ON THE THERMISTOR PROBLEM WITH TEMPERATURE DEPENDENT CONDUCTIVITY

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1 Introduction

▶ **Problem** The one dimensional *thermistor* problem: $\Omega=(-L,+L)$

$$\begin{cases} u_t-(\kappa(u)u_x)_x=\sigma(u)(v_x)^2 & \text{in} \quad \Omega\times(0,T), \\ (\sigma(u)v_x)_x=0 & \text{in} \quad \Omega\times(0,T), \\ v=v_D,\ u=u_D\geq 0 & \text{on} \quad \Gamma_D\times(0,T), \\ \sigma(u)\frac{\partial v}{\partial n}=0,\ \kappa(u)\frac{\partial u}{\partial n}=0 & \text{on} \quad \Gamma_N\times(0,T), \\ u(x,0)=u_0(x)\geq 0 & \text{on} \quad \Omega. \end{cases} \tag{1}$$

▶ n is the outpointing normal vector

▶ $\Gamma_D\cup\Gamma_N=\partial\Omega$, $\Gamma_D\cap\Gamma_N=\phi$, the possibility $\Gamma_D=\phi$ (the empty set), or $\Gamma_N=\phi$, being not excluded.

▶ This problem models the diffusion of heat produced by Joule's effect in a one dimensional conductor (see for instance Kohlrausch 1900, Cimati 1988).

▶ u is the temperature, $\kappa(u)$ the thermal conductivity of the medium, v is the inside potential and $\sigma(u)$ the electric conductivity which (as κ as well) is supposed to depend on the temperature.

▶Metallic conduction, the Wiedemann-Franz law $k(u)=k_0u\sigma(u)\Longrightarrow$the temperature equation degenerates where $u=0$.

▶Many results in the literature but: lack of existence and uniqueness results if $k(0) = 0$ [degenerate equation, finite speed of propagations]. Case of $\sigma(u)$ degenerate ($\sigma(0) = 0$).

▶New formulation:

$$\varphi(s) = \int_0^s \kappa(\tau) d\tau$$

$$\begin{cases} u_t - \varphi(u)_{xx} = \sigma(u)(v_x)^2 & \text{in} \quad \Omega \times (0,T), \\ (\sigma(u)v_x)_x = 0 & \text{in} \quad \Omega \times (0,T), \\ v = v_D,\ \varphi(u) = \varphi(u_D) \geq 0 & \text{on} \quad \Gamma_D \times (0,T), \\ \sigma(u)\frac{\partial v}{\partial n} = 0,\ \frac{\partial \varphi(u)}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{on} \quad \Omega. \end{cases} \tag{2}$$

▶No classical solution does not exist in general if $\kappa(0) = 0$. Quadratic growth of the right hand side.

▶**General assumptions**

$$\sigma \quad \text{is Lipschitz continuous,} \tag{3}$$

$$\begin{cases} \text{there exists a bounded strictly increasing function } \sigma_0(u), \text{ with } \sigma_0(0) \geq 0 \text{ and } \sigma_1 > 0 \\ \text{such that } \sigma_0(u) \leq \sigma(u) < \sigma_1 \qquad \forall\ u \geq 0, \end{cases} \tag{4}$$

$$\varphi \in C^1([0,+\infty)) \cap C^2((0,+\infty)), \tag{5}$$

$$\varphi'(0) \geq 0, \qquad \varphi'(r) > 0 \quad \forall\, r > 0, \tag{6}$$

$$\left\{ \begin{array}{c} \text{there exists } V_D \in L^\infty((0\ T); H^1(\Omega)) \text{ such that} \\ V_D = v_D \text{ on } \Gamma_D \times (0,T) \text{ and } \frac{\partial V_D}{\partial n} = 0 \text{ on } \Gamma_N \times (0,T), \end{array} \right. \tag{7}$$

$$\left\{ \begin{array}{c} \text{there exists } U_D \text{ such that } \varphi(U_D) \in H^1((0\ T); H^1(\Omega)) \\ \varphi(U_D) = \varphi(u_D) \geq 0 \text{ on } \Gamma_D \times (0,T) \text{ and } \frac{\partial \varphi(U_D)}{\partial n} = 0 \text{ on } \Gamma_N \times (0,T), \end{array} \right. \tag{8}$$

$$u_0 \in L^\infty(\Omega),\ 0 \leq u_0 \leq M, \tag{9}$$

▶Notice that the Wiedemann-Franz law and the assumption (4) imply

$$k_0 \int_0^u \sigma_0(s) s ds \leq \varphi(u) < \widetilde{C} u^2 \qquad \forall\, u \geq 0, \tag{10}$$

with $\widetilde{C} = \frac{k_0 \sigma_1}{2}$.

▶Our existence result will require the additional condition

$$\left\{ \begin{array}{c} \sigma_0(0) > 0 \\ \text{or} \\ \varphi(u)^\alpha \leq \sigma_0(u) \text{ for any } u \in [0,\delta],\ \text{for some } \alpha \in (0,1) \text{ and } \delta > 0. \end{array} \right. \tag{11}$$

▶The great generality allowed on $\sigma(u)$ requires to spend some words on the way in which the boundary conditions are satisfied. We shall show that $\sigma(u)v_x \in L^\infty(Q)$ and that $\varphi(u(.,t))$ is continuous. Then the assumption

$$\sigma(u_D(x,t)) > 0 \text{ on } \Gamma_D \times [0,T] \tag{12}$$

implies that the trace of v on $\Gamma_D \times (0,T)$ is well defined.
▶It turns out that a function which plays a crucial role in the study of the system is the function

$$J := \sigma(u)v_x,$$

which corresponds to *the current density*. Notice that the second equation of (1) implies that J is independent of x, i.e., for a.e. $t \in (0,T)$

$$\sigma(u(x,t))v_x(x,t) = J(t) \text{ for a.e. } x \in \Omega. \tag{13}$$

▶Since the first equation can be, equivalently written as

$$u_t - \varphi(u)_{xx} = Jv_x \text{ in } \Omega \times (0,T),$$

if $J(t) \equiv 0$ on some subinterval $(t_{1,}t_2) \subset (0,T)$ then the equations of system (1) are not coupled on $\Omega \times (t_{1,}t_2)$.

▶Notice also that $J(t) \equiv 0$ on $(0, T)$ if $inf_{x\in\Omega} |v_x(x,t)| = 0$ (case, for instance, of $\Gamma_N \neq \phi$) or $\min_{x\in\overline{\Omega}} \sigma(u(x,t)) = 0$ (case, for instance, of $\Gamma_D \neq \phi$, $\sigma_0(0) = 0$ and $u_D(t,x) = 0$).
▶Moreover, if $\min_{x\in\overline{\Omega}} \sigma(u(x,t)) > 0$ we have

$$v_x(x,t) = \frac{J(t)}{\sigma(u(x,t))} \text{ a.e. } x \in \Omega.$$

Then, a simple integration shows that, for a.e. $t \in (0, T)$

$$v(L,t) - v(-L,t) = J(t) \int_\Omega \frac{dx}{\sigma(u(x,t))}, \tag{14}$$

which will play an important role in our proof of the existence of solutions and also can be understood as a weak sense in which the Dirichlet condition holds (notice that if $J(t) \equiv 0$ and, both, $\Gamma_N \neq \phi$ and $\Gamma_D \neq \phi$ then, necessarily, $v(x,t) = v_D(x,t)$ on $\Gamma_N \times (0,T)$).
▶Notice also that if $J(t) = 0$ and $\sigma(u(x,t)) > 0$ for any $x \in \Omega$, we get that $v_x \equiv 0$. Finally, if $\Gamma_N = \phi$ as $\int_\Omega \sigma(u(x,t))dx > 0$ for a.e. $t \in (0,T)$,we get from (14) that $J(t) = 0$ (respectively $J(t) > 0$ or $J(t) < 0$) if and only if $v_D(L,t) - v_D(-L,t) = 0$ (respectively $v_D(L,t) - v_D(-L,t) > 0$ or $v_D(L,t) - v_D(-L,t) < 0$).

▶The uniqueness of a weak solution will be obtained for the cases in which $\Gamma_D = \phi$ or $u_D(t,x) > 0$ on $\Gamma_D \times (0,T)$ (notice that the possible vanishing of u_0 maintains the degenerate character to the parabolic equation).

▶The last section of the paper is devoted to the study of a qualitative property which is peculiar to the case $\varphi'(0) = 0$. It concerns with the occurrence of a *free boundary* (given by the boundary of the support of u).

▶When $\sigma_0(0) > 0$ the vanishing set of the solution can be reduced to some curves in Q.

▶Nevertheless, if $\sigma_0(0) = 0$, we show the, so called, *finite speed of propagations* property: if $u_0(x) = 0$ on $B_{\rho_0}(x_0) := (x_0 - \rho_0, x_0 + \rho_0)$ for some $x_0 \in \Omega$ and $\rho_0 > 0$ then there exists $t^* > 0$ and a function $\rho(t) : [0, t^*) \mapsto [0, \infty)$, with $\rho(0) \leq \rho_0$, such that $u(x,t) = 0$ a.e. in $B_{\rho(t)}(x_0), \forall\, t \in [0, t^*)$.

▶This result opens the possibility of further studies on the properties (and regularity) on this free boundary.

2 Existence of a weak solution

Assumed (12), by a **weak solution** to problem (2) we mean a couple of functions (u,v) such that

$$\varphi(u)\in L^2(0,T;H^1(\Omega)), u\geq 0, u\in C([0,T];L^1(\Omega))\cap L^\infty(Q),$$

(15)

$$v\in L^\infty(Q), \tag{16}$$

$$\sigma(u)v_x\in L^1(0,T;L^1(\Omega)), \sigma(u)\left|v_x\right|^2\in L^1(0,T;L^1(\Omega)), \tag{17}$$

the boundary conditions $v=v_D$, $\varphi(u)=\varphi(u_D)$ and $\sigma(u)\frac{\partial v}{\partial n}=0,\frac{\partial\varphi(u)}{\partial n}=0$ hold on $\Gamma_D\times(0,T)$ and $\Gamma_N\times(0,T)$ respectively, $u(.,0)=u_0$ in $L^1(\Omega)$ and

$$\int_\Omega u(x,T)\xi(x,T)dx-\int_\Omega u_0(x)\,\xi(x,0)dx=\int_0^T\int_\Omega u\,\xi_t dtdx \\ -\int_0^T\int_\Omega \varphi(u)_x\xi_x dtdx-\int_0^T\int_\Omega \sigma(u)\left|v_x\right|^2\xi dtdx,$$

(18)

$$\int_\Omega \sigma(u) v_x \zeta_x dx = 0 \text{ a.e. } t \in (0,T),$$

(19)

for all $\xi, \zeta \in C^1(\overline{Q})$ such that $\xi(x,t), \zeta(x,t) = 0$ on $\Gamma_D \times (0,T)$..
Theorem 1. *Under the assumption (12) there exists, at least, one weak solution to the problem (2). Moreover,* $J(t){:=}\sigma(u(x,t))v_x(x,t)$ *is a bounded (constant in* x*) function on* $(0,T)$ *and if* $\sigma_0(0) > 0$ *then* $v_x \in L^\infty(Q)$.
Proof. We can always assume that

$$v_D(L,t) \neq v_D(-L,t) \text{ a.e. } t \in (0,T). \tag{20}$$

Indeed, otherwise, as pointed out at the Introduction, $J(t) \equiv 0$ on $(0,T)$ and the system is reduced to two uncoupled equations for which the existence of solutions is well-known in the literature.

▶Notice that the same appears if $\Gamma_N = \phi$, nevertheless we shall not assume this condition in the rest of the proof in order to recall an approximation argument which will be used in the proof of the uniqueness.
▶The process of proof consists in three steps.
▶*Step 1: Approximation.* The method consists in approximating the solution (u, v) by (u^ϵ, v^ϵ) the solution of

$$\begin{cases} u_t^\epsilon - \varphi(u^\epsilon)_{xx} = \sigma(u^\epsilon)(v_x^\epsilon)^2 & \text{in} \quad \Omega \times (0,T), \\ (\sigma(u^\epsilon)v_x^\epsilon)_x = 0 & \text{in} \quad \Omega \times (0,T), \\ v^\epsilon = v_D,\ \varphi(u^\epsilon) = \varphi(\max(u_D, \epsilon)) & \text{on} \quad \Gamma_D \times (0,T), \\ \frac{\partial v^\epsilon}{\partial n} = 0,\ \frac{\partial \varphi(u^\epsilon)}{\partial n} = 0 & \text{on} \quad \Gamma_N \times (0,T), \\ u^\epsilon(.,0) = u_0 + \epsilon & \text{on} \quad \Omega. \end{cases} \tag{21}$$

▶As $u_t^\epsilon - \varphi(u^\epsilon)_{xx} \geq 0$, we get

$$u^\epsilon \geq \epsilon \text{ a.e. on } \Omega \times (0,T). \tag{22}$$

Thus, $\varphi'(u^\epsilon) > 0$, the operator is, now, **uniformly parabolic** and so a solution (u^ϵ, v^ϵ) to problem (21) is known to exist (see, e.g., Cimatti 1988).
Step 2: A priori estimates. We show that $\sigma(u^\epsilon)(v_x^\epsilon)^2$ (respectively u^ϵ and $\sigma(u)v_x \in L^\infty(Q)$) is bounded in $L^1(Q)$ (respectively in $L^\infty(Q)$) independently of ϵ. [For that,

multiply the equation of v^ϵ in problem (21) by $v^\epsilon - V_D$, integrate by parts, applying the Cauchy-Schwarz inequality,

▶ $\sigma(u^\epsilon)(v_x^\epsilon)^2$ bounded in $L^\infty(0,T : L^1(\Omega))$ implies that $u_t^\epsilon - \varphi(u^\epsilon)_{xx} = f^\epsilon(t,x)$ with f^ϵ uniformly bounded in $L^1(Q)$ and so we have from Kawanago (1993) u^ϵ is uniformly bounded in $L^\infty(Q)$.

▶On the other hand, from the equation of v^ϵ we have

$$\sigma(u^\epsilon)v_x^\epsilon = J^\epsilon(t)$$

and hence

$$\frac{|J^\epsilon(t)|}{\sigma_1} \leq |v_x^\epsilon| \tag{23}$$

Plugging this into (??) we obtain

$$|J^\epsilon(t)|^2 \leq C(T)^2\sigma_1^2 ess \sup_{t\in[0,T]} \left(\frac{1}{\int_\Omega \sigma(u^\epsilon(x,t)dx}\right) \tag{24}$$

where $C(T)$ denotes some constant independent of ϵ and so

$$|J^\epsilon(t)| \leq C^*(T) \tag{25}$$

for some positive constant independent of ϵ.

►It is easy to get a $L^\infty(Q)$ a priori estimate on v^ϵ since, if $\Gamma_D \neq \phi$ then, by the maximum principle,

$$|v^\epsilon(x,t)| \leq \|v_D\|_{L^\infty(Q)}, \text{ for a.e. } x \in \Omega \text{ and any } t \in [0,T].$$

In the case $\Gamma_D = \phi$ the function $v_x^\epsilon(x,t) = 0$ and since v^ϵ is determined up a constant we can take $v^\epsilon(x,t) = 0$ for a.e. $x \in \Omega$ and any $t \in [0,T]$.
►If $J_0(t) \neq 0$ for any $t \in [0,T]$

$$\int_0^T \int_\Omega |v_x^\epsilon|\, dxdt \leq \frac{\int_0^T \int_\Omega \sigma(u^\epsilon)(v_x^\epsilon)^2 dxdt}{\min_{t\in[0,T]} |J^\epsilon(t)|} \leq C(T). \tag{26}$$

To get other *a priori* estimates we see that by multiplying the equation of u^ϵ in problem (21) by $\varphi(u^\epsilon) - \varphi(\max(U_D,\epsilon)) \in L^2(0,T;H^1(\Omega))$ and integrating over Ω, we get that if

$$B(s) = \int_0^s \varphi(u)du$$

and integrating over $(0,t)$ we obtain for some new constants $\widehat{C}(T)$ and

$$\int_\Omega B(u^\epsilon(x,t))dx + \frac{1}{2}\int_0^t \int_\Omega (\varphi(u^\epsilon)_x)^2 dt dx \tag{27}$$
$$\leq \widehat{C}(T) + \int_\Omega B(u_0)dx,$$

►Then, since $B(s) \geq 0$ for $s \geq 0$, there exists some constant $C = C(T)$, independent of ϵ, such that

$$\int_\Omega B(u^\epsilon(.,t)) \leq C(T) \; \forall t \in (0,T),$$

$$||\varphi(u^\epsilon)||_{L^2(0,T;H^1(\Omega))} \leq C(T),$$

and (from the equation of u^ϵ)

$$||u_t^\epsilon||_{L^2(0,T;H^{-1}(\Omega))} \leq C(T).$$

Moreover, since

$$\varphi(u^\epsilon)_t = \varphi'(u^\epsilon)u_t^\epsilon,$$

it follows that

$$||\varphi(u^\epsilon)_t||_{L^2(0,T;H^{-1}(\Omega))} \leq C(T).$$

Step 3 : Passage to the limit. Using a classical compactness argument, (see [30]) and the monotonicity of φ, we can extract a "subsequence" that for simplicity we still label by "ϵ" such that

$$\varphi(u^\epsilon) \rightharpoonup l_1 \text{ in } L^2(0,T;H^1(\Omega)), \tag{28}$$

$$\varphi(u^\epsilon) \to l_1 \text{ in } L^2(Q), \tag{29}$$

$$u^\epsilon \to u \text{ in } L^\infty(Q), \tag{30}$$

$$(u^\epsilon)_t \rightharpoonup l_2 \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)), \tag{31}$$

$$J^\epsilon(t) \rightharpoonup J(t) \text{ weakly-star in } L^\infty(0,T), \tag{32}$$

$$v^\epsilon \rightharpoonup v \text{ weakly-star in } L^\infty(0,T:L^\infty(\Omega)), \tag{33}$$

$$\sigma(u^\epsilon)\left(v_x^\epsilon\right)^2 = J^\epsilon(t)v_x^\epsilon \rightharpoonup l_3 \text{ weakly-star in } L^\infty(0,T:L^1(\Omega)). \tag{34}$$

►Clearly, one deduces that $l_1 = \varphi(u)$, $l_2 = u_t$, . In order to prove the regularity $u \in C([0,T];L^1(\Omega))$ it suffices to multiply the equation of u^ϵ by $sign(\varphi(u^\epsilon))$. Then,

$$\frac{d}{dt}\int_\Omega |u^\epsilon(x,t)|\,dx \leq \int_\Omega \left|\sigma(u^\epsilon(x,t))(v_x^\epsilon(x,t))^2\right|dx$$

and in the limit

$$\frac{d}{dt}\int_\Omega |u(x,t)|\,dx \leq \int_\Omega |l_3(x,t)|\,dx$$

which proves that $u \in C([0,T]; L^1(\Omega))$.

▶Since $\min_{x\in\overline{\Omega}} \sigma(u^\epsilon(x,t)) > 0$, by using the identity

$$v_D(L,t) - v_D(-L,t) = J^\epsilon(t) \int_\Omega \frac{dx}{\sigma(u^\epsilon(x,t))}, \tag{35}$$

we deduce that, for a.e. $t \in (0,T)$,

$$J^\epsilon(t) \to J(t) \text{ in } \mathbb{R} \tag{36}$$

since for a.e. $t \in (0,T)$

$$\frac{v_D(L,t) - v_D(-L,t)}{\int_\Omega \frac{dx}{\sigma(u^\epsilon(x,t))}} = J^\epsilon(t), \tag{37}$$

$\sigma(u^\epsilon(x,t)) \to \sigma(u(x,t))$ for any $x \in \Omega$ (recall that $\varphi(u^\epsilon) \rightharpoonup \varphi(u)$ in $L^2(0,T;H^1(\Omega))$ implies that $\varphi(u^\epsilon(.,t)) \to \varphi(u(.,t))$ in $C(\overline{\Omega})$ for a.e. $t \in (0,T)$) and notice that if $\min_{x\in\overline{\Omega}} \sigma(u(x,t)) = 0$ then $J(t) = 0$ and (36) is reduced to $J^\epsilon(t) \to 0$ in $\mathbb{R}$.

▶Thus,we conclude that for any $\xi \in C^1(\overline{Q})$ such that $\xi(x,t) = 0$ on $\Gamma_D \times (0,T)$

$$\begin{aligned}\int_\Omega \sigma(u^\epsilon(x,t)) \left|v_x^\epsilon(x,t)\right|^2 \xi(x,t)dx &= J^\epsilon(t) \int_\Omega v_x^\epsilon(x,t)\xi(x,t)dx \\ &= -J^\epsilon(t) \int_\Omega v^\epsilon(x,t)\xi_x(x,t)dx.\end{aligned}$$

Using (33), (34) and (36) we can pass to the limit to deduce that

$$\int_{\Omega} \sigma(u^{\epsilon}(x,t)) \left|v_x^{\epsilon}(x,t)\right|^2 \xi(x,t)dx \to -J(t) \int_{\Omega} v(x,t)\xi_x(x,t)dx = \int_{\Omega} l_3(x,t)\xi(x,t)dx.$$

▶Then, we can pass to the limit in the boundary conditions and in the equations to get that

$$\begin{array}{c} \int_{\Omega} u(x,T)\xi(x,T)dx - \int_{\Omega} u_0(x)\xi(x,0)dx - \int_0^T \int_{\Omega} u\xi_t dtdx \\ + \int_0^T \int_{\Omega} \varphi(u)_x \xi_x dtdx = \int_0^T \int_{\Omega} \sigma(u)(v_x)^2 \xi dtdx, \end{array} \tag{38}$$

$$\int_{\Omega} \sigma(u) v_x \zeta_x dx = 0 \text{ a.e. } t \in (0,T), \tag{39}$$

to get the existence result.■

3 Uniqueness of solutions

▶Our main idea will consist in proving that any possible weak solution must coincide with the solution constructed in the previous section by using a method that, coming from the Holmgren duality method, it was first adapted to degenerate equations by A.S. Kalashnikov (1979) and then refined in Díaz-Kersner (1993)[16].

Theorem 2. *Assume* $\Gamma_D = \phi$ *or* $u_D(t,x) > 0$ *on* $\Gamma_D \times (0,T)$. *Then problem (2) has a unique weak solution* (u,v) *such that* $v \in L^1(0,T;W^{1,1}(\Omega))$ *(*$v(.,t)$ *being univocally determined in* Ω *unless a constant in the case of* $\Gamma_D = \phi$ *and arbitrary on the set* $\{(x,t) \in Q,\, u(t,x) = 0\}$*).*

▶Before giving the proof of this theorem let us introduce some notation. Let (u^ϵ, v^ϵ) be as before. Let (w,z) be any weak solution to problem (2). By subtracting and using that

$$\int_0^T \int_\Omega (\sigma(w)\,(z_x)^2\,\xi dtdx = \int_0^T \int_\Omega ((\sigma(w) z z_x)_x \xi dtdx = -\int_0^T \int_\Omega \sigma(w) z_x z \xi_x dtdx,$$

which is justified since $\sigma(w)z_x \in L^1(Q)$ and $z \in L^\infty(Q)$, we have

$$\int_\Omega (w-u^\epsilon)(x,T)\xi(x,T)dx - \int_\Omega (w-u^\epsilon)(x,0)\xi(x,0)dx = \int_0^T \int_\Omega (w-u^\epsilon)\xi_t dtdx$$
$$+\int_0^T \int_\Omega (\varphi(w)-\varphi(u^\epsilon))\xi_{xx} dtdx - \int_0^T \int_\Omega (\sigma(w) z z_x - \sigma(u^\epsilon) v^\epsilon v^\epsilon_x)\xi_x dtdx$$
$$+\int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t)))\frac{\partial \xi}{\partial n}(s,t)dtds + \int_0^T \int_\Omega (\sigma(w)z_x - \sigma(u^\epsilon)v^\epsilon_x)\zeta_x dtdx \tag{40}$$

for all $\xi, \zeta \in C^1(\overline{Q}) \cap C([0,T] : C^2(\overline{\Omega}))$ such that $\xi(x,t), \zeta(x,t) = 0$ on $\Gamma_D \times (0,T)$.

▶Here the term $\int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t))) \frac{\partial \xi}{\partial n}(s,t) dt ds$ must be understood in the usual onedimensional integration by parts sense. So if, for instance, $\Gamma_D = \{-L\} \cup \{L\}$ we have that

$$\int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t))) \frac{\partial \xi}{\partial n}(s,t) dt ds$$

$$= \int_0^T [(\varphi(\max(U_D(L,t),\epsilon)) - \varphi(u_D(L,t)))\xi_x(L,t) - (\varphi(\max(U_D(-L,t),\epsilon)) - \varphi(u_D(-L,$$

▶Let us denote by I the left hand side of (40). Thus

$$I = \int_0^T \int_\Omega (w - u^\epsilon)(\xi_t + \frac{\varphi(w) - \varphi(u^\epsilon)}{w - u^\epsilon} \xi_{xx} - \frac{\sigma(w) - \sigma(u^\epsilon)}{w - u^\epsilon} z z_x \xi_x +$$

$$+ \frac{\sigma(w) - \sigma(u^\epsilon)}{w - u^\epsilon} z_x \zeta_x) dt dx + \int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t))) \frac{\partial \xi}{\partial n}(s,t) dt ds$$

$$- \int_0^T \int_\Omega (z - v^\epsilon) \left(\sigma(u^\epsilon) z_x \xi_x - (\sigma(u^\epsilon) v^\epsilon \xi_x)_x + (\sigma(u^\epsilon)\zeta_x)_x\right) dt dx. \qquad (41)$$

▶Let us set

$$A_\epsilon = A_\epsilon(x,t) = \frac{\varphi(w)-\varphi(u^\epsilon)}{w-u^\epsilon}, \qquad B_\epsilon = B_\epsilon(x,t) = \frac{\sigma(w)-\sigma(u^\epsilon)}{w-u^\epsilon} z z_x \tag{42}$$

$$C_\epsilon = C_\epsilon(x,t) = \frac{\sigma(w)-\sigma(u^\epsilon)}{w-u^\epsilon} z_x \;, \qquad D_\epsilon = D_\epsilon(x,t) = \sigma(u^\epsilon) z_x \tag{43}$$

$$E_\epsilon = E_\epsilon(x,t) = \sigma(u^\epsilon)v^\epsilon, \qquad F_\epsilon = F_\epsilon(x,t) = \sigma(u^\epsilon). \tag{44}$$

Thus (41) reads now :

$$I = \int_0^T \int_\Omega (w-u^\epsilon)\{\xi_t + A_\epsilon \xi_{xx} - B_\epsilon \xi_x + C_\epsilon \zeta_x\} dt dx + \int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t))$$

$$- \int_0^T \int_\Omega (z - v^\epsilon)\{D_\epsilon \xi_x - (E_\epsilon \xi_x)_x + (F_\epsilon \zeta_x)_x\} dt dx.$$

Lemma 1. *There exist three positive constants* m_ϵ , M_ϵ *and M* (M* independent of* ϵ*) such that* $m_\epsilon \leq A_\epsilon(x,t) \leq M_\epsilon \quad \forall (x,t) \in Q$, $|B_\epsilon(x,t)|,\ |C_\epsilon(x,t)| \leq M^* \quad \forall (x,t) \in Q.$ ■

▶Assume that we extend $A_\epsilon,\ B_\epsilon,\ C_\epsilon$ to the whole $\mathbb{R}^2$ respectively by $m_\epsilon, 0, 0$ and denote again these extensions by $A_\epsilon,\ B_\epsilon,\ C_\epsilon$. Let ρ be a function of class C^∞ with support in the ball $B(0,1)$ of center $\mathbf{0}$ and radius 1 of $\mathbb{R}^2$ and such that

$$\int_{B(0,1)} \rho dtdx = 1.$$

Set

$$\rho_n(x,t) = n^2 \rho(nx, nt)$$

and

$$A_\epsilon^n = \rho_n * A_\epsilon, \quad B_\epsilon^n = \rho_n * B_\epsilon, \quad C_\epsilon^n = \rho_n * C_\epsilon,$$

where $*$ denotes the usual convolution of functions. Clearly, these functions are of class C^∞ in $\mathbb{R}^2$. Moreover, one has

$$m_\epsilon \leq A_\epsilon^n(x,t) \leq M_\epsilon \quad \forall (x,t) \in Q, \forall n \tag{45}$$

$$|B_\epsilon^n|, \ \ |C_\epsilon^n| \leq \ M^* \quad \forall (x,t) \in Q, \ \forall n. \tag{46}$$

Thus, equation (40) reads now

$$I = \int_0^T \int_\Omega (w - u^\epsilon)\{\xi_t + A_\epsilon^n \xi_{xx} - B_\epsilon^n \xi_x + C_\epsilon^n \zeta_x\} dtdx +$$
$$\int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t), \epsilon)) - \varphi(u_D(s,t))) \frac{\partial \xi}{\partial n}(s,t) dtds$$

$$+\int_0^T\int_\Omega (w-u^\epsilon)\xi_{xx}(A_\epsilon - A_\epsilon^n)dtdx - \int_0^T\int_\Omega (w-u^\epsilon)\xi_x(B_\epsilon - B_\epsilon^n)dtdx$$

$$+\int_0^T\int_\Omega (w-u^\epsilon)\zeta_x(C_\epsilon - C_\epsilon^n)dtdx - \int_0^T\int_\Omega (z-v^\epsilon)\{D_\epsilon\xi_x - (E_\epsilon\xi_x)_x + (F_\epsilon\zeta_x)_x\}dtdx. \tag{47}$$

A similar argument must be applied if the coefficients D_ϵ, E_ϵ and F_ϵ are not bounded (we leave the details to the reader).

Now we construct a "dual system" which plays a crucial role in the proof of Theorem 2.

Lemma 2. *There exists a unique smooth solution* $(\xi,\zeta) = (\xi_\epsilon^{n,m}, \zeta_\epsilon^{n,m})$ *to the system*

$$\begin{cases} \xi_t + A_\epsilon^n\xi_{xx} - B_\epsilon^n\xi_x + C_\epsilon^n\zeta_x = 0 & \text{in} \quad Q, \\ -(F_\epsilon\zeta_x)_x = (E_\epsilon\xi_x)_x - D_\epsilon\xi_x & \text{in} \quad Q, \\ \zeta = 0, \quad \xi = 0 & \text{on} \quad \Gamma_D\times(0,T), \\ \frac{\partial\zeta}{\partial n} = 0,\ \frac{\partial\xi}{\partial n} = 0 & \text{on} \quad \Gamma_N\times(0,T), \\ \xi(.,T) = w^m & \text{on} \quad (0,L), \end{cases} \tag{48}$$

where $w^m \in C_0^\infty(\Omega)$ *is such that* $|w^m(x)| \le 1$ for any $x \in (-L,L)$ and

$$w^m \to sign(w(x,T) - u^\epsilon(x,T)) \quad \text{in} \quad L^2(\Omega), \text{when } m\to\infty \tag{49}$$

(here $sign$ denotes the sign$_0$ function, i.e., $sign(x) = x/|x|$ if $x \neq 0$, and 0 if $x = 0$).

Next, we show some estimates which we shall need later.

Lemma 3. *Let $(\xi_\epsilon^{n,m}, \zeta_\epsilon^{n,m})$ be the solution to problem (48). Then, there exists a constant C_ϵ independent of n and m such that*

$$\left\|\xi^{n,m}_{\epsilon,xx}\right\|_{2,Q}, \quad \left\|\xi^{n,m}_{\epsilon,x}\right\|_{2,Q}, \quad \left\|\zeta^{n,m}_{\epsilon,x}\right\|_{2,Q} \leq C_\epsilon \tag{50}$$

($\|\|_{2,Q}$ denotes the usual L^2-norm on $L^2(Q)$).

Proof of Theorem 2. In (40), (47) choose $(\xi, \zeta) = (\xi_\epsilon^{n,m}, \zeta_\epsilon^{n,m})$ solution to (48) where w^m satisfies (49). Then expressions (40) and (47) leads to

$$\int_\Omega (w - u^\epsilon)(x,T) w^m(x) dx - \int_\Omega (w - u^\epsilon)(x,0)\xi(x,0) dx \tag{51}$$

$$= \int_0^T \int_{\Gamma_D} (\varphi(\max(U_D(s,t),\epsilon)) - \varphi(u_D(s,t))) \frac{\partial \xi}{\partial n}(s,t) dt ds$$

$$+ \int_0^T \int_\Omega (w - u_\epsilon)\xi_{xx}(A_\epsilon - A_\epsilon^n) dx dt \tag{52}$$

$$- \int_0^T \int_\Omega (w - u_\epsilon)\xi_x(B_\epsilon - B_\epsilon^n) dx dt + \int_0^T \int_\Omega (w - u_\epsilon)\zeta_x(C_\epsilon - C_\epsilon^n) dx dt.$$

Notice that the assumption $u_D > 0$ implies that for $\epsilon > 0$ small enough we get that $\max(U_D(s,t),\epsilon)) = u_D(s,t))$ and so the first term of the right hand side disappears (this is also the case of $\Gamma_D = \phi$). Then, by passing to the limit (first in $\varepsilon \to 0$, then in $n \to \infty$ and finally in $m \to \infty$) we get that $\int_\Omega |w(x,T) - u^\epsilon(x,T)|\, dx = 0$. Since T is arbitrary we get that $w \equiv u^\epsilon$ and then, obviously, $z \equiv v$ on Q if $\Gamma_D \neq \phi$. ▶When $\Gamma_D = \phi$ we deduce that, for any $t \in [0.T]$ there exists a constant $C(t)$ such that $z(.,t) - v(.,t) = C(t)$ on Ω. On the set $\{(x,t) \in Q,\, u(t,x) = 0\}$ z and v may be different without any consequence on the rest of points of Q.■

Remark 2. The case $u_D = 0$ is more delicate since the first term of the right hand side of (51) becomes

$$\int_0^T \int_{\Gamma_D} \varphi(\epsilon) \frac{\partial \xi}{\partial n}(s,t) dt ds$$

and the passing to the limit requires sharper estimates obtained under additional assumptions (see, for instance, Díaz-Kersner 1993 for the case of a single scalar equation). Nevertheless, we conjecture that, as in the scalar case the uniqueness of weak solutions holds also for $u_D = 0$ and general functions φ.

4 On the existence of the free boundary

►The assumption $\varphi'(0)=0$ and a suitable growth assumption lead to the existence of a free boundary given as the boundary of the support of the solution. It is the *finite speed of propagation property*: if $u_0(x)=0$ on $B_{\rho_0}(x_0):=(x_0-\rho_0,x_0+\rho_0)$ for some $x_0\in\Omega$ and $\rho_0>0$ then there exists $t^*>0$ and a function $\rho(t):[0,t^*)\mapsto[0,\infty)$, with $\rho(0)\leq\rho_0$, such that $u(x,t)=0$ a.e. in $B_{\rho(t)}(x_0),\forall\, t\in[0,t^*)$.

►When $\Gamma_N\neq\phi$ we know that the system becomes uncoupled (see the Introduction) and, so, the criterium for the finite speed of propagation is well known (see, e.g., the surveys Kalashnikov (1987) and Antontsev-Díaz-Shmarev 2002).

►Nevertheless, if $\Gamma_N=\phi$ and $\sigma_0(0)>0$ the vanishing set of the solution can be reduced (at most) to some curves in Q since, if we assume that $u(.,t)$ is a convex function of x then

$$u_t\geq\sigma_0(v_x)^2$$

and thus

$$u(x,t)\geq\sigma_0\int_0^t v_x(x,s)^2ds+u_0(x).$$

Then $\int_0^t v_x(x_0,s)^2ds>0$ implies that $u(x_0,t)>0$. Notice also that, by the strong maximum principle, $v_x(x,.)$ can not be zero on a subset of Ω of positive measure (for any fixed $t \in [0,T]$).

▶On the other hand, if $\sigma(u(x,t))\,(v_x(x,t))^2 > 0$ on Q, it is impossible to get solutions $u(x,t)$ vanishing on an open subset ω of Q since we would reach a contradiction on ω trough the equation of (1).

▶The case $\sigma_0(0) = 0$ (and $\Gamma_N = \phi$) is different. More precisely we have:

Theorem 3. *Assume φ satisfying*

$$\int_{0^+} \frac{\varphi'(s)}{s} ds < \infty, \tag{53}$$

$\sigma_0(0) = 0$ *and* $\Gamma_N = \phi$. *Then, if suppu_0 is a non empty compact subset of Ω the same happens with supp$u(.,t)$ for any $t \in [0,t^*)$, for some $t^* \in (0,T]$. Moreover, if $t^* < T$ then $u(x,t) > 0$ for any $t \in (t^*,T]$.*

Proof. Consider w as the solution of the scalar homogeneous problem

$$\begin{cases} w_t - \varphi(w)_{xx} = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(w) = \varphi(u_D) \geq 0 & \text{on} \ \ \Gamma_D \times (0,T), \\ w(x,0) = u_0(x) \geq 0 & \text{on} \qquad\qquad \Omega. \end{cases} \tag{54}$$

(remember that now $\Gamma_D = \partial\Omega$). Thanks to the assumption (53) we know that there exists $t^* \in (0,T]$ such that supp$w(.,t)$ is a compact subset of the open set Ω for any $t \in [0, t^*)$ and that if $t^* < T$ then $w(x, t) > 0$ for any $t \in (t^*, T]$. It is easy to see that, necessarily, $w(., t)$ must coincide with $u(., t)$ for any $t \in [0, t^*)$. Indeed, as $\sigma(u(x, t))v_x(x, t)$ must be a constant (in x) $J(t)$ we get that, necessarily $J(t) = 0$ if $\sigma(u(x_0, t)) = 0$ for some $x_0 \in \Omega$. Then, as $\sigma(w(x_0, t)) = 0$, for some $x_0 \in \Omega$ if $t \in [0, t^*)$, we can take $v = z$ as the unique function solution of

$$\begin{cases} z_x = 0 & \text{in} \quad \Omega \times (0, t^*), \\ z = v_D, & \text{on} \quad \Gamma_D \times (0, t^*), \\ \frac{\partial z}{\partial n} = 0, & \text{on} \quad \Gamma_N \times (0, t^*), \end{cases}$$

and we get that (w, z) satisfies problem (1) on $[0, t^*)$ (notice that $\sigma(w(x, t))(z_x(x, t))^2 = J(t)z_x(x, t) = 0$ on $\Omega \times (0, t^*)$). By the uniqueness of solutions for problem (1) we conclude that $(u, v) = (w, z)$ on $\Omega \times [0, t^*)$. Moreover, as $\sigma(u(x, t))(v_x(x, t))^2 \geq 0$ on $\Omega \times (0, T)$ we conclude (by the maximum principle for problem (54)) that $u(x, t) \geq w(x, t) \geq 0$ on $\Omega \times [0, T]$ and then $u(x, t) > 0$ on $\Omega \times (t^*, T]$.■

Remark 4. Notice that assumption (53) holds under the Wiedemann-Franz law $k(u) = k_0 u \sigma(u)$ (remember (10)).

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