Rearrangement techniques
(and finite speed of propagation)
for a chemotaxis system

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1. Introduction

Two main goals concerning

$$\begin{cases} u_t = \operatorname{div}(\nabla u^m - \chi u^p \nabla u), & x \in \mathbb{R}^N,\ t > 0, \\ -\Delta v + \gamma v = \alpha u & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N. \end{cases}$$

$m > 0,\ \alpha > 0.$

A. New symmetrization results for the system for any m>0 and p=1 **(Section2).**

Previous results (for m=1):

J. I. Díaz, T. Nagai. Symmetrization in a parabolic-elliptic system related to chemotaxis. *Advances in Mathematical Sciences and Applications*, Vol.5, No 2, 659-680,1995. (bounded domain and Dirichlet conditions for u)

J. I. Díaz, T. Nagai, J. M. Rakotoson. Symmetrization techniques on unboubded domains: application to a Chemotaxis system on R^N. *Journal of Differential Equations*, 145, No.1, 156-183.1998

T. Nagai, Global existence and decay estimates of solutions to a parabolic-elliptic system of drift-diffusion type in R^2. To appear in Diff. Integral Eqs.

B. *Finite speed of propagation if m>1 (p>0 arbitrary) .*

Previous results:

Y. Sugiyama, Finite speed of propagation in 1-D degenerate Keller-Segel system, to appear in Math. Nachr.

Y. Sugiyama, Interfaces for 1-D degenerate Keller-Segel systems, J. Evol. Equ., 9, 123-142, 2009.

N-dimensional *Energy Methods:*

S. Antontsev, J. I. Díaz, S. Shmarev*: Energy methods for free boundary problems. Applications to nonlinear PDEs and Fluid Mechanics*, Birkäuser, Boston, 2002,

J. I. Díaz, G. Galiano, A. Jüngel, On a quasilinear degenerate system arising in semiconductor theory. Part II: Localization of vacuum solutions, Nonlinear Analysis, 36, 569-594, 1999.

2. New symmetrization results

2.1. On the general method.

2.2. The semilinear case.

2.3. The quasilinear case.

2.1. On the general method.

Isoperimetric inequality

Let $E \subset \mathbb{R}^n$ be a measuble set of finite measure, then:

$$n\omega_n^{1/n}|E|^{1-1/n} \leq Per(E).$$

Equality holds true if and only if E is a ball.
(ω_n denotes the measure of the unit ball in $\mathbb{R}^n$)

In other words, it holds:

$$Per(E^\sharp) \leq Per(E),$$

where $E^\sharp$ is a ball such that $|E^\sharp| = |E|$.

[E. De Giorgi, 1954]

When $n = 2$ it reads as:

$$2\sqrt{\pi|E|} \leq Per(E).$$

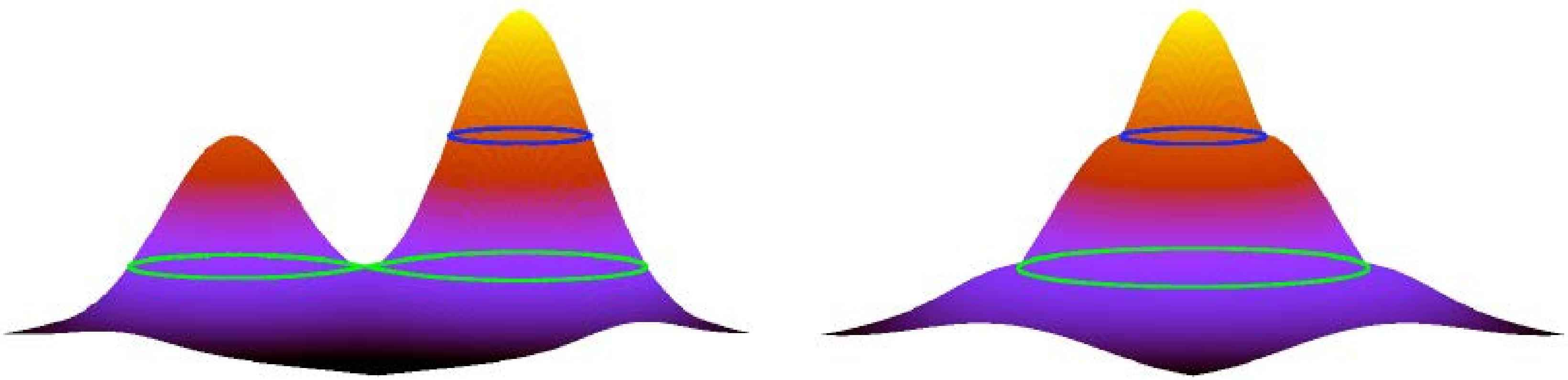
Symmetrization of a function

Let Ω be an open bounded set of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be a measurable function.

Schwarz symmetrization of f (spherically symmetric decreasing rearrangement), denoted by $f^\sharp$, is such that its level sets are balls of $\mathbb{R}^n$ centered at the origin having the same measure of the corresponding level sets of f.
In other words, we have:

$$\{x \in \Omega^\sharp : f^\sharp(x) > t\} = \{x \in \Omega : |f(x)| > t\}^\sharp \qquad t \geq 0.$$

Symmetrization of a function



f $\qquad$ $f^\sharp$

The decreasing rearrangement of f is the function f^* of one variable such that

$$f^\sharp(x) = f^*(\omega_n |x|^n), \qquad x \in \Omega^\sharp.$$

Some properties of rearrangements

It is not difficult to see thet, roughly speaking,

$$f^* = \mu_f^{-1}$$

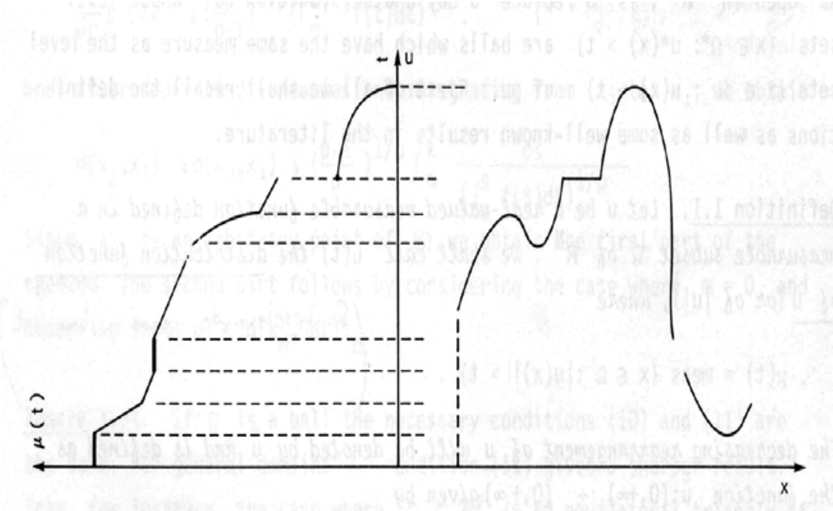
where μ_f is the distribution function of f

$$\mu_f(t) = |\{x \in \Omega : |f(x)| > t\}|, \qquad t \geq 0.$$

- f, $f^\sharp$ and f^* are equimeasurable, that is,

$$\mu_f(t) = \mu_{f^\sharp}(t) = \mu_{f^*}(t), \qquad t \geq 0.$$

- $\|f\|_{L^p(\Omega)} = \|f^\sharp\|_{L^p(\Omega^\sharp)} = \|f^*\|_{L^p(0,|\Omega|)} \quad 1 \leq p \leq \infty$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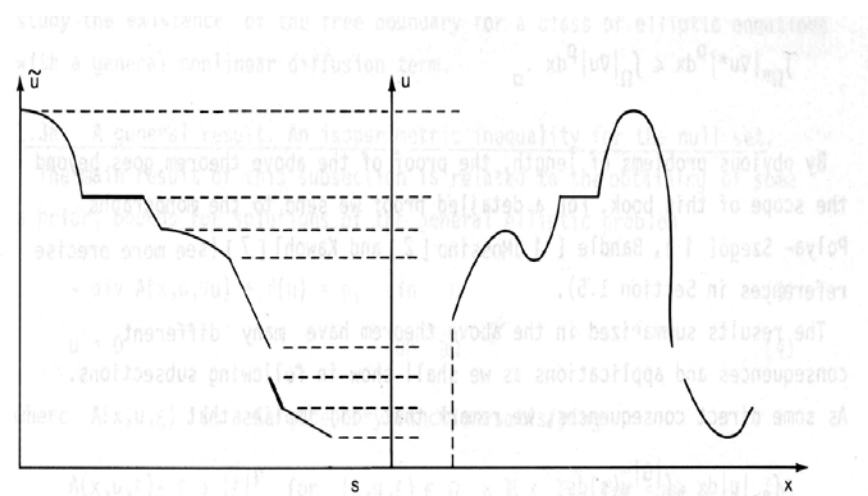
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- $$\int_E |f|\,dx \le \int_0^{|E|} f^*(s)\,ds, \qquad E \subset \Omega$$
- $$\int_{|f|>t} |f|\,dx = \int_0^{\mu_f(t)} f^*(s)\,ds$$
- Hardy-Littlewood inequality:

$$\int_\Omega |f\,g|\,dx \le \int_0^{|\Omega|} f^*(s)g^*(s)\,ds$$

- $$\int_0^s f^*(r)\,dr \le \int_0^s g^*(r)\,dr, \qquad s \in (0, |\Omega|)$$

$$\Downarrow$$

$$\|f\|_{L^p(\Omega)} \le \|g\|_{L^p(\Omega)} \qquad 1 \le p \le \infty$$

Applications of isoperimetric inequality

Pólya-Szegö inequality

Let $u \in W^{1,p}(\mathbb{R}^n)$ be a non-negative function with compact support, then:

$$\int_{\mathbb{R}^n} |Du^\sharp|^p \, dx \le \int_{\mathbb{R}^n} |Du|^p \, dx.$$

Comparison results for elliptic equations

Consider the problems ($a_{ij}\xi_i\xi_j \ge |\xi|^2$)

$$\begin{cases} -(a_{ij}u_{x_i})_{x_j} = f & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } E^\sharp \\ v = 0 & \text{on } \partial E^\sharp \end{cases}$$

then

$$u^\sharp(x) \le v(x), \qquad \text{in } E^\sharp.$$

The equality case has been characterized.

[H.F. Weinberger, 1962], [V.G. Maz'ja, 1969], [G. Talenti, 1976], [C.Bandle 1976], [J.L. Vázquez 1982], [J. Mossino 1984], [B. Kawhol 1985],[J.I. Díaz 1985] [A. Alvino -P.-L. Lions - G. Trombetti, 1986], [V.Ferone - M.R. Posteraro,1991], [J. Mossino-J.M. Rakotoson 1992], [A. Alvino –J.I. Diaz-P.-L. Lions - G. Trombetti, 1993], …[J.M. Rakotoson 2009],

Comparison result

For the sake of simplicity we assume that the data are regular and furthermore, $f \geq 0$, then $u \geq 0$.

Main ingredients of the proof

- Integration on the levels of u
- Gauss theorem
- Co-area formula
- Isoperimetric inequality
- Rearrangements properties

$$
\begin{aligned}
\int_{u>t} f &= \int_{u>t} -(a_{ij} u_{x_j})_{x_i} = && \text{(Gauss)} \\
&= \int_{u=t} a_{ij} u_{x_j} \frac{u_{x_i}}{|\nabla u|} \geq && \text{(ellipticity)} \\
&\geq \int_{u=t} |\nabla u| \geq && \text{(Cauchy-Schwarz)} \\
&\geq \frac{\left(\int_{u=t} 1\right)^2}{\int_{u=t} \frac{1}{|\nabla u|}}
\end{aligned}
$$

$$\int_{u>t} f \geq \frac{\left(\int_{u=t} 1\right)^2}{\int_{u=t} \frac{1}{|\nabla u|}} = \qquad \text{(co-area + isop. ineq.)}$$

$$\geq \frac{\left(n\omega_n^{1/n}\mu_u(t)^{1-1/n}\right)^2}{-\mu_u'(t)} \qquad \mu_u(t) = |\{u > t\}|$$

Hardy-Littlewood gives

$$\int_{u>t} f \leq \int_0^{\mu_u(t)} f^*$$

then

$$\frac{n^2\omega_n^{2/n}\mu_u(t)^{2-2/n}}{-\mu_u'(t)} \leq \int_0^{\mu_u(t)} f^*$$

$$(-u^*(s))' \le \frac{1}{n^2\omega_n^{2/n} s^{2-2/n}} \int_0^s f^*$$

$$u^\sharp(x) = u^*(\omega_n|x|^n) \le \frac{1}{n^2\omega_n^{2/n}} \int_{\omega_n|x|^n}^{|E|} \frac{1}{r^{2-2/n}} \int_0^r f^* = v(x)$$

where $v(x)$ is the solution of the problem

$$\begin{cases} -\Delta v = f^\sharp & \text{in } E^\sharp \\ \\ v = 0 & \text{on } \partial E^\sharp \end{cases}$$

2.2. The semilinear case.

The main idea: *a Burgers type auxilary function*

Define the function K on $\overline{Q_T^*}$ by $$K(s,t) = \int_0^s U^*(\sigma,t)\,d\sigma.$$

We then have the following lemma.

Lemma 6. *K and V^* satisfy the following:*

(i) $K \in L^\infty(Q_T^*) \cap H^1(0,T;W^{1,p}(\Omega^*)) \cap \bigcap_{\delta>0} L^2(0,T;W^{2,p}(\delta,|\Omega|))$,

(ii) $V^*(\cdot,t) \in C^1(\Omega^*)$ *for* $t>0$,

(iii) $d(s)\dfrac{\partial V^*}{\partial s} + \alpha K = 0$ *in* Q^*,

(iv) $\dfrac{\partial K}{\partial t} - d(s)\dfrac{\partial^2 K}{\partial s^2} - \alpha\chi K\dfrac{\partial K}{\partial s} = 0$ *in* Q^*,

(v) $K(s,0) = k(s,0)$ *on* Ω^* *and* $K(0,t)=0$, $\dfrac{\partial K}{\partial s}(|\Omega|,t) = 0$ *for any* $t \in [0,\ T]$.

Proof. For $\hat{\psi} \in C_0^\infty(\Omega^*)$ let us define

$$\psi(s) = \int_s^{|\Omega|} \hat{\psi}(\sigma)d\sigma \quad \text{and} \quad \varphi(x) = \psi(\kappa_N|x|^N).$$

Multiply (1.6) by $\varphi \in C_0^1(\tilde{\Omega})$ and integrate over $\tilde{\Omega}$. Integrating by parts and using $U^* = \partial K/\partial s$ gives

$$\begin{aligned} 0 &= \int_{\tilde{\Omega}} (\nabla V \cdot \nabla \varphi - \alpha U \varphi) dx = \int_{\Omega^*} \left(d(s) \frac{\partial V^*}{\partial s} \frac{d\psi}{ds} - \alpha U^* \psi \right) ds \\ &= -\int_{\Omega^*} \left(d(s) \frac{\partial V^*}{\partial s} + \alpha K \right) \hat{\psi} ds, \end{aligned}$$

which implies (iii). Next, multiply (1.5) by φ and integrate over $\tilde{\Omega}$. Using integration by parts, we obtain

$$\begin{aligned} 0 &= \int_{\tilde{\Omega}} \left\{ \frac{\partial U}{\partial t} \varphi + (\nabla U - \chi U \nabla V) \cdot \nabla \varphi \right\} dx \\ &= \int_{\Omega^*} \left\{ \frac{\partial U^*}{\partial t} \psi + d(s) \left(\frac{\partial U^*}{\partial s} - \chi U \frac{\partial V^*}{\partial s} \right) \frac{d\psi}{ds} \right\} ds \\ &= -\int_{\Omega^*} \left\{ -\frac{\partial K}{\partial t} + d(s) \left(\frac{\partial^2 K}{\partial s^2} - \chi \frac{\partial K}{\partial s} \frac{\partial V^*}{\partial s} \right) \right\} \hat{\psi} ds, \end{aligned}$$

from which together with (iii) we get (iv). □

APPLICATION OF SYMMETRIZATION TECHNIQUES TO THE PROBLEM (P)

Let u_0 satisfy (1.1) and let (u, v) be the corresponding solution of (P). Define the function $k(t, s)$ on $[0, T_{\max}) \times [0, \infty)$ by

$$k(t, s) = \int_0^s u_*(t, \sigma)\, d\sigma,$$

where $T_{\max}$ is the maximal existence time of (u, v) and $u_*(t, s)$ is the decreasing rearrangement of $u(t, x)$ with respect to x. Then we have the following.

PROPOSITION 3.1. *The function $k(t, s)$ satisfies*

$$k \in L^\infty([0, T_{\max}) \times (0, +\infty)) \cap H^1(0, T_{\max}: W^{1,p}_{loc}(0, +\infty))$$
$$\cap L^2(0, T: W^{2,p}_{loc}(0, +\infty)).$$

$$
\begin{cases}
\dfrac{\partial k}{\partial t} - d(s)\,\dfrac{\partial^2 k}{\partial s^2} - \alpha\chi k\,\dfrac{\partial k}{\partial s} \leqslant 0 & \textit{a.e. in} \quad (0, T_{\max}) \times (0, +\infty), \\[2ex]
k(t, 0) = 0, \qquad k(t, +\infty) = \displaystyle\int_{\mathbb{R}^N} u_0\, dx & \textit{for} \quad t \in [0, T_{\max}), \\[2ex]
k(0, s) = \displaystyle\int_0^s u_{0*}(\sigma)\, d\sigma & \textit{for} \quad s \geqslant 0,
\end{cases} \tag{3.1}
$$

where $d(s) = N^2 \kappa_N^{2/N} s^{2(N-1)/N}$ *and* κ_N *is the volume of the unit ball in* $\mathbb{R}^N$.

The proof of the differential inequality in (3.1) as well as of the rest of properties follows as in Diaz–Nagai [Lemma 4 in [11]] once the formula (1.2) is established on unbounded domains.

For the proof of the boundedness of solutions (u, v) to (P) on $\mathbb{R}^2$, we need a comparison principle for functions satisfying the differential inequalities in Proposition 3.1. In the following proposition, $C(t)$ denotes a generic positive function in $L^2(0, T)$.

PROPOSITION 3.2. *Let f and g be functions on $Q_T^* = [0, T] \times (0, +\infty)$ such that*

(i) $f, g \in L^\infty(Q_T^*) \cap L^2(0, T; W_{loc}^{2,2}(0, +\infty))$, $\partial f/\partial t$, $\partial g/\partial t \in {}^2(0, T; L_{loc}^2(0, +\infty))$,

(ii) $|\partial f/\partial s(t, s)| \leqslant C(t)$ *and* $|\partial g/\partial s(t, s)| \leqslant C(t) \max\{s^{-\ell}, 1\}$, *where ℓ is a constant satisfying* $0 \leqslant \ell < 1$.

If f and g satisfy the following

$$\begin{cases} \dfrac{\partial f}{\partial t} - d(s)\dfrac{\partial^2 f}{\partial s^2} - \alpha\chi f\dfrac{\partial f}{\partial s} \leqslant \dfrac{\partial g}{\partial t} - d(s)\dfrac{\partial^2 g}{\partial s^2} - \alpha\chi g\dfrac{\partial g}{\partial s} \qquad \text{a.e. } \textit{in } Q_T^*, \\ 0 = f(t, 0) \leqslant g(t, 0) \quad \text{and} \quad f(t, +\infty) \leqslant g(t, +\infty) \quad \textit{for any} \quad t \in [0, T], \\ f(0, s) \leqslant g(0, s) \quad \textit{on } (0, +\infty) \quad \textit{and} \quad g(t, s) \geqslant 0 \quad \textit{on } Q_T^*, \end{cases}$$

then $f \leqslant g$ on Q_T^*.

Proof. Put $w = f - g$, which satisfies

$$\frac{\partial w}{\partial t} - d(s)\frac{\partial^2 w}{\partial s^2} - \alpha\chi\left(w\frac{\partial f}{\partial s} + g\frac{\partial w}{\partial s}\right) \leqslant 0 \qquad \text{a.e. in } Q_T^*. \tag{3.2}$$

By multiplying (3.2) by $s^{2(1-N)/N}w_+$ and integrating over (δ, L) $(0 < \delta < 1 < L)$, the integration by parts and $|\partial f/\partial s| \leqslant C(t)$ yield that

$$\frac{1}{2}\frac{d}{dt}\int_\delta^L s^{2(1-N)/N}(w_+)^2\,ds + N^2\kappa_N^{2/N}\int_\delta^L\left(\frac{\partial w_+}{\partial s}\right)^2 ds$$

$$\leqslant \alpha\chi\int_\delta^L s^{2(1-N)/N}\left\{(w_+)^2\frac{\partial f}{\partial s} + w_+\frac{\partial w}{\partial s}\,g\right\}ds + G(t, \delta, L)$$

$$\leqslant C(t)\int_\delta^L s^{2(1-N)/N}(w_+)^2\,ds + \alpha\chi\int_\delta^L s^{2(1-N)/N}w_+\frac{\partial w}{\partial s}\,g\,ds + G(t, \delta, L)$$

a.e. in Q_T^*, where

$$G(t, \delta, L) = Const.\left\{\left|\frac{\partial w}{\partial s}(t, \delta)\right| w_+(t, \delta) + \left|\frac{\partial w}{\partial s}(t, L)\right| w_+(t, L)\right\}$$

which satisfies

$$G(t, \delta, L) \leqslant C(t)^2, \qquad G(t, \delta, L) \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{and} \quad L \to \infty.$$

By $f(t,0)=0$ and (ii), we see that $f^+(t,s) \leqslant C(t)s$ on $(0,\infty)$, from which together with $f \in L^\infty(Q_T^*)$ it follows that $s^{(1-N)/N} f^+(t,s) \leqslant C(t)$ *on* $(0,T)\times(0,+\infty)$. We then obtain

$$\alpha\chi \int_\delta^L s^{2(1-N)/N} w_+ \frac{\partial w}{\partial s} g \, ds$$

$$\leqslant \alpha\chi \int_\delta^L s^{2(1-N)/N} w_+ \left| \frac{\partial w_+}{\partial s} \right| f^+ \, ds$$

$$\leqslant \frac{1}{2} N^2 \kappa_N^{2/N} \int_\delta^L \left(\frac{\partial w_+}{\partial s} \right)^2 ds + C(t)^2 \int_\delta^L s^{2(1-N)/N} (w_+)^2 \, ds.$$

Hence, for $t \in (0,T)$,

$$\frac{d}{dt} \int_\delta^L s^{2(1-N)/N} w_+^2 \, ds \leqslant C(t)^2 \int_\delta^L s^{2(1-N)/N} w_+^2 \, ds + G(t,\delta,L),$$

from which together with $w_+(0, s) = 0$ on $(0, +\infty)$ it follows that

$$\int_\delta^L s^{2(1-N)/N} w_+^2 \, ds \leqslant e^{\int_0^t C(\tau)^2 \, d\tau} \int_0^t e^{\int_0^\tau C(\sigma)^2 \, d\sigma} G(t, \delta, L) \, d\tau \tag{3.3}$$

for $t \in (0, T)$. Letting $\delta \to 0$ and $L \to \infty$ in (3.3) yields that

$$\int_0^\infty s^{2(1-N)/N} w_+^2 \, ds = 0 \qquad \text{for} \quad t \in (0, T),$$

which implies $w_+ = 0$ in Q_T^*. Hence, $f \leqslant g$. ∎

As an application of Proposition 3.1, we give the boundedness of solutions (u, v) to (P) in $\mathbb{R}^2$.

THEOREM 3.1. *Let u_0 be a function on $\mathbb{R}^2$ satisfying* (1.1) *and* (u, v) *the corresponding solution of* (P). *If* $\alpha\chi \int_{\mathbb{R}^2} u_0 \, dx < 8\pi$, *then* $T_{\max} = \infty$ *and*

$$\|u(t)\|_{L^p(\mathbb{R}^2)} \leqslant L(u_0, \alpha, \chi, p), \qquad \|v(t)\|_{L^p(\mathbb{R}^2)} \leqslant \frac{\alpha}{\gamma} L(u_0, \alpha, \chi, p) \tag{3.4}$$

for any $t \geqslant 0$ *and any* $p \in [1, +\infty]$, *where*

$$L(u_0, \alpha, \chi, p) = \begin{cases} \dfrac{8\pi}{\alpha\chi} (2p-1)^{-1/p} \, \|u_0\|_{L^\infty(\mathbb{R}^2)}^{1-1/p} \left(\dfrac{8\pi}{\alpha\chi} - \|u_0\|_{L^1(\mathbb{R}^2)} \right)^{1/p - 1} \\ \qquad \text{if} \quad p \geqslant 1, \\ \dfrac{8\pi}{\alpha\chi} \|u_0\|_{L^\infty(\mathbb{R}^2)} \left(\dfrac{8\pi}{\alpha\chi} - \|u_0\|_{L^1(\mathbb{R}^2)} \right)^{-1} \\ \qquad \text{if} \quad p = +\infty. \end{cases}$$

Proof. By Proposition 3.1 and $d(s) = 4\pi s$, the function $k(t, s) = \int_0^s u_*(t, \sigma) \, d\sigma$ satisfies

$$\frac{\partial k}{\partial t} - 4\pi s \frac{\partial^2 k}{\partial s^2} - \alpha\chi k \frac{\partial k}{\partial s} \leqslant 0 \qquad \text{in} \quad (0, T_{\max}) \times (0, +\infty).$$

Let us define the function $w(s)$ by

$$w(s) = \frac{8\pi q s}{\alpha\chi(1+qs)} \qquad \text{for} \quad s \geqslant 0,$$

where q is a positive constant determined below. The function w satisfies

$$4\pi s w'' + \alpha\chi w w' = 0 \qquad \text{on} \quad (0, +\infty).$$

By noting that $k(0, +\infty) = \int_{\mathbb{R}^2} u_0 \, dx < 8\pi/(\alpha\chi) = w(+\infty)$, it is shown that $k(0, s) \leqslant w(s)$ on $[0, +\infty)$ whenever

$$q \geqslant \|u_0\|_{L^\infty(\mathbb{R}^2)} \left(\frac{8\pi}{\alpha\chi} - \|u_0\|_{L^1(\mathbb{R}^2)} \right)^{-1}. \tag{3.5}$$

For any $t > 0$ we also have $k(t, +\infty) = \int_{\mathbb{R}^2} u_0 \, dx < w(+\infty)$. Hence, applying Proposition 3.2 gives.

$$k(t, s) \leqslant w(s) \qquad \text{on} \quad (0, T_{\max}) \times (0, +\infty). \tag{3.6}$$

By a simple application of a lemma in [2, p. 74] or Lemma 1.33 in [8], (3.6) implies

$$\|u_*(t)\|_{L^p(0,\infty)} \leqslant \|w'\|_{L^p(0,\infty)} \qquad (0 \leqslant t < T_{\max}) \tag{3.7}$$

for any $p \in [1, +\infty]$. We take q as the equal sign in (3.5) so that $\|w'\|_{L^p(0,\infty)} = L(u_0, \alpha, \chi, p)$. Then it follows from (3.7) and $\|u_*(t)\|_{L^p(0,\infty)} = \|u(t)\|_{L^p(\mathbb{R}^2)}$ that the desired inequality on u in (3.4) is derived. Finally, the inequality on v in (3.4) is derived from the inequality on u in (3.4) and the following inequality

$$\|v(t)\|_{L^p(\mathbb{R}^2)} \leqslant \frac{\alpha}{\gamma} \|u(t)\|_{L^p(\mathbb{R}^2)} \qquad (0 \leqslant t < T_{\max})$$

for any $p \in [1, +\infty]$. Therefore, $T_{\max} = +\infty$ and the proof is complete. ∎

2.3. The quasilinear case.

- Problem

$$\begin{cases} u_t = \operatorname{div}(\nabla u^m - \chi u \nabla v) & \text{in } Q_T = (0,T)\times\mathbb{R}^N, \\ -\Delta v + \gamma v = \alpha u & \text{in } Q_T \\ u(0,x) = u_0(x), & \text{on } \mathbb{R}^N. \end{cases}$$

- (U, V) solution for a $U_0 = \tilde{U}_0$ and $\gamma \equiv 0$.

- $k(t,s) = \int_0^s u_*(t,\sigma)\,d\sigma$, $K(t,s) = \int_0^s U_*(t,\sigma)\,d\sigma$

- $$\begin{cases} \dfrac{\partial K}{\partial t} - d(s)\left(\left|\dfrac{\partial K}{\partial s}\right|^{m-1}\dfrac{\partial K}{\partial s}\right)_s - \alpha\chi K\dfrac{\partial K}{\partial s} = 0 & \text{a.e. in } (0,T^U_{max})\times(0,+\infty), \\ K(t,0)=0\ ,\ K(t,+\infty)=\int_{\mathbb{R}^N} U_0(x)\,dx & t\in(0,T^U_{max}), \\ K(0,s)=\int_0^s U_{0,*}(\sigma)\,d\sigma & s\geq 0, \end{cases}$$

- $$\begin{cases} \dfrac{\partial k}{\partial t} - d(s)\left(\left|\dfrac{\partial k}{\partial s}\right|^{m-1}\dfrac{\partial k}{\partial s}\right)_s - \alpha\chi k\dfrac{\partial k}{\partial s} \leq 0 & \text{a.e in } (0,T^u_{max})\times(0,+\infty) \\ k(t,0)=0\ ,\ k(t,+\infty)=\int_{\mathbb{R}^N} u_0(x)\,dx & t\in(0,T^u_{max}) \\ k(0,s)=\int_0^s u_{0,*}(\sigma)\,d\sigma \end{cases}$$

- New argument:

$$\frac{\partial k}{\partial s} \in BC([t_0,T)\times(0,+\infty)) \cap L^\infty(0,T:L^1(0,\infty)) \qquad \forall\ T<T^u_{max}.$$

• Comparison ($\Leftarrow L^1$- continuous dependence estimate).

<u>Proposition</u>. Assume $\int_{\mathbb{R}^N} u_0(x) \leq \int_{\mathbb{R}^N} U_0(x)$. Then

$$(*) \quad \| (k(t,\cdot) - K(t,\cdot))_+ \|_{L^1(0,+\infty)} \leq \| (k(0,\cdot) - K(0,\cdot))_+ \|_{L^1(0,+\infty)} .$$

<u>Idea of the proof</u>: Multiply by a regular approximation of

$$\mathrm{sign}_+(k - K) = \begin{cases} 1 & \text{if } k > K \\ 0 & \text{if } k \leq K . \end{cases}$$

• If $U_{0,*} = u_{0,*}$ then $(*)$ implies

$$\| u(t,\cdot) \|_{L^p(\mathbb{R}^N)} \leq \| U(t,\cdot) \|_{L^p(\mathbb{R}^N)} \quad \forall\, p \in [1, +\infty] .$$

In particular:

a) $\| U(t,\cdot) \|_{L^p(\Omega)} < +\infty \Rightarrow$ Global existence

b) isoperimetric inequality of the blow-up time

$$T^u_{max} \leq T^{symm}_{max}$$

- Other (new) estimate:

$$\|v(t,\cdot)\|_{L^p(\mathbb{R}^N)} \leq \|V(t,\cdot)\|_{L^p(\mathbb{R}^N)} \quad \forall p \in [1,+\infty].$$

- Comparison with K self-similar solution of

$$\begin{cases} \dfrac{\partial K}{\partial t} - d(s)\left(\left|\dfrac{\partial K}{\partial s}\right|^{m-1}\dfrac{\partial K}{\partial s}\right)_s - \alpha\chi K\dfrac{\partial K}{\partial s} = 0 \\ K(t,0)=0, \quad K(t,+\infty)=M \\ K(0,s)\equiv M \end{cases}$$

P. Biler (1995), Y. Naito and T. Suzuki (2004) $[m \equiv 1]$.

Thanks for your attention