ON THE DEGENERATE THERMISTOR PROBLEM[1]

M. Chipot(1)**, J. I. Díaz**(2)**and R. Kersner**(3)
(1) Institut für Mathematik, Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich, (Switzerland).
chipot@amath.unizh.ch
(2) Departamento de Matematica Aplicada,
Universidad Complutense de Madrid, 28040 Madrid (Spain).
ildefonso_diaz@mat.ucm.es
(3) Computer and Automation Institute,
Hungarian Academy of Sciences, H-1502 Budapest,
P.O. Box 63 Kende U. 13-17 (Hungary).
kersner@sztaki.hu

1 Introduction

This is a preliminar presentation of a set of results by the authors ([12]) on the transient thermistor problem. We study here the following one-dimensional thermistor formulation: given the interval $(-L, +L)$, which we denote as Ω, we consider the system

$$\begin{cases} u_t - (k(u)u_x)_x = \sigma(u)(v_x)^2 & \text{in} \quad \Omega \times (0,T), \\ (\sigma(u)v_x)_x = 0 & \text{in} \quad \Omega \times (0,T), \\ v = v_D,\, u = u_D \geq 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \sigma(u)\frac{\partial v}{\partial n} = 0,\, k(u)\frac{\partial u}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{on} \quad \Omega. \end{cases} \tag{1}$$

Here n is the outward unit normal vector and Γ_D, Γ_N are subsets of $\partial\Omega$ with $\Gamma_D \cup \Gamma_N = \partial\Omega$, the possibility $\Gamma_D = \phi$ (the empty set), or $\Gamma_N = \phi$, being not excluded. Obviously, we assume that $\Gamma_D \cap \Gamma_N = \phi$. This problem models the diffusion of heat produced by Joule's effect in a one dimensional conductor (see for instance [28], [14]): u is the temperature, $k(u)$ the thermal conductivity of the medium, v is the inside potential and $\sigma(u)$ the electric conductivity which (as k as well) is supposed to depend on the temperature.

We point out that the physically important case of the metallic conduction $k(u)$ is given by the Wiedemann-Franz law $k(u) = k_0 u\sigma(u)$, where k_0 is a positive constant and so the temperature equation becomes degenerate where $u = 0$. In spite of its relevance in the applications, to the best of our knowledge, the study of the case where the parabolic equation is degenerate is not completely well-known in our days. Indeed: several authors considered the case in which electric conductivity $\sigma(u)$ degenerate (becoming identically zero for $u \geq u^*$, for some $u^* > 0$) but always under the assumption $k(u) > 0$ (see [8] and [9], [10], [35], [37], [39]). Other authors considered the case in which there is a change of phase in the temperature and then $k(u)$ is taken as in the usual Stefan problem (see [36], [40], [33], [41] and [20]). The case in which the thermal conductivity k depends (even in a degenerate way) of the gradient of the temperature $k = k(\nabla u)$ was considered in [38]). The non-degenerate problem is studied in [1], [2]. For further reading see [16], [24], [25], [29] and [34]. In spite the long period since our research was started (some of the present results were already mentioned in the papers [2] and [1]), concerning the case $k(0) = 0$ we merely are aware of the existence of solutions proved in [22], [23] (in both cases for $\sigma(u) > 0$). Our treatment will include the case in which $\sigma(u)$ may degenerate at $u = 0$. We mention the lack of results on the uniqueness of solutions for the case $k(0) = 0$ and that, even under nondegeneracy assumption on k, very few is said about the uniqueness of solutions in the above list of works.

If we set

$$\varphi(s) = \int_0^s k(\tau)d\tau$$

problem (1) reads

$$\begin{cases} u_t - \varphi(u)_{xx} = \sigma(u)(v_x)^2 & \text{in} \quad \Omega \times (0,T), \\ (\sigma(u)v_x)_x = 0 & \text{in} \quad \Omega \times (0,T), \\ v = v_D,\, \varphi(u) = \varphi(u_D) \geq 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \sigma(u)\frac{\partial v}{\partial n} = 0,\, \frac{\partial \varphi(u)}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{on} \quad \Omega. \end{cases} \tag{2}$$

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The main goal of our results is to prove that problem (2) has one and only one weak solution under suitable conditions in the degenerate case. It is well known that for such a problem a classical solution does not exist in general. Notice that the case which interests us here is when k is allowed to vanish at 0 and that the set where $u = 0$ may be non empty according the assumptions made on u_D and/or on $u_0(x)$. Notice also that besides the possible degeneracy of φ, an interesting feature of our system is also the quadratic growth of its second right hand side.

We will set $Q = \Omega \times (0, T)$, moreover we will assume

$$\sigma \text{ is Lipschitz continuous,} \tag{3}$$

$$\begin{cases} \text{there exists a bounded increasing function } \sigma_0(u), \text{ with } \sigma_0(0) \geq 0 \text{ and } \sigma_1 > 0 \\ \text{such that } \sigma_0(u) \leq \sigma(u) < \sigma_1 \quad \forall\, u \geq 0, \end{cases} \tag{4}$$

$$\varphi \in C^1([0, +\infty)) \cap C^2((0, +\infty)), \tag{5}$$

$$\varphi'(0) \geq 0, \qquad \varphi'(r) > 0 \quad \forall\, r > 0, \tag{6}$$

$$\begin{cases} \text{there exists } V_D \in L^\infty((0, T); H^1(\Omega)) \text{ such that} \\ V_D = v_D \text{ on } \Gamma_D \times (0, T) \text{ and } \frac{\partial V_D}{\partial n} = 0 \text{ on } \Gamma_N \times (0, T), \end{cases} \tag{7}$$

$$\begin{cases} \text{there exists } U_D \text{ such that } \varphi(U_D) \in H^1((0, T); H^1(\Omega)) \\ \varphi(U_D) = \varphi(u_D) \geq 0 \text{ on } \Gamma_D \times (0, T) \text{ and } \frac{\partial \varphi(U_D)}{\partial n} = 0 \text{ on } \Gamma_N \times (0, T), \end{cases} \tag{8}$$

$$u_0 \in L^\infty(\Omega),\ 0 \leq u_0 \leq M, \tag{9}$$

where M is some positive constant. We point out that the case in which the constant σ_1 is replaced by an increasing function $\sigma_1(u) \geq u^m$ for some $m > 1$ was treated in [1] and [2] where it was proved the blow up of the solution in a finite time. Notice also that the Wiedemann-Franz law and the assumption (4) imply that

$$k_0 \int_0^u \sigma_0(s) s ds \leq \varphi(u) < \widetilde{C} u^2 \qquad \forall\, u \geq 0, \tag{10}$$

with $\widetilde{C} = \frac{k_0 \sigma_1}{2}$.

Our existence result will require the following additional condition:

$$\begin{cases} \sigma_0(0) > 0 \\ \text{or} \\ \varphi(u)^\alpha \leq \sigma_0(u) \text{ for any } u \in [0, \delta], \text{ for some } \alpha \in (0, 1),\, \delta > 0 \text{ and,} \\ \text{if } \Gamma_N \neq \phi \text{ then } \{t \in [0, T] : V_D(L, t) = V_D(-L, t)\} = (\cup_i^N I_i) \cup I \text{ with} \\ N \in \mathbb{N} \cup \{0\},\ I_i \text{ subinterval of } [0, T] \text{ and } I \text{ a zero measure subset of } [0, T]. \end{cases} \tag{11}$$

Notice that the great generality allowed on $\sigma(u)$ requires to say some words on the way the boundary conditions are satisfied. We shall show that $\sigma(u)v_x \in L^\infty(Q)$ and that $\varphi(u(.,t))$ is continuous. Then the assumption

$$\sigma(u_D(x, t)) > 0 \text{ on } \Gamma_D \times [0, T] \tag{12}$$

implies that the trace of v on $\Gamma_D \times (0, T)$ is well defined. It turns out that a function which plays a crucial role in the study of the system is the function

$$J := \sigma(u) v_x,$$

which corresponds to the current density. Notice that the second equation of (1) implies that J is independent of x, i.e., for a.e. $t \in (0, T)$

$$\sigma(u(x, t)) v_x(x, t) = J(t) \text{ for a.e. } x \in \Omega. \tag{13}$$

Since the first equation can be, equivalently written as

$$u_t - \varphi(u)_{xx} = J v_x \quad \text{in} \quad \Omega \times (0, T),$$

if $J(t) \equiv 0$ on some subinterval $(t_1, t_2) \subset (0, T)$ then the equations of system (1) are not coupled on $\Omega \times (t_1, t_2)$. Notice also that $J(t) \equiv 0$ on $(0, T)$ if $inf_{x \in \Omega}\ |v_x(x, t)| = 0$ (case, for instance, of

$\Gamma_N \neq \phi$) or $\min_{x\in\overline{\Omega}} \sigma(u(x,t)) = 0$ (case, for instance, of $\Gamma_D \neq \phi$, $\sigma_0(0) = 0$ and $u_D(t,x) = 0$). Moreover, if $\min_{x\in\overline{\Omega}} \sigma(u(x,t)) > 0$ we have

$$v_x(x,t) = \frac{J(t)}{\sigma(u(x,t))} \text{ a.e. } x \in \Omega.$$

Then, a simple integration shows that, for a.e. $t \in (0,T)$

$$v(L,t) - v(-L,t) = J(t) \int_\Omega \frac{dx}{\sigma(u(x,t))}, \tag{14}$$

which will play an important role in our proof of the existence of solutions and also can be understood as a weak sense in which the Dirichlet condition holds (notice that if $J(t) \equiv 0$ and, both, $\Gamma_N \neq \phi$ and $\Gamma_D \neq \phi$ then, necessarily, $v(x,t) = v_D(x,t)$ on $\Gamma_N \times (0,T)$). Notice also that if $J(t) = 0$ and $\sigma(u(x,t)) > 0$ for any $x \in \Omega$, we get that $v_x \equiv 0$. Finally, if $\Gamma_N = \phi$ as $\int_\Omega \frac{dx}{\sigma(u(x,t))} > 0$ for a.e. $t \in (0,T)$,we get from (14) that $J(t) = 0$ (respectively $J(t) > 0$ or $J(t) < 0$) if and only if $v_D(L,t) - v_D(-L,t) = 0$ (respectively $v_D(L,t) - v_D(-L,t) > 0$ or $v_D(L,t) - v_D(-L,t) < 0$).

The uniqueness of a weak solution will be obtained for the cases in which $\Gamma_D = \phi$ or $u_D(t,x) > 0$ on $\Gamma_D \times (0,T)$ (notice that the possible vanishing of u_0 maintains the degenerate character to the parabolic equation). We point out that the uniqueness technique of this paper applies to more general formulations and, in particular, to N-dimensional problems (see Remark 3).

The last section of the paper is devoted to the study of a qualitative property which is peculiar to the case $\varphi'(0) = 0$ (and a suitable growth condition which is satisfied when (10) holds). It concerns with the occurrence of a *free boundary* (given by the boundary of the support of u). When $\sigma_0(0) > 0$ the vanishing set of the solution can be reduced to some curves in Q. Nevertheless, if $\sigma_0(0) = 0$, we show the, so called, *finite speed of propagations* property: if $u_0(x) = 0$ on $B_{\rho_0}(x_0) := (x_0 - \rho_0, x_0 + \rho_0)$ for some $x_0 \in \Omega$ and $\rho_0 > 0$ then there exists $t^* > 0$ and a function $\rho(t) : [0,t^*) \mapsto [0,\infty)$, with $\rho(0) \leq \rho_0$, such that $u(x,t) = 0$ a.e. in $B_{\rho(t)}(x_0), \forall\, t \in [0,t^*)$. We shall prove that this property holds under the assumption (10). This result opens the possibility of further studies on the properties (and regularity) on this free boundary.

2 Existence of a weak solution

Definition 1 *Assumed (12), by a weak solution to problem (2) we mean a couple of functions (u,v) such that*

$$\varphi(u) \in L^2(0,T;H^1(\Omega)), u \geq 0, u \in C([0,T];L^1(\Omega)) \cap L^\infty(Q), \tag{15}$$

$$v \in L^\infty(Q), \tag{16}$$

$$\sigma(u)v_x \in L^1(0,T;L^1(\Omega)), \sigma(u)\left|v_x\right|^2 \in L^1(0,T;L^1(\Omega)), \tag{17}$$

the boundary conditions $v = v_D$, $\varphi(u) = \varphi(u_D)$ and $\sigma(u)\frac{\partial v}{\partial n} = 0, \frac{\partial \varphi(u)}{\partial n} = 0$ hold on $\Gamma_D \times (0,T)$ and $\Gamma_N \times (0,T)$ respectively, $u(.,0) = u_0$ in $L^1(\Omega)$ and

$$\begin{aligned} \int_\Omega u(x,T)\xi(x,T)dx - \int_\Omega u_0(x)\ \xi(x,0)dx = \int_0^T \int_\Omega u\ \xi_t dtdx \\ - \int_0^T \int_\Omega \varphi(u)_x \xi_x dtdx - \int_0^T \int_\Omega \sigma(u)\left|v_x\right|^2 \xi dtdx, \end{aligned} \tag{18}$$

$$\int_\Omega \sigma(u)v_x\zeta_x dx = 0 \quad a.e.\ t \in (0,T), \tag{19}$$

for all $\xi, \zeta \in C^1(\overline{Q})$ such that $\xi(x,t), \zeta(x,t) = 0$ on $\Gamma_D \times (0,T)$.

The following existence result is proved in an entirely different way than for the existence results of [22], [23]. The special approximation procedure will be used in our proof of the uniqueness of solution of next section.

Theorem 1. *Under the assumptions (11) and (12) there exists, at least, one weak solution to the problem (2). Moreover, $J(t){:=}\sigma(u(x,t))v_x(x,t)$ is a bounded (constant in x) function on $(0,T)$ and if $\sigma_0(0) > 0$ then $v_x \in L^\infty(Q)$.*

Proof. By assumption (11), without lost of generality we can assume that $\sigma_0(0) > 0$ or

$$v_D(L,t) \neq v_D(-L,t) \text{ a.e. } \; t \in (0,T). \tag{20}$$

Indeed, let us assume that $v(L,t) = v(-L,t)$ (and so $J(t) \equiv 0$) on some subinterval, for instance, $I_1 = [0,t_2) \subset (0,T)$ and that $J(t) \neq 0$ on some other interval $(t_2,t_3) \subset (0,T)$ (other possibilities are treated in a similar way). Then, since the equations of system are not coupled on $\Omega \times (0,t_2)$, we can take $v(t,x) \equiv v(L,t)$ on $\Omega \times (0,t_2)$ and u as the unique solution of the homogeneous porous medium equation

$$\begin{cases} w_t - \varphi(w)_{xx} = 0 & \text{in} \quad \Omega \times (0,t_2), \\ \varphi(w) = \varphi(u_D) \geq 0 & \text{on} \quad \Gamma_D \times (0,t_2), \\ w(x,0) = u_0(x) \geq 0 & \text{on} \quad \Omega. \end{cases} \tag{21}$$

Then, we are reduced to the conditions announced in (20) on the interval (t_2,t_3) with the new initial condition for u given by $w(x,t_2)$. Notice that the same arises when $\Gamma_N = \phi$, nevertheless we shall not assume this condition in the rest of the proof in order to recall an approximation argument which will be used in the proof of the uniqueness of solutions.

The process of proof under condition (20) consists in three different steps.

Step 1: Approximation of u by a sequence $u_\epsilon > 0$. The method consists in approximating the solution (u,v) by (u^ϵ, v^ϵ) the solution of the problem

$$\begin{cases} u^\epsilon_t - \varphi(u^\epsilon)_{xx} = \sigma(u^\epsilon)(v^\epsilon_x)^2 & \text{in} \quad \Omega \times (0,T), \\ (\sigma(u^\epsilon)v^\epsilon_x)_x = 0 & \text{in} \quad \Omega \times (0,T), \\ v^\epsilon = v_D, \, \varphi(u^\epsilon) = \varphi(\max(u_D,\epsilon)) & \text{on} \quad \Gamma_D \times (0,T), \\ \frac{\partial v^\epsilon}{\partial n} = 0, \, \frac{\partial \varphi(u^\epsilon)}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ u^\epsilon(.,0) = u_0 + \epsilon & \text{on} \quad \Omega. \end{cases} \tag{22}$$

As $u^\epsilon_t - \varphi(u^\epsilon)_{xx} \geq 0$, and from the assumptions (8) and (9) we deduce, from the maximum principle, that any solution u^ϵ must verify

$$u^\epsilon \geq \epsilon \text{ a.e. on } \Omega \times (0,T). \tag{23}$$

Thus, $\varphi'(u^\epsilon) > 0$, the operator is, now, uniformly parabolic and so a solution (u^ϵ, v^ϵ) to problem (22) is known to exist (see, e.g., [14]). Moreover, this solution is smooth provided that our data are smooth.

Step 2: A priori estimates. First, let us show that $\sigma(u^\epsilon)(v^\epsilon_x)^2$, u^ϵ and $\sigma(u)v_x$ are bounded, independently of ϵ, in $L^1(Q)$, $L^\infty(Q)$ and $L^\infty(Q)$, respectively. For that, multiply the equation of v^ϵ in problem (22) by $v^\epsilon - V_D$ and integrate by parts. We get :

$$\int_\Omega \sigma(u^\epsilon)v^\epsilon_x(v^\epsilon - V_D)_x dx = 0 \text{ a.e. } \; t \in (0,T).$$

Hence,

$$\int_\Omega \sigma(u^\epsilon)(v^\epsilon_x)^2 dx \leq \int_\Omega \sqrt{\sigma(u^\epsilon)}\,|v^\epsilon_x|\,\sqrt{\sigma(u^\epsilon)}\,|(V_D)_x|\,dx.$$

Applying the Cauchy-Schwarz inequality, we get

$$\int_\Omega \sigma(u^\epsilon)(v^\epsilon_x)^2 dx \leq C(T) \tag{24}$$

thanks to (7). Now, since $\sigma(u^\epsilon)(v^\epsilon_x)^2$ is bounded in $L^\infty(0,T;L^1(\Omega))$ we get that $u^\epsilon_t - \varphi(u^\epsilon)_{xx} = f^\epsilon(t,x)$ with f^ϵ uniformly bounded in $L^\infty(0,T;L^1(\Omega))$ and so we have from [27] that u^ϵ is uniformly bounded in $L^\infty(Q)$. On the other hand, from the equation of v^ϵ in problem (22) we have

$$\sigma(u^\epsilon)v^\epsilon_x = J^\epsilon(t)$$

where $J^\epsilon(t)$ is a continuous function of t and hence

$$\frac{|J^\epsilon(t)|}{\sigma_1} \leq |v^\epsilon_x| \tag{25}$$

Plugging this into (24) we obtain

$$|J^\epsilon(t)|^2 \leq C(T)\sigma_1^2 ess \sup_{t\in[0,T]} \left(\frac{1}{\int_\Omega \sigma(u^\epsilon(x,t))dx} \right) \tag{26}$$

where $C(T)$ denotes some constant independent of ϵ. We shall prove later, in the third step of the proof that this implies that

$$|J^\epsilon(t)| \leq C^*(T) \tag{27}$$

for some positive constant independent of ϵ.
Notice that if $\sigma_0(0) > 0$ then v_x^ϵ is bounded in $L^\infty(Q)$ independently of ϵ. Indeed, in that case

$$\int_\Omega \sigma_0(v_x^\epsilon)^2 dx \leq \int_\Omega \sigma(u^\epsilon)(v_x^\epsilon)^2 dx \leq \int_\Omega \sigma_1 v_x^\epsilon (V_D)_x dx.$$

Applying the Cauchy-Schwarz inequality, we get now that

$$\int_\Omega (v_x^\epsilon)^2 dx \leq C(T). \tag{28}$$

From the equation of v^ϵ in problem (22) we deduce that

$$\frac{|J^\epsilon(t)|}{\sigma_1} \leq |v_x^\epsilon| \leq \frac{|J^\epsilon(t)|}{\sigma_0(0)} \tag{29}$$

Plugging this into (28) we obtain

$$|J^\epsilon(t)|^2 \leq C(T)\frac{\sigma_1^2}{|\Omega|} \tag{30}$$

where $C(T)$ denotes some constant independent of ϵ and $|\Omega|$ is the measure of Ω. The $L^\infty(Q)$-estimate follows then by (29). Notice also that, in fact, for general functions $\sigma_0(u)$ (with $\sigma_0(0) = 0$) if we consider the subset

$$[\delta < u^\varepsilon] := \{(x,t) \in Q : \delta < u^\varepsilon(x,t)\},$$

for some $\delta > 0$ then we also have that

$$|v_x^\epsilon(x,t)| \leq \frac{|C_1(T)|}{\sigma_0(\delta)} \text{ for any } (x,t) \in [\delta < u^\varepsilon],$$

for some $C_1(T)$.
It is easy to get a $L^\infty(Q)$ a priori estimate on v^ϵ since, if $\Gamma_D \neq \phi$ then, from the maximum principle,

$$|v^\epsilon(x,t)| \leq \|v_D\|_{L^\infty(Q)}, \text{ for a.e. } x \in \Omega \text{ and any } t \in [0,T].$$

In the case $\Gamma_D = \phi$ the function $v_x^\epsilon(x,t) = 0$ and since v^ϵ is determined up a constant we can take $v^\epsilon(x,t) = 0$ for a.e. $x \in \Omega$ and any $t \in [0,T]$.
We also point out that if $\sigma_0(0) > 0$ we get that $u_t^\epsilon - \varphi(u^\epsilon)_{xx} = f(t,x)$ with $f \in L^\infty(Q)$ and so we have, alternatively to the result of [27] (by the maximum principle and using the assumptions (8) and (9) and that $H^1((0\ T); H^1(\Omega)) \subset C([0\ T]; C(\overline{\Omega})) = C(\overline{Q})$) that there exists a positive constant (which we denote again by $C(T)$) such that

$$\epsilon \ \leq u^\epsilon \leq C(T) \text{ on } \Omega \times [0,T]. \tag{31}$$

Notice that

$$\sigma(u^\epsilon)v_x^\epsilon = \frac{\sigma(u^\epsilon)(v_x^\epsilon)^2}{v_x^\epsilon} = J^\epsilon(t),$$

and from inequality (24)

$$\int_\Omega |v_x^\epsilon(x,t)|\, dx \leq \frac{\int_\Omega \sigma(u^\epsilon)(v_x^\epsilon)^2 dxdt}{|J^\epsilon(t)|} \leq \frac{C(T)}{|J^\epsilon(t)|} \tag{32}$$

In particular, if $J_0(t) \neq 0$ for any $t \in [0,T]$

$$\int_0^T \int_\Omega |v_x^\epsilon|\, dxdt \leq \frac{\int_0^T \int_\Omega \sigma(u^\epsilon)(v_x^\epsilon)^2 dxdt}{\min_{t\in[0,T]} |J^\epsilon(t)|} \leq C(T). \tag{33}$$

To get other *a priori* estimates we see that by multiplying the equation of u^ϵ in problem (22) by $\varphi(u^\epsilon)-\varphi(\max(U_D,\epsilon))\in L^2(0,T;H^1(\Omega))$ and integrating over Ω, we get,

$$\int_\Omega(\varphi(u^\epsilon)-\varphi(\max(U_D,\epsilon))u^\epsilon_t dx+\int_\Omega\varphi(u^\epsilon)_x(\varphi(u^\epsilon)-\varphi(\max(U_D,\epsilon)))_x dx$$
$$=\int_\Omega\sigma(u^\epsilon)(v^\epsilon_x)^2(\varphi(u^\epsilon)-\varphi(\max(U_D,\epsilon)))dx.$$

Setting the real function

$$B(s)=\int_0^s\varphi(u)du$$

and integrating over $(0,t)$ we obtain, for a new constant $\widehat{C}(T)$, that

$$\int_\Omega B(u^\epsilon(x,t))dx+\frac{1}{2}\int_0^t\int_\Omega(\varphi(u^\epsilon)_x)^2dtdx \tag{34}$$
$$\leq\widehat{C}(T)+\int_\Omega B(u_0)dx,$$

where we used that

$$\int_0^t\int_\Omega\varphi(\max(U_D,\epsilon))u^\epsilon_t dxdt=\int_\Omega\varphi(\max(U_D(x,t),\epsilon))u^\epsilon(t,x)dx$$
$$-\int_\Omega\varphi(\max(U_D(0,x),\epsilon))(u_0(x)+\epsilon)dx-\int_0^t\int_\Omega(\varphi(\max(U_D,\epsilon)))_t u^\epsilon dxdt$$

as well as that, from assumption (11), the term $\int_\Omega\sigma(u^\epsilon)(v^\epsilon_x)^2(\varphi(u^\epsilon)-\varphi(\max(U_D,\epsilon)))dx$ is bounded if $\sigma_0(0)>0$ and that if $\sigma_0(0)=0$ then

$$\int_{[0\leq u\leq\delta]}\sigma(u^\epsilon)(v^\epsilon_x)^2\varphi(u^\epsilon)dx+\int_{[\delta<u]}\sigma(u^\epsilon)(v^\epsilon_x)^2\varphi(u^\epsilon)dx$$
$$\leq\int_{[0\leq u\leq\delta]}(\varphi(u^\epsilon))^{1-\alpha}dx+C(T)\leq\int_\Omega(\varphi(u^\epsilon))^{1-\alpha}dx+C(T).$$

So, in this case, it suffices to apply Poincaré and Young inequalities to get (34).
Then, since $B(s)\geq 0$ for $s\geq 0$, there exists some constant $C=C(T)$, independent of ϵ, such that

$$\int_\Omega B(u^\epsilon(.,t))\leq C(T)\ \forall t\in(0,T),$$

$$||\varphi(u^\epsilon)||_{L^2(0,T;H^1(\Omega))}\leq C(T),$$

and (from the equation of u^ϵ)

$$||u^\epsilon_t||_{L^2(0,T;H^{-1}(\Omega))}\leq C(T).$$

Moreover, since

$$\varphi(u^\epsilon)_t=\varphi'(u^\epsilon)u^\epsilon_t,$$

it follows that

$$||\varphi(u^\epsilon)_t||_{L^2(0,T;H^{-1}(\Omega))}\leq C(T).$$

Step 3 : Passage to the limit. Using a classical compactness argument (see [31]) and the monotonicity of φ, we can extract a "subsequence" that for simplicity we still label by "ϵ" such that

$$\varphi(u^\epsilon)\rightharpoonup l_1\ \text{ in } L^2(0,T;H^1(\Omega)), \tag{35}$$

$$\varphi(u^\epsilon)\to l_1\ \text{ in } L^2(Q), \tag{36}$$

$$u^\epsilon\to u\ \text{ in } L^\infty(Q), \tag{37}$$

$$(u^\epsilon)_t\rightharpoonup l_2\quad\text{in}\quad L^2(0,T;H^{-1}(\Omega)), \tag{38}$$

Clearly, one deduces that $l_1=\varphi(u)$, $l_2=u_t$. To prove estimate (27) we use estimate (26), the fact that, since σ is Lipschitz, $\sigma(u^\epsilon)\to\sigma(u)$ in $L^2(Q)$, and finally that $\int_\Omega\frac{dx}{\sigma(u(x,t))}<\infty$ for $t\in(0,T]$.

This last property can be shown easily since we know that $u_t^\epsilon - \varphi(u^\epsilon)_{xx} = f(t,x) \geq 0$ and so, by the maximum principle, $u^\epsilon \geq w$ with w solution of (21). Then, in the limit, $u \geq w$ and, on the other hand, by the well-known results (see, e. g. [26]) on problem (21) we know that $w(t,x) \geq 0$ *a.e.* on Q and that $\int_\Omega w(t,x)dx > 0$ for any $t \in (0,T]$. In conclusion, we get the estimate (27) and then

$$J^\epsilon(t) \rightharpoonup J(t) \text{ weakly-star in } \ L^\infty(0,T), \tag{39}$$

$$v^\epsilon \rightharpoonup v \text{ weakly-star in } \ L^\infty(0,T;L^\infty(\Omega)). \tag{40}$$

In order to prove the regularity $u \in C([0,T];L^1(\Omega))$ it suffices to multiply the equation of u^ϵ by $sign(\varphi(u^\epsilon))$ (which can be justified by taking a sequence of regular functions $p_n(r)$ converging to the function $sign(r)$ when $n \to \infty$ and such that $p_n(r) = 1$ if $r > \frac{\epsilon}{2}$). Then, by well-known results, we get that

$$\frac{d}{dt}\int_\Omega |u^\epsilon(x,t)|\,dx \leq \int_\Omega \left|\sigma(u^\epsilon(x,t))(v_x^\epsilon(x,t))^2\right| dx$$

and, since, $\sigma(u^\epsilon)(v_x^\epsilon)^2$ is bounded in $L^1(Q)$ independently of ϵ (recall (24)), the modulus of continuity of $u^\epsilon \in C([0,T];L^1(\Omega))$ is uniformly bounded in ϵ. Then, in the limit

$$\frac{d}{dt}\int_\Omega |u(x,t)|\,dx \leq C(T)$$

with $C(T)$ given by (24) which proves that $u \in C([0,T];L^1(\Omega))$. Since $\min_{x\in\overline{\Omega}} \sigma(u^\epsilon(x,t)) > 0$, by using the identity

$$v_D(L,t) - v_D(-L,t) = J^\epsilon(t)\int_\Omega \frac{dx}{\sigma(u^\epsilon(x,t))}, \tag{41}$$

(recall the arguments mentioned at the Introduction) we deduce that, for a.e. $t \in (0,T)$,

$$J^\epsilon(t) \to J(t) \text{ in } \mathbb{R} \tag{42}$$

since for a.e. $t \in (0,T)$

$$\frac{v_D(L,t) - v_D(-L,t)}{\int_\Omega \frac{dx}{\sigma(u^\epsilon(x,t))}} = J^\epsilon(t), \tag{43}$$

$\sigma(u^\epsilon(x,t)) \to \sigma(u(x,t))$ for any $x \in \Omega$ (recall that $\varphi(u^\epsilon) \rightharpoonup \varphi(u)$ in $L^2(0,T;H^1(\Omega))$ implies that $\varphi(u^\epsilon(.,t)) \to \varphi(u(.,t))$ in $C(\overline{\Omega})$ for a.e. $t \in (0,T)$) and notice that if $\min_{x\in\overline{\Omega}} \sigma(u(x,t)) = 0$ then $J(t) = 0$ and (42) is reduced to $J^\epsilon(t) \to 0$ in $\mathbb{R}$.
Thus, we conclude that for any $\xi \in C^1(\overline{Q})$ such that $\xi(x,t) = 0$ on $\Gamma_D \times (0,T)$

$$\begin{aligned}\int_\Omega \sigma(u^\epsilon(x,t))\left|v_x^\epsilon(x,t)\right|^2 \xi(x,t)dx &= J^\epsilon(t)\int_\Omega v_x^\epsilon(x,t)\xi(x,t)dx \\ &= -J^\epsilon(t)\int_\Omega v^\epsilon(x,t)\xi_x(x,t)dx.\end{aligned}$$

Using (40) and (42) we can pass to the limit to deduce that

$$\int_\Omega \sigma(u^\epsilon(x,t))\left|v_x^\epsilon(x,t)\right|^2 \xi(x,t)dx \to -J(t)\int_\Omega v(x,t)\xi_x(x,t)dx$$

Then, we can pass to the limit in the boundary conditions and in the equations to get that

$$\begin{array}{c}\int_\Omega u(x,T)\xi(x,T)dx - \int_\Omega u_0(x)\xi(x,0)dx - \int_0^T\int_\Omega u\xi_t dtdx \\ + \int_0^T\int_\Omega \varphi(u)_x\xi_x dtdx = \int_0^T\int_\Omega \sigma(u)(v_x)^2\xi dtdx,\end{array} \tag{44}$$

$$\int_\Omega \sigma(u)v_x\zeta_x dx = 0 \text{ a.e. } t \in (0,T), \tag{45}$$

to get the existence result.∎
Remark 1. It is possible to get some additional regularity on u and v by using the Bernstein type regularity method (see, to this respect, Theorem 3, the regularity results for the homogeneous porous medium equation mentioned at [26] or the results, for the non-homogeneous case, by [32]). Arguing as in the approximate step we have that if $J(t) \neq 0$ for any $t \in [t_1,t_2] \subset [0,T]$ then

$$\int_0^T\int_\Omega |v_x|\,dxdt \leq \frac{\int_0^T\int_\Omega \sigma(u)(v_x)^2}{\min_{t\in[t_1,t_2]}|J(t)|} \leq C(T). \tag{46}$$

3 Uniqueness of solutions

In this section we prove that the weak solution to problem (2) is unique. Our main idea will consist in proving that any possible weak solution must coincide with the solution constructed in the previous section by using a method that, coming from the Holmgren duality method, it was first adapted to degenerate equations by A.S. Kalashnikov (see references in [26]) and then refined in [17]. Here the difficulty comes from the fact that we are dealing with a coupled system of equations and not merely with a scalar degenerate equation (see the uniqueness results of [1], [4], [11], [18] and [19] for some related systems).

Theorem 2. *Assume $\Gamma_D = \phi$ or $u_D(t,x) > 0$ on $\Gamma_D \times (0,T)$. Then problem (2) has a unique weak solution (u,v) such that $v \in L^1(0,T;W^{1,1}(\Omega))$ ($v(.,t)$ being univocally determined (up to a constant if $\Gamma_D = \phi$) and taking arbitrary values on the set $\{(x,t) \in Q,\ \sigma(u(t,x)) = 0\}$).*

Before giving the proof of this theorem let us introduce some notation. Let (u^ϵ, v^ϵ) be the solution to problem (22) introduced in the previous section. Let (w,z) be any weak solution to problem (2). By subtracting and using that

$$\int_0^T\int_\Omega (\sigma(w)\,(z_x)^2\,\xi dtdx = \int_0^T\int_\Omega ((\sigma(w)zz_x)_x\xi dtdx = -\int_0^T\int_\Omega \sigma(w)z_xz\xi_x dtdx,$$

which is justified since $\sigma(w)z_x \in L^1(Q)$ and $z \in L^\infty(Q)$, we have

$$\int_\Omega (w-u^\epsilon)(x,T)\xi(x,T)dx - \int_\Omega (w-u^\epsilon)(x,0)\xi(x,0)dx = \int_0^T\int_\Omega (w-u^\epsilon)\xi_t dtdx$$

$$+\int_0^T\int_\Omega (\varphi(w)-\varphi(u^\epsilon))\xi_{xx}dtdx - \int_0^T\int_\Omega (\sigma(w)zz_x - \sigma(u^\epsilon)v^\epsilon v^\epsilon_x)\xi_x dtdx$$

$$+\int_0^T\int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t)))\frac{\partial\xi}{\partial n}(s,t)dtds + \int_0^T\int_\Omega (\sigma(w)z_x - \sigma(u^\epsilon)v^\epsilon_x)\zeta_x dtdx \quad (47)$$

for all $\xi,\zeta \in C^1(\overline{Q}) \cap C([0,T] : C^2(\overline{\Omega}))$ such that $\xi(x,t), \zeta(x,t) = 0$ on $\Gamma_D \times (0,T)$. Here the term $\int_0^T\int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t)))\frac{\partial\xi}{\partial n}(s,t)dtds$ must be understood in the usual one-dimensional integration by parts sense. So if, for instance, $\Gamma_D = \{-L\} \cup \{L\}$ we have that

$$\int_0^T\int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t)))\frac{\partial\xi}{\partial n}(s,t)dtds$$
$$= \int_0^T [(\varphi(\max(U_D(L,t),\epsilon)) - \varphi(u_D(L,t)))\xi_x(L,t)$$
$$- (\varphi(\max(U_D(-L,t),\epsilon)) - \varphi(u_D(-L,t)))\xi_x(-L,t)]dt.$$

Let us denote by I the left hand side of (47). Notice that and using that $v, z \in L^1(0,T;W^{1,1}(\Omega))$

$$\int_0^T\int_\Omega (\sigma(w)zz_x - \sigma(u^\epsilon)v^\epsilon v^\epsilon_x)\xi_x dtdx =$$

$$\int_0^T\int_\Omega (\sigma(w)-\sigma(u^\epsilon))zz_x\xi_x dtdx + \int_0^T\int_\Omega \sigma(u^\epsilon)(zz_x - v^\epsilon v^\epsilon_x)\xi_x dtdx =$$

$$\int_0^T\int_\Omega (\sigma(w)-\sigma(u^\epsilon))zz_x\xi_x dtdx + \int_0^T\int_\Omega \sigma(u^\epsilon)(z-v^\epsilon)z_x\xi_x dtdx$$

$$+\int_0^T\int_\Omega \sigma(u^\epsilon)v^\epsilon(z-v^\epsilon)_x\xi_x dtdx = \int_0^T\int_\Omega (\sigma(w)-\sigma(u^\epsilon))zz_x\xi_x dtdx$$

$$+\int_0^T\int_\Omega \sigma(u^\epsilon)(z-v^\epsilon)z_x\xi_x dtdx - \int_0^T\int_\Omega (\sigma(u^\epsilon)v^\epsilon\xi_x)_x(z-v^\epsilon)dtdx$$

and

$$\int_0^T\int_\Omega (\sigma(w)z_x - \sigma(u^\epsilon)v^\epsilon_x)\zeta_x dtdx =$$

$$\int_0^T\int_\Omega(\sigma(w)-\sigma(u^\epsilon))z_x\zeta_x dtdx+\int_0^T\int_\Omega\sigma(u^\epsilon)(z-v^\epsilon)_x\zeta_x dtdx=$$
$$\int_0^T\int_\Omega(\sigma(w)-\sigma(u^\epsilon))z_x\zeta_x dtdx-\int_0^T\int_\Omega(\sigma(u^\epsilon)\zeta_x)_x(z-v^\epsilon)dtdx.$$

Thus combining this with (47) we obtain

$$I=\int_0^T\int_\Omega(w-u^\epsilon)(\xi_t+\frac{\varphi(w)-\varphi(u^\epsilon)}{w-u^\epsilon}\xi_{xx}-\frac{\sigma(w)-\sigma(u^\epsilon)}{w-u^\epsilon}zz_x\xi_x+$$
$$+\frac{\sigma(w)-\sigma(u^\epsilon)}{w-u^\epsilon}z_x\zeta_x)dtdx+\int_0^T\int_{\Gamma_D}(\varphi(\max(U_D(s,t),\epsilon))-\varphi(u_D(s,t)))\frac{\partial\xi}{\partial n}(s,t)dtds$$
$$-\int_0^T\int_\Omega(z-v^\epsilon)\left(\sigma(u^\epsilon)z_x\xi_x-(\sigma(u^\epsilon)v^\epsilon\xi_x)_x+(\sigma(u^\epsilon)\zeta_x)_x\right)dtdx. \tag{48}$$

Let us set

$$A_\epsilon=A_\epsilon(x,t)=\frac{\varphi(w)-\varphi(u^\epsilon)}{w-u^\epsilon},\qquad B_\epsilon=B_\epsilon(x,t)=\frac{\sigma(w)-\sigma(u^\epsilon)}{w-u^\epsilon}zz_x \tag{49}$$
$$C_\epsilon=C_\epsilon(x,t)=\frac{\sigma(w)-\sigma(u^\epsilon)}{w-u^\epsilon}z_x\ ,\qquad D_\epsilon=D_\epsilon(x,t)=\sigma(u^\epsilon)z_x \tag{50}$$
$$E_\epsilon=E_\epsilon(x,t)=\sigma(u^\epsilon)v^\epsilon,\qquad F_\epsilon=F_\epsilon(x,t)=\sigma(u^\epsilon). \tag{51}$$

Thus (48) reads now :

$$\begin{aligned}I\ &=\ \int_0^T\int_\Omega(w-u^\epsilon)\{\xi_t+A_\epsilon\xi_{xx}-B_\epsilon\xi_x+C_\epsilon\zeta_x\}dtdx\\&+\int_0^T\int_{\Gamma_D}(\varphi(\max(U_D(s,t),\epsilon))-\varphi(u_D(s,t)))\frac{\partial\xi}{\partial n}(s,t)dtds\\&-\int_0^T\int_\Omega(z-v^\epsilon)\{D_\epsilon\xi_x-(E_\epsilon\xi_x)_x+(F_\epsilon\zeta_x)_x\}dtdx.\end{aligned}$$

Lemma 1. *There exist three positive constants m_ϵ , M_ϵ and M^* (M^* independent of ϵ) such that*

$$m_\epsilon\le A_\epsilon(x,t)\le M_\epsilon\qquad\forall(x,t)\in Q \tag{52}$$
$$|B_\epsilon(x,t)|,\ |C_\epsilon(x,t)|\le M^*\qquad\forall(x,t)\in Q. \tag{53}$$

Proof. Estimate (53) results from the fact that σ is supposed to be Lipschitz continuous and from (25), (26). To prove (52) recall that we are considering here a bounded solution w to problem (2). So, there exists a constant $\widetilde{M}$ such that

$$0\le w(x,t),\ u_\epsilon(x,t)\le\widetilde{M}\qquad\forall(x,t)\in Q.$$

By the mean value theorem one has

$$A_\epsilon(x,t)=\frac{\varphi(w)-\varphi(u^\epsilon)}{w-u^\epsilon}=\varphi'(\theta(x,t))$$

where $\theta(x,t)$ belongs to the interval $(w(x,t)\ u_\epsilon(x,t))$. Since v and u_ϵ are bounded from above, θ is bounded from above and so does $A_\epsilon(x,t)$.
If (x,t) is such that

$$|w(x,t)-u_\epsilon(x,t)|\le\frac{\epsilon}{2}$$

then, since by the maximum principle, $u_\epsilon\ge\epsilon$

$$w(x,t)=u_\epsilon(x,t)+w(x,t)-u_\epsilon(x,t)\ge\epsilon-\frac{\epsilon}{2}=\frac{\epsilon}{2}$$

So, $\theta\ge\frac{\epsilon}{2}$ and by (6) $A_\epsilon(x,t)$ is bounded from below.
Next consider the case where

$$|w(x,t)-u_\epsilon(x,t)|>\frac{\epsilon}{2}$$

i.e.

$$w(x,t) > u_\epsilon(x,t) + \frac{\epsilon}{2} \quad \text{or} \quad w(x,t) < u_\epsilon(x,t) - \frac{\epsilon}{2}.$$

In the case where $w > u_\epsilon + \frac{\epsilon}{2}$ using the monotonicity of φ and the fact that $|w - u_\epsilon| \leq 2M$ we get

$$A_\epsilon(x,t) = \frac{\varphi(w) - \varphi(u^\epsilon)}{w - u^\epsilon} > \frac{\varphi(u_\epsilon + \frac{\epsilon}{2}) - \varphi(u^\epsilon)}{2M} = \frac{\varphi'(\theta)}{2M}$$

where $\theta \geq \epsilon$. Thus it results from (6) that $A_\epsilon(x,t)$ is bounded from below.
In the case where $w < u_\epsilon - \frac{\epsilon}{2}$ using the monotonicity of φ we obtain

$$A_\epsilon(x,t) = \frac{\varphi(u^\epsilon) - \varphi(w)}{u^\epsilon - w} > \frac{\varphi(u^\epsilon) - \varphi(u_\epsilon - \frac{\epsilon}{2})}{2M} = \frac{\varphi'(\theta)}{2M}$$

where $\theta \geq \frac{\epsilon}{2}$. Thus it results from (6) that in this case $A_\epsilon(x,t)$ is bounded from below and we can take

$$m_\epsilon = \min\{\varphi'(\theta) : \theta \in [\frac{\epsilon}{2}, M]\} \min\{1, 1/2M\}. \tag{54}$$

This completes the proof of the Lemma.■

Assume that we extend $A_\epsilon,\ B_\epsilon,\ C_\epsilon$ to the whole $\mathbb{R}^2$ respectively by $m_\epsilon, 0, 0$ and denote again these extensions by $A_\epsilon,\ B_\epsilon,\ C_\epsilon$. Let ρ be a function of class C^∞ with support in the ball $B(0,1)$ of center $\mathbf{0}$ and radius 1 of $\mathbb{R}^2$ and such that

$$\int_{B(0,1)} \rho dt dx = 1.$$

Set

$$\rho_n(x,t) = n^2 \rho(nx, nt)$$

and

$$A^n_\epsilon = \rho_n * A_\epsilon, \quad B^n_\epsilon = \rho_n * B_\epsilon, \quad C^n_\epsilon = \rho_n * C_\epsilon,$$

where $*$ denotes the usual convolution operator. Clearly, these functions are of class C^∞ in $\mathbb{R}^2$. Moreover, one has

$$m_\epsilon \leq A^n_\epsilon(x,t) \leq M_\epsilon \quad \forall (x,t) \in Q, \forall n \tag{55}$$

$$|B^n_\epsilon|, \quad |C^n_\epsilon| \leq \ M^* \quad \forall (x,t) \in Q,\ \forall n. \tag{56}$$

Thus, equation (47) reads now

$$\begin{aligned} I \quad = \quad & \int_0^T \int_\Omega (w - u^\epsilon)\{\xi_t + A^n_\epsilon \xi_{xx} - B^n_\epsilon \xi_x + C^n_\epsilon \zeta_x\} dt dx \\ & + \int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t))) \frac{\partial \xi}{\partial n}(s,t) dt ds \\ & + \int_0^T \int_\Omega (w - u^\epsilon) \xi_{xx}(A_\epsilon - A^n_\epsilon) dt dx - \int_0^T \int_\Omega (w - u^\epsilon) \xi_x (B_\epsilon - B^n_\epsilon) dt dx \\ & + \int_0^T \int_\Omega (w - u^\epsilon) \zeta_x (C_\epsilon - C^n_\epsilon) dt dx - \int_0^T \int_\Omega (z - v^\epsilon)\{D_\epsilon \xi_x - (E_\epsilon \xi_x)_x + (F_\epsilon \zeta_x)_x\} dt dx. \end{aligned} \tag{57}$$

A similar argument must be applied if the coefficients D_ϵ, E_ϵ and F_ϵ are not bounded (we leave the details to the reader).

Now we construct a "dual system" which plays a crucial role in the proof of Theorem 2.
Lemma 2. *There exists a unique smooth solution* $(\xi, \zeta) = (\xi^{n,m}_\epsilon, \zeta^{n,m}_\epsilon)$ *to the system*

$$\begin{cases} \xi_t + A^n_\epsilon \xi_{xx} - B^n_\epsilon \xi_x + C^n_\epsilon \zeta_x = 0 & \text{in} \quad Q, \\ -(F_\epsilon \zeta_x)_x = (E_\epsilon \xi_x)_x - D_\epsilon \xi_x & \text{in} \quad Q, \\ \zeta = 0, \quad \xi = 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \frac{\partial \zeta}{\partial n} = 0,\ \frac{\partial \xi}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ \xi(.,T) = w^m & \text{on} \quad (0,L), \end{cases} \tag{58}$$

where $w^m \in C_0^\infty(\Omega)$ is such that $|w^m(x)| \leq 1$ for any $x \in (-L, L)$ and

$$w^m \to sign(w(x,T) - u^\epsilon(x,T)) \quad \text{in} \qquad L^2(\Omega), \text{when } m \to \infty \tag{59}$$

(here sign denotes the $sign_0$ function, i.e., $sign(x) = x/|x|$ if $x \neq 0$, and 0 if $x = 0$).
Proof. First we make the change $t \to T - t$ in such a way that the system becomes (with obvious notation) to find (ξ, ζ) such that

$$\begin{cases} \xi_t = A_\epsilon^n \xi_{xx} - B_\epsilon^n \xi_x + C_\epsilon^n \zeta_x & \text{in} \quad Q, \\ -(F_\epsilon \zeta_x)_x = (E_\epsilon \xi_x)_x - D_\epsilon \xi_x & \text{in} \quad Q, \\ \zeta = 0, \quad \xi = 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \frac{\partial \zeta}{\partial n} = 0, \ \frac{\partial \xi}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ \xi(.,0) = w^m & \text{on} \quad (0,L). \end{cases} \tag{60}$$

Next looking for (ξ, ζ) of the form $(e^{-\lambda t}\xi, e^{-\lambda t}\zeta)$ the system reduces to find a solution to a system of the type

$$\begin{cases} \xi_t = A_\epsilon^n \xi_{xx} - B_\epsilon^n \xi_x + C_\epsilon^n \zeta_x + \lambda \xi & \text{in} \quad Q, \\ -(F_\epsilon \zeta_x)_x = (E_\epsilon \xi_x)_x - D_\epsilon \xi_x & \text{in} \quad Q, \\ \zeta = 0, \quad \xi = 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \frac{\partial \zeta}{\partial n} = 0, \ \frac{\partial \xi}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ \xi(.,0) = w^m & \text{on} \quad (0,L). \end{cases} \tag{61}$$

Writing

$$A_\epsilon^n \xi_{xx} = (A_\epsilon^n \xi_x)_x - (A_\epsilon^n)_x \xi_x$$

one can assume, without lost of generality, that the system can be reformulated as

$$\begin{cases} \xi_t = (A_\epsilon^n \xi_x)_x - B_\epsilon^n \xi_x + C_\epsilon^n \zeta_x + \lambda \xi & \text{in} \quad Q, \\ -(F_\epsilon \zeta_x)_x = (E_\epsilon \xi_x)_x - D_\epsilon \xi_x & \text{in} \quad Q, \\ \zeta = 0, \quad \xi = 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \frac{\partial \zeta}{\partial n} = 0, \ \frac{\partial \xi}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ \xi(.,0) = w^m & \text{on} \quad (0,L). \end{cases} \tag{62}$$

Let us introduce the space $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, let V' its dual and denote by $\mathcal{L}$ the operator which to $f \in V'$ associates $u = \mathcal{L}(f)$ the solution to

$$\begin{cases} -(F_\epsilon u_x)_x = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma_D, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \Gamma_N. \end{cases}$$

Then the second equation of problem (62) reads

$$\zeta = \mathcal{L}((E_\epsilon \xi_x)_x - D_\epsilon \xi_x). \tag{63}$$

Hence the first one becomes

$$\xi_t = (A_\epsilon^n \xi_x)_x - B_\epsilon^n \xi_x + C_\epsilon^n \left(\mathcal{L}((E_\epsilon \xi_x)_x - D_\epsilon \xi_x)\right)_x + \lambda \xi$$

i.e.

$$\xi_t = (A_\epsilon^n \xi_x)_x - B_\epsilon^n \xi_x + \lambda \xi + \mathrm{L}(\xi)$$

where L is a bounded linear operator from V into $L^2(\Omega)$. Taking into account (63) the system reduces to solve the problem

$$\begin{cases} \xi_t = (A_\epsilon^n \xi_x)_x - B_\epsilon^n \xi_x + \lambda \xi + \mathrm{L}(\xi) & \text{in} \quad Q, \\ \xi = 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \frac{\partial \xi}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ \xi(.,0) = w^m & \text{on} \quad (0,L). \end{cases} \tag{64}$$

Setting

$$a_\epsilon^n(t; u, v) = \int_\Omega A_\epsilon^n u_x v_x dx + \int_\Omega B_\epsilon^n u_x v dx + \lambda \int_\Omega uv dx + \int_\Omega \mathrm{L}(u) v dx$$

it is easy to see that for λ selected large enough one has

$$a_\epsilon^n(t;u,u) \geq \nu_\epsilon ||u||_V$$

for some constant $\nu_\epsilon > 0$ (independently of n, even if we apply the regularization process to D_ϵ, E_ϵ and F_ϵ) and where $||u||_V$ denotes the usual norm in $H^1(\Omega)$ (or the equivalent norm

$$||u||_V = (\int_\Omega u_x^2 dx)^{\frac{1}{2}}$$

in the case of $\Gamma_D \neq \phi$). Existence of a solution ξ to problem (64) follows then from a standard Galerkin scheme (see for instance [6], Theorem 6.1).
We have also

$$\xi \in L^2(0,T;V), \quad \xi_t \in L^2(0,T;V')$$

and

$$\|\xi\|_{L^2(0,T;V)} \leq C(\epsilon) \tag{65}$$

where $C(\epsilon)$ is a constant independent of n (see [6]). Moreover

$$\zeta \in L^2(0,T;V).$$

Using standard arguments we can show then that since our data are smooth so is our solution.■

Next, we show some estimates which we shall need later.
Lemma 3. *Let $(\xi_\epsilon^{n,m}, \zeta_\epsilon^{n,m})$ be the solution to problem (58). Then, there exists a constant C_ϵ independent of n and m such that*

$$\left\|\xi_{\epsilon,xx}^{n,m}\right\|_{2,Q}, \quad \left\|\xi_{\epsilon,x}^{n,m}\right\|_{2,Q}, \quad \left\|\zeta_{\epsilon,x}^{n,m}\right\|_{2,Q} \leq C_\epsilon \tag{66}$$

($\|\|_{2,Q}$ denotes the usual L^2-norm on $L^2(Q)$).
Proof. For simplicity in the notation we drop the m- dependence. From (65) we have clearly

$$\|\xi_x\|_{2,Q} \leq C_\epsilon \tag{67}$$

where C_ϵ is independent of n and m. Multiplying the second equation of (62) by ζ and integrating by parts over Ω, we then easily deduce

$$\|\zeta_x\|_{2,\Omega} \leq C_\epsilon \|\xi_x\|_{2,\Omega} \tag{68}$$

for some constant C_ϵ independent of n and m. Then, it follows from (67) that

$$\|\zeta_x\|_{2,Q} \leq C_\epsilon. \tag{69}$$

Next, if we multiply the first equation of (61) by ξ_{xx} and integrate over Ω we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega \xi_x^2 dx = \int_\Omega A_\epsilon^n \xi_{xx}^2 dx - \int_\Omega B_\epsilon^n \xi_x \xi_{xx} dx + \int_\Omega B_\epsilon^n \zeta_x \xi_{xx} dx.$$

Hence, integrating over $(0,T)$ and recalling (52) we get

$$m_\epsilon \int_0^T \int_\Omega \xi_{xx}^2 dx \leq \int_\Omega \xi_x^2(x,0)dx + \int_0^T \int_\Omega |B_\epsilon^n||\xi_x|\xi_{xx}|dx + \int_0^T \int_\Omega |C_\epsilon^n||\xi_x|\xi_{xx}|dx.$$

Using Young's Inequality and (53) in the two last integrals (66) follows.■

We are now able to complete the proof of Theorem 2.
Proof of Theorem 2. In (47), (57) choose $(\xi,\zeta) = (\xi_\epsilon^{n,m}, \zeta_\epsilon^{n,m})$ solution to (58) where w^m satisfies (59). Then expressions (47) and (57) leads to

$$\begin{aligned}
&\int_\Omega (w-u^\epsilon)(x,T)w^m(x)dx - \int_\Omega (w-u^\epsilon)(x,0)\xi(x,0)dx \qquad (70)\\
&= \int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t)))\frac{\partial \xi}{\partial n}(s,t)dtds + \int_0^T \int_\Omega (w-u_\epsilon)\xi_{xx}(A_\epsilon - A_\epsilon^n)dxdt \\
&- \int_0^T \int_\Omega (w-u_\epsilon)\xi_x(B_\epsilon - B_\epsilon^n)dxdt + \int_0^T \int_\Omega (w-u_\epsilon)\zeta_x(C_\epsilon - C_\epsilon^n)dxdt.
\end{aligned}$$

Notice that the assumption $u_D > 0$ implies that for $\epsilon > 0$ small enough we get that $\max(U_D(s,t), \epsilon) = u_D(s,t))$ and so the first term of the right hand side disappears (this is also the case of $\Gamma_D = \phi$). Then, by passing to the limit (first in $\varepsilon \to 0$, then in $n \to \infty$ and finally in $m \to \infty$) we get that $\int_\Omega |w(x,T) - u(x,T)|\, dx = 0$. Since T is arbitrary we get that $w \equiv u^\epsilon$ and then, obviously, $z \equiv v$ on Q if $\Gamma_D \neq \phi$. When $\Gamma_D = \phi$ we deduce that, for any $t \in [0.T]$ there exists a constant $C(t)$ such that $z(.,t) - v(.,t) = C(t)$ on Ω. On the set $\{(x,t) \in Q, \sigma(u(t,x)) = 0\}$ z and v may be different without any consequence on the rest of points of Q.■

As in [17], the technique used for the proof of the uniqueness of weak solutions also leads to the following continuous dependence result

Corollary 1. *Let* $(u,v), (u^*, v^*)$ *be the solutions corresponding to the initial data* u_0, u_0^* *and the same boundary conditions. Then we have the continuous dependence property*

$$\|u(t) - u^*(t)\|_{L^1(\Omega)} \leq K_2^* \|u_0 - u_0^*\|_{L^1(\Omega)}$$

for any $t \in [0,T]$ *where* $K_2^* = \inf\{K_2(\epsilon)\}$ *with* $K_2(\epsilon) := \sup_{n,m} \|\xi_\epsilon^{n,m}\|_{L^\infty(\Omega)}$ *and* $\xi_\epsilon^{n,m}$ *given in the proof of Lemma 4.*

Remark 2. The case $u_D = 0$ is more delicate since the first term of the right hand side of (70) becomes

$$\int_0^T \int_{\Gamma_D} \varphi(\epsilon) \frac{\partial \xi}{\partial n}(s,t) dt ds$$

and the passing to the limit requires sharper estimates obtained under additional assumptions (see, for instance, [17] for the case of a single scalar equation). Nevertheless, we conjecture that, as in the scalar case (see also [21]), the uniqueness of weak solutions holds also for $u_D = 0$ and general functions φ.

Remark 3. The above results (existence and uniqueness of weak solutions) can be obtained for higher dimensions when Ω is assumed to be a bounded domain with smooth boundary and satisfying the exterior sphere condition. For extensions of the technique introduced in [17] to$N$-dimensional equations see the papers [7], [18] and [19]. Notice also that the introduction of the linear operator, $u = \mathcal{L}(f)$, in the proof of the above theorem, could be applied to a more general class of coupled systems.

4 On the existence of the free boundary

The assumption $\varphi'(0) = 0$ and a suitable growth assumption lead to the existence of a free boundary given as the boundary of the support of the solution. In the case of evolution problems, as the one under consideration, such property is usually denoted as the *finite speed of propagation property*: if $u_0(x) = 0$ on $B_{\rho_0}(x_0) := (x_0 - \rho_0, x_0 + \rho_0)$ for some $x_0 \in \Omega$ and $\rho_0 > 0$ then there exists $t^* > 0$ and a function $\rho(t) : [0, t^*) \mapsto [0, \infty)$, with $\rho(0) \leq \rho_0$, such that $u(x,t) = 0$ a.e. in $B_{\rho(t)}(x_0), \forall\, t \in [0, t^*)$.

When $\Gamma_N \neq \phi$ we know that the system becomes uncoupled (see the Introduction) and, so, the criterium for the finite speed of propagation is well known (see, e.g., the surveys [26] and [5]). Nevertheless, if $\Gamma_N = \phi$ and $\sigma_0(0) > 0$ the vanishing set of the solution can be reduced (at most) to some curves in Q since, if we assume that $u(.,t)$ is a convex function of x then

$$u_t \geq \sigma_0 (v_x)^2$$

and thus

$$u(x,t) \geq \sigma_0 \int_0^t v_x(x,s)^2 ds + u_0(x).$$

Then $\int_0^t v_x(x_0,s)^2 ds > 0$ implies that $u(x_0,t) > 0$. Notice also that, from the strong maximum principle, $v_x(x,.)$ can not be zero on a subset of Ω of positive measure (for any fixed $t \in [0,T]$). On the other hand, if $\sigma(u(x,t)) \left(v_x(x,t)\right)^2 > 0$ on Q, it is impossible to get solutions $u(x,t)$ vanishing on an open subset ω of Q since we would reach a contradiction on ω trough the equation of (1).

The case $\sigma_0(0) = 0$ (and $\Gamma_N = \phi$) is different. More precisely we have:

Theorem 3. *Assume* φ *satisfying*

$$\int_{0^+} \frac{\varphi'(s)}{s} ds < \infty, \tag{71}$$

$\sigma_0(0) = 0$ *and* $\Gamma_N = \phi$. *Then, if* $suppu_0$ *is a non empty compact subset of* Ω *the same happens with* $suppu(.,t)$ *for any* $t \in [0, t^*)$, *for some* $t^* \in (0, T]$. *Moreover, if* $t^* < T$ *then* $u(x,t) > 0$ *for any* $t \in (t^*, T]$.

Proof. Consider w as the solution of the scalar homogeneous problem (21) (remember that now $\Gamma_D = \partial\Omega$). Thanks to the assumption (71) we know that there exists $t^* \in (0, T]$ such that $\text{supp}w(.,t)$ is a compact subset of the open set Ω for any $t \in [0, t^*)$ and that if $t^* < t < T$ then $w(x,t) > 0$ for any $t \in (t^*, T]$. It is easy to see that, necessarily, $w(.,t)$ must coincide with $u(.,t)$ for any $t \in [0, t^*)$. Indeed, as $\sigma(u(x,t))v_x(x,t)$ must be a constant (in x) $J(t)$ we get that, necessarily $J(t) = 0$ if $\sigma(u(x_0,t)) = 0$ for some $x_0 \in \Omega$. Then, as $\sigma(w(x_0,t)) = 0$, for some $x_0 \in \Omega$ if $t \in [0, t^*)$, we can take $v = z$ as the unique function solution of

$$\begin{cases} z_x = 0 & \text{in} \quad \Omega \times (0, t^*), \\ z = v_D, & \text{on} \quad \Gamma_D \times (0, t^*), \\ \frac{\partial z}{\partial n} = 0, & \text{on} \quad \Gamma_N \times (0, t^*), \end{cases}$$

and we get that (w, z) satisfies problem (1) on $[0, t^*)$ (notice that $\sigma(w(x,t))(z_x(x,t))^2 = J(t)z_x(x,t) = 0$ on $\Omega \times (0, t^*)$). By the uniqueness of solutions for problem (1) we conclude that $(u, v) = (w, z)$ on $\Omega \times [0, t^*)$. Moreover, as $\sigma(u(x,t))(v_x(x,t))^2 \geq 0$ on $\Omega \times (0, T)$ we conclude (by the maximum principle for problem (21)) that $u(x,t) \geq w(x,t) \geq 0$ on $\Omega \times [0, T]$ and then $u(x,t) > 0$ on $\Omega \times (t^*, T]$.$\blacksquare$

Remark 4. Notice that assumption (71) holds under the Wiedemann-Franz law $k(u) = k_0 u\sigma(u)$ (remember (10)). We also point out that, in spite the great resemblance between system (1) and the one arising in the study of the secondary recuperation of petroleum in a porous medium (see, e.g. [3] and [4]), there are important differences in the study of the free boundary. In this case the systems can be reformulated as

$$\begin{cases} u_t - (k(u)u_x)_x = (\theta(u)\sigma(u)vv_x)_x & \text{in} \quad \Omega \times (0, T), \\ (\sigma(u)v_x)_x = 0 & \text{in} \quad \Omega \times (0, T), \end{cases} \tag{72}$$

for some suitable function $\theta(u)$ such that $\theta'(u) \approx u\varphi(u)$ which allows the application of some suitable energy method (see, [3] and Chapter 4 of [5]). The applicability of that method to our case (in which $\theta(u) \equiv 1$) is highly doubtful.$\blacksquare$

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References

[1] S. N. Antontsev, M. Chipot, The Thermistor Problem: Existence, Smoothness, Uniqueness, Blow up. *SIAM J. Math. Analysis*, **25**, (1994), 1128-1156.

[2] S. N. Antontsev, M. Chipot, The analysis of blow up for the Thermistor problem. *Siberian Mathematical Journal,* **38,** (1997), 827-841.

[3] S.N. Antontsev and J.I. Díaz, Space and time localization in the flow of two immiscible fluids through a porous medium: Energy methods applied to systems, *Nonlinear Analysis*,**16**, 4, (1991), 299-313.

[4] S. N. Antontsev, J. I. Díaz, and A. V. Domanski,Stability and stabilization of generalized solutions of degenerate problems of two-phase filtration, Dokl. Akad. Nauk, 325 (1992), pp. 1151–1155(English translation in "Soviet Phys. Dokl.", v.37, n.8, (1993), 411-413).

[5] S.N. Antontsev, J.I. Díaz, S.I. Shmarev, *Energy Methods for Free Boundary Problems. Applications to nonlinear PDEs and fluid mechanics.* Progress in Nonlinear Differential Equations and Their Applications, **48**. Birkhäuser, Boston, 2002.

[6] A. Bensoussan and J. L. Lions, *Application des Inéquations Variationnelles en Contrôle Stochastique.* Dunod, Paris, (1978).

[7] M. Bertsch and D. Hilhorst, A density dependent diffusion in Population Dynamics: stabilization to equilibrium, *SIAM J. Math. Anal.,* **17,** 4, (1986), 863-883.

[8] X. Chen, Existence and regularity of nonuniform elliptic system arising from a thermistor problem, *J. Partial Differential Equations,* **7,** (1994), 19-34.

[9] X. Chen and A. Friedman, A free boundary problem for a nonlinear degenerate elliptic system modeling a thermistor, *Ann.Sc. Norm. Sup. Pisa,* **19,** (1993), 615-636.

[10] X. Chen and A. Friedman, The thermistor problem for conductivity which vanishes at large temperature, *Quaterly of Appl. Math., **51,*** (1993), 101-115.

[11] M. Chipot, G. Cimatti, A uniqueness result for the Thermistor problem. *European Journal of Applied Math.*, **2**, (1991), 97-103.

[12] M. Chipot, J. I. Díaz, R. Kersner On the thermistor problem with temperature dependent conductivity, in preparation.

[13] G. Cimatti, Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor, *Ann. Mat. Pura Appl.,* **162,** (1992), 33-42.

[14] G. Cimatti, A bound for the temperature in the thermistor problem. *J. of Applied Math.* **40**, (1988), 15-22.

[15] G. Cimatti, Remark on existence and uniqueness for the thermistor problem. *Quaterly of Applied Math.* **74**, (1989), 117-121.

[16] G. Cimatti, G. Prodi, Existence results for a nonlinear elliptic system modelling a temperature dependent electrical resistor. *Ann. Mat. Pura Appl.* **152**, (1989), 227-236.

[17] J. I. Diaz, R. Kersner, On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium. *J. Differential Equations,* **69**, (1987), 368-403.

[18] J.I. Díaz, G. Galiano, Existence and uniqueness of solutions of the Boussinesq system with nonlinear thermal diffusion, *Topological Methods in Nonlinear Analysis,* **11,** (special issue dedicated to O. Ladyzhenskaya), No1, (1998), 59-82.

[19] J.I. Díaz, G. Galiano, A. Jüngel, On a quasilinear degenerate system arising in semiconductors theory. Part I: Existence and uniqueness of solutions, *Nonlinear Analysis: Real World Applications,* **2,** (2001), 305-336.

[20] R.F. Gariepy, M. Shillor, X. Xu: Existence of generalized weak solutions to a model for in situ vitrification, *Euro. Jnl. of Applied Mathematics,* **9,** (1998), 543-559.

[21] B. H. Gilding, Improved theory for a nonlinear degenerate parabolic equation. *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.* (4), **16,** (1989), 165-224.

[22] M.T. González Montesinos, F. Ortegón Gallego, The evolution thermistor problem with degenerate thermal conductivity, *Communications on Pure and Applied Analysis,* **1,** (2002), 29-41.

[23] M.T. González Montesinos, F. Ortegón Gallego, Renormalized solutions to a nonlinear parabolic-elliptic system, *preprint* (2003).

[24] S. D. Howison, A remark on the Thermistor problem in two space dimensions. *Quaterly Applied Math.* **47**, (1982), 509-512.

[25] S. D. Howison, J. F. Rodrigues, M. Shilor, Stationary solution to the Thermistor problem. *Journ. Math. Anal. Appl.*,**174,** (1993), 573-588.

[26] A. S. Kalashnikov, Some problems of the qualitative theory of nonlinear degenerate second order parabolic equations. *Russian Math. Surveys,* **42**, (1987), 169-222.

[27] T. Kawanago, Estimation $L^p - L^q$ des solutions de $u_t = \Delta\phi(u) + f$, *C.R. Acad. Sci. Paris,* **317,** (1993), 821-824.

[28] F. Kohlrausch, Über den stationären Temperatur-zustand eines electrisch geheiz ten Leiters. *Ann. Physics,* **1**, (1900), 132-158.

[29] A. Lacey, Thermal Runaway in a Non-Local Problem Modelling Ohmic Heating, 1: Model Derivation and Some Special Cases. *European J. Appl. Math.*, **6**, (1995), 127-144.

[30] A. Lacey, Thermal Runaway in a Non-Local Problem Modelling Ohmic Heating, 2: General Proofs of Blow up and asymptotics of Runaway, *European J. Appl. Math.*, **6,** (1995), 127-144.

[31] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, (1969).

[32] M. S. Moulay and M. Pierre, Regularizing effect for a nonlinear diffusion with irregular source-term, *Nonlinear Analysis,* **13,** (1989), 645-655.

[33] P. Shi, M. Shillor, X. Xu: Existence of a solution to the Stefan problem with Joule's heating, *J. Differential Eqns.,* **105,** (1993), 239-263.

[34] W. Xie, On the existence and uniqueness for the thermistor *Advances in math. Sciences and Appl.,* **2,** (1993), 63-73.

[35] H. Xie and W. Allegretto, $C^{\alpha}(\Omega)$ solutions of a class of nonlinear degenerate elliptic systems arising in the thermistor problem, *SIAM, J. Math. Anal.,* **22,** (1991), 1491-1499.

[36] X. Xu, A degenerate Stefan-like problem with Joule's heating, *SIAM, J. Math. Anal.,* **23,** (1992), 1417-1438.

[37] X. Xu: Existence theory for a strongly degenerate parabolic system, *Rocky Mountain Journ. Mathematics,* **23,** (1993), 1555-1580.

[38] X. Xu: A strongly degenerate system involving an equation of parabolic type and an equation of elliptic type, *Commun. in Partial Differential Equations,* **18,** (1993), 199-213.

[39] X.Xu: The thermistor problem with conductivity vanishing for large temperature, *Proc. Royal Soc. Edinburgh,* **124A,** (1994), 1-21.

[40] X. Xu: A compactness theorem and its application to a system of partial differential equations, *Differential and Integral Equations,* **9,** (1996), 119-136.

[41] X. Xu and M. Shillor: The Stefan problem with convection and Joule's heating, *Adv. Diff. Eqns., 2,* (1997), 667-691.