.g., in [30] and references therein.

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