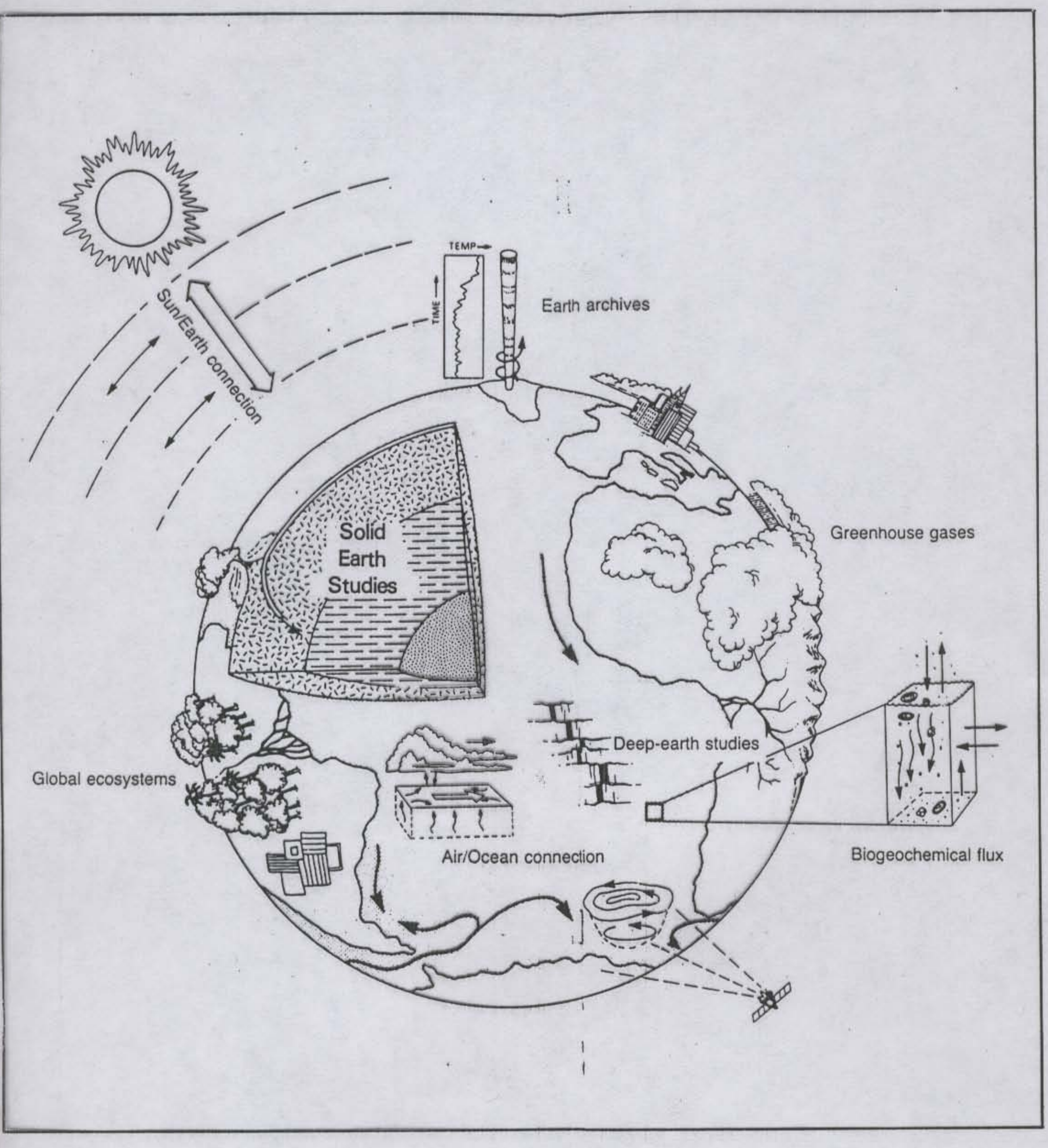

Sun/Earth connection
TEMP
TIME
Earth archives
Greenhouse gases
Solid
Earth
Studies
Deep-earth studies
Global ecosystems
Air/Ocean connection
Biogeochemical flux

ENERGY (RADIATION) BALANCE AND THE GREENHOUSE EFFECT

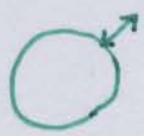
Main ingredients:

$R_a \equiv$ Solar radiation absorbed by the Earth
(short-wave energy from the Sun)

$R_e \equiv$ Emitted radiation (by the Earth)
(long-wave radiation escaping into Space)

Other features define a HIERARCHY OF MODELS

- Time
  - equilibrium (stationary)
  - transient (dynamics)

- Space
  - 0-d zero-dimensional (homogeneous models)
  - 1-d one-dimensional (distributed models)
    - Latitudinal models 
    - vertical models (radiative models) 
  - 2-d
    - horizontal (surface) models 
    - meridional plane models
  - 3-d
    - General Circulation Models 

- Coupling models (Glaceology, Geophysics, Celestial Mechanics..)

- Pioneer paper: Svante ARRHENIUS (1896) On the influence of carbonic acid in the air upon the temperature of the ground, Philosh. Mag, 41, 237-271

- Precursor papers

  W.D. SELLERS (1969) A global climatic model based on the energy balance of the earth-atmosphere system, J. Appl. Meteorol. 8, 392-400

  M.I. BUDYKO (1969) The effects of solar radiation variations on the climate of the Earth, Tellus, 21, 611-619

- Modeling

  - simple balance arguments

$$c\frac{\partial u}{\partial t} = R_a - R_e + D$$

$u(t,x)$:= annually or seasonally averaged surface temperature

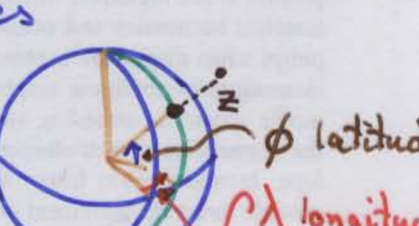
c := heat capacity

D := diffusion operator (redistributi

  - averaging the atmosphere primitive equations (see e.g. J.T. Kiehl (1992))

Different averaging processes

$$\xi = \xi(t, z, \lambda, \phi)$$



$$\hat{\xi} := \int_0^{z_0} \xi(t, z, \lambda, \phi)\,\rho\,dz = \int_0^{p_s} \xi(t, p, \lambda, \phi)\frac{dp}{g}$$

p_s = surface pressure

$$[\xi] := \frac{1}{2\pi}\int_0^{2\pi} \xi(t, z, \lambda, \phi)\,d\lambda$$

$$\langle \xi \rangle := \frac{1}{2\pi^2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \xi(t, z, \lambda, \phi)\cos\phi\,d\lambda\,d\phi$$

$$\overline{\xi} := \frac{1}{2\tau}\int_{t-\tau}^{t+\tau} \xi(s, z, \lambda, \phi)\,ds$$

Thermodynamic equation for the atmosphere in flux form

$$(1) \qquad c_p \frac{\partial T}{\partial t} = -c_p \nabla\cdot(\vec{v}T) - c_p\frac{\partial(wT)}{\partial p} + c_p \kappa \frac{wT}{p} + \tilde{Q}_{rad} - \tilde{Q}_{con}$$

T := temperature

[Other forms: Lions-Temam-Wang (1992, 93, ...)

$\vec{v}$:= horizontal wind vector, w vertical p-velocity

$\kappa := \frac{R}{c_p} \approx 0.286$

$\tilde{Q}_{rad}$:= net radiative heating

$\tilde{Q}_{con}$:= heating due to condensational processes

HIERARCHY

0-d model $\qquad u(t) := \langle \hat{\overline{T}} \rangle (t)$

1-d model $\quad u(t,\phi) := [\hat{\overline{T}}](t,\phi)$

Examples : $\tau = 5$ years

$$u(t,\phi) := \frac{1}{20\pi}\int_{-5}^{5}\int_{0}^{2\pi}\int_{0}^{p_s} T(t+s,p,\lambda,\phi)\frac{dp}{g}\,d\lambda\,ds$$

$\tau = 1/4$ year (seasonal model)

$$\underset{\text{latitude}}{u(t,\phi)} := \frac{2}{11\pi}\sum_{j=-5}^{5}\int_{-1/8}^{1/8}\int_{0}^{2\pi}\int_{0}^{p_s} T(t+s+j,p,\lambda,\phi)\frac{dp}{g}\,d\lambda\,ds$$

$$c_p\frac{\partial[\hat{\overline{T}}]}{\partial t} = -\frac{c_p}{a\cos\phi}\,\frac{\partial([\hat{\overline{vT}}]\cos\phi)}{\partial\phi} + [R_a] - [R_e]$$

$a :=$ the radius of the Earth

$v :=$ the meridional (northward) wind velocity

2-d model $\quad u(t,\lambda,\phi) := \hat{\overline{T}}(t,\lambda,\phi)$

$$c_p\frac{\partial\hat{\overline{T}}}{\partial t} = -c_p\,\mathrm{div}\left(\hat{\overline{\vec{v}T}}\right) + R_a - R_e$$

REQUIRED : relation between the horizontal heat transport term $c_p\nabla\cdot(\hat{\overline{\vec{v}T}})$ and $\hat{\overline{T}}$

$$\left([\hat{\overline{vT}}] \quad \text{and} \quad [\hat{\overline{T}}] \right)$$

- simple approximation (Sellers (69), Budyko(6

$$-[\hat{\overline{vT}}] = K_e\frac{\partial[\hat{\overline{T}}]}{\partial\phi} \qquad K_e = \text{eddy diffusion coefficient}$$

- $$-(\hat{\overline{\vec{v}T}}) = K_e\,\nabla\hat{\overline{T}}$$

- nonlinear approximation (P.H.STONE (1972))

  K_e is much larger on the average in the tropic than in mid-latitudes

$$K_e = K_e^*\left|\frac{\partial[\hat{\overline{T}}]}{\partial\phi}\right|\cos\phi, \quad K_e = K^*|\nabla\hat{\overline{T}}| \Rightarrow \text{NON LINEAR DIFFUSIONS}$$

$$\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(K\left|\frac{\partial u}{\partial\phi}\right|\frac{\partial u}{\partial\phi}\cos^2\phi\right)$$

Defining $x := \sin\phi \Rightarrow x \in (-1, 1)$

$$\frac{\partial}{\partial x}\left(K(1-x^2)^2\left|\frac{\partial u}{\partial x}\right|\frac{\partial u}{\partial x}\right)$$

More in general: for $p \geq 2$

$$\frac{\partial}{\partial x}\left(K(1-x^2)^{p-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)$$

($p=2$ Sellers-Budyko)
($p=3$ Stone, Held-Suarez (1974), (1981), ...)

- TWO DIMENSIONAL MODEL

$$\operatorname{div}\left(K|\nabla u|^{p-2}\nabla u\right)$$

div and ∇ over $S^2_a \equiv$ sphere of radious a

- SPATIAL DOMAIN

S^2_a

$M \equiv$ two dimensional Riemannian compact manifold <u>without boundary</u>

($I = (-1, 1)$ one-dimensional model)
degenerating weight on $x = \pm 1 \Rightarrow$ <u>no boundary conditions</u>

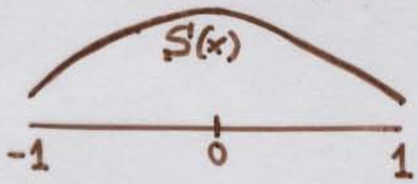
ESTRUCTURE ASSUMPTIONS on R_a and R_e

$$\boxed{R_a \equiv Q S(x) \beta(u)}$$

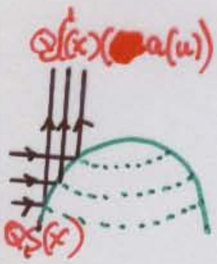
$Q \equiv$ Solar constant (~ 1.370 W/m^2)

$S(x) \equiv$ distribution of insolation

$$S \in C^\infty(M), \quad S(x) \geq S_0 > 0$$



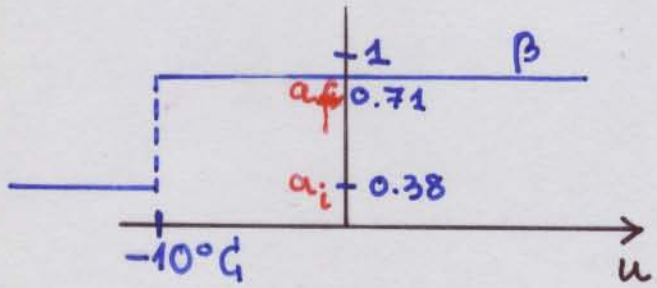
$\left(S(t,x) \geq 0 \text{ in seasonal models: "the polar night"} \right)$



$\beta(u) := (1 - a(u))$

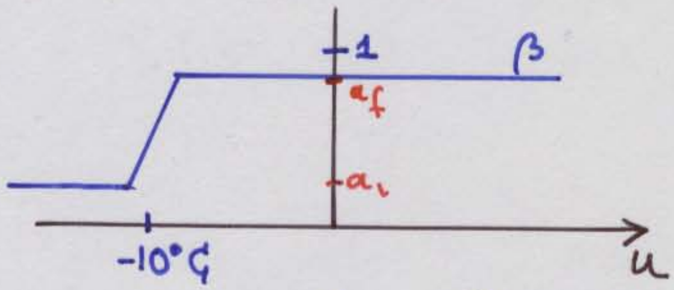
$a(u) \equiv$ planetary albedo ($\beta(u) \equiv$ coalbedo)
(aborbed fraction)

BUDYKO



β discontinuous

SELLERS



β regular (Lipschitz cont.)

(H_β) β is a bounded maximal monotone graph of $\mathbb{R}^2$

In Section: S-shaped curve diagr.

$$(H_\beta^*) \begin{cases} \beta(r) = \{m\} & \forall r \in (-\infty, -10-\varepsilon) \\ \beta(r) = \{M\} & \forall r \in (-10+\varepsilon, +\infty) \\ \text{for some } \varepsilon > 0 & 0 < m < M \end{cases}$$

($m = a_i$)
($M = a_f$)

$$\boxed{R_e \equiv G(u) - f(t,x)}$$

- empirical relation
- depends on the ammount of greenhouse gases, clouds, water vapor in the atmosphere ("internal variables") (include anthropogenerate changes)
- Stefan-Boltzmann law

$$R_e \sim \sigma T^4 \qquad (T \text{ in Kelvin})$$

BUDYKO

(linear approximation)

$\langle \hat{T} \rangle \sim 15°C$

$$R_e = A(t,x) + Bu$$

$A \sim 210\ W/m^2$ $(\equiv -f)$

$B \sim 1.9\ W/m^2 °C$

SELLERS

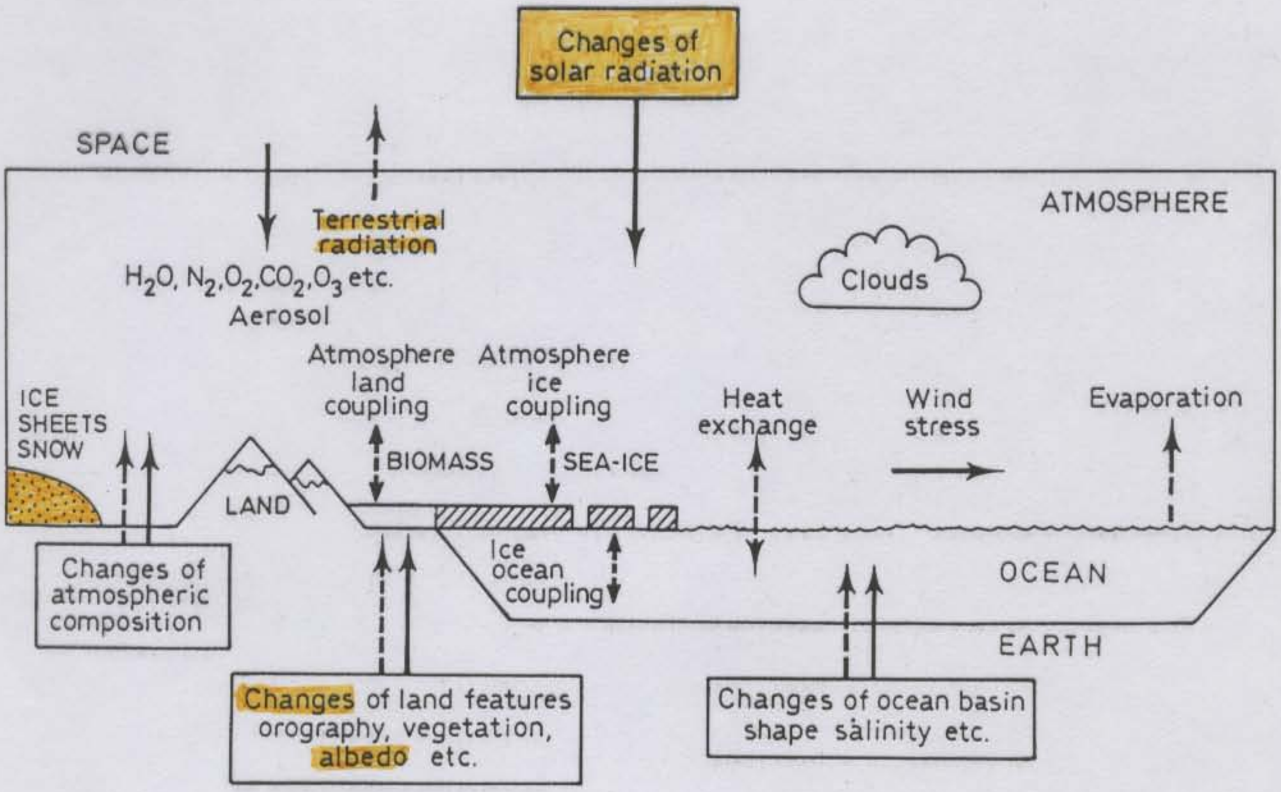
$$R_e \sim \sigma T^4$$

$\sigma > 0$ emmisivity

$$(H_G)\ \begin{cases} G : \mathbb{R} \to \mathbb{R} \text{ continuous} \\ |G(u)| \geq C|u|^r \text{ for some } r \geq 1 \\ G \text{ strictly increasing} \end{cases}$$

for simplicity

$$(H_f) \quad f \in L^\infty((0,\infty) \times M)$$



Schematic illustration of the components of the climate system. Full arrows are examples of external processes and dashed arrows are examples of internal processes.

Mechanisms of climate change

- EXTERNAL CAUSES OF CLIMATE CHANGE
  - Luminosity of the Sun (increased by 20%~40% during the $4,5 \times 10^9$ years of the Earth)
  - Astronomical relationship between the Earth and the Sun : MILANKOVITCH THEORY
    - eccentricity
    - precession of the location of the perihelion
    - obliquity
- INTERNAL CAUSES
  - Volcanoes
    - eruptions 15-25 Km
    - particles have a long residence time in the stratosphere
    - cooling effect
  - Aerosols
    - cooling effect
  - Clouds
    - cirrus cloud lead to climatic warming
  - Carbon dioxide
    - increased during last 100 years
    - greenhouse effect
    - warming (controversial)

4. Most of the papers on EBM by climatologi
based on numerical approach

R.North [Bowman , Kim , Cahalan, ...]
S.H. Schneider [....]
M.Ghil [...]
I. Held , M.Suarez .... A.Pielke
S.de Gregorio, G.A.Dalu, ...
G.Hetzer , H.Jarausch , W.Mackens

A.Henderson-Sellers , K. Mcguffie

R.Bermejo (finite elements method ,
convergence estimates ....)

J.Atmos. Sci.
J.Climate
Climate Dynamics
Tellus
J.Appl. Meteorol.
J.Geophys. Res.
Global and Planetary Change
Quart. J.Roy. Meteor. Soc.
Mon. Wea. Rev.

DIFFUSIVE 2-d ENERGY BALANCE MODELS

Transient model

$$
(\mathrm{P})\quad \begin{cases} c(x)\,u_t - \operatorname{div}\left(k(x)|\nabla u|^{p-2}\nabla u\right) + \mathcal{G}(u) \in QS(x)\beta(u) + f & \text{in } (0,\infty)\times\mathcal{M} \\ u(0,x) = u_0(x) & \text{on } \mathcal{M} \end{cases}
$$

- $\mathcal{M}$ bidimensional compact Riemannian manifold without bounda[ry]
- (H_β) β bounded maximal monotone graph of $\mathbb{R}^2$
- $p \geq 2$ (for simplicity) [$p=3$ Stone [1972]....]
- $(H\mathcal{G})$ $\mathcal{G}: \mathbb{R} \to \mathbb{R}$ cont. strictly increasing, $|\mathcal{G}(u)| \geq C|u|^r$, $r \geq 1$
- $c \in L^\infty(\mathcal{M})$, $c(x) \geq c_0 > 0$
- $k \in C^\infty(\mathcal{M})$, $k(x) \geq k_0 > 0$
- (H_f) $f \in L^\infty((0,\infty)\times\mathcal{M})$
- (H_{u_0}) $u_0 \in L^\infty(\mathcal{M})$

Stationary problem

$$
P_Q \quad \left\{ -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \mathcal{G}(u) \in QS(x)\beta(u) + f(x) \quad \text{in } \mathcal{M} \right.
$$

2. THE EVOLUTION MODEL

2.1. On the existence of weak solutions

- Energy space

$$V := \{ u \in L^2(M) : \nabla u \in L^p(TM) \}$$

$$TM := \bigcup_{p \in M} T_p M \quad \text{tangent bundle}$$

- Existence results ($u_0 \in L^\infty(M)$)
  - via fixed point theorems (D 93, D-Tello 94, ..)
  - via approximation by classical solutions

$$c u_t - \mathrm{div}\left(\kappa \left[\varepsilon + |\nabla u|^2\right]^{\frac{p-2}{2}} \nabla u\right) + G_\varepsilon(u) =$$
$$= Q S(x) \beta_\varepsilon(u) + f_\varepsilon$$

(Xu 91, Tolksdorff 83 ($M \equiv \Omega$), D 93,..)
Feireisl-Norbury 91,
Gianni - Hulshof 92, Gianni 95

(• Time periodical solutions (Badii-D 99))

(• Memory terms (D-Hetzer 98))

2. EXISTENCE OF SOLUTIONS

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u + g(u) \in QS(x)\beta(u) + f(t,x) & \text{in } (0,\infty)\times M \\ u(0,x)=u_0(x) & \text{on } M. \end{cases}$$

· Notion of solutions:

· classical solutions ($u, u_t, \nabla u, \Delta_p u \in C((0,\infty)\times M)$

· OK for $p=2$ (linear diffusion) and Sellers type coalbedo function [i.e. β Lipschitz]

· FALSE for $p>2$ (degenerated diffusion)

· FALSE for Budyko coalbedo type functions (i.e. β discontinuous or multivalued), for any value of $p>1$.

· weak solutions

· Formal considerations:

M compact manifold without boundary similar to the cases

(A) · $\Omega = \mathbb{R}^2$

either

(B) · $\Omega \subset \mathbb{R}^2$ open bounded regular set

+ Neumann homogeneous boundary conditions on $\partial\Omega$: $\dfrac{\partial u}{\partial n} = 0$ on $(0,\infty)\times\partial\Omega$

· Notion of weak solutions for case (B).

Let $v(t,x)$ be a "test function" (to be well-defined at the end of the process). Multiplying the pde by v and "integrating by parts" on $(0,T)\times M$, $T>0$ arbitrary:

Integral identity [(I.I.) in the following]

$$\int_M u(T,x)v(T,x)\,dA - \int_0^T\int_M u(t,x)v_t(t,x)\,dA\,dt \;+$$

$$+\int_0^T\int_M |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dA\,dt$$

$$= \int_0^T\int_M \big(QS(x)\beta(u)+f(t,x)-g(u)\big)v\,dA\,dt$$

$$+\int_M u_0(x)v(0,x)\,dA \quad + \underbrace{\int_0^T\int_{\partial M}|\nabla u|^{p-2}\frac{\partial u}{\partial n}v\,ds\,dt}_{=0}$$

$\partial M=\emptyset$ case of manifolds without boundary

$\frac{\partial u}{\partial n}\equiv 0$ case of homogeneous Neumann bound. cond and $M\equiv\Omega$ open bounded set of $\mathbb{R}^2$

- **Key idea**: The own solution, u, "must" be a test function $\Rightarrow$

$$\forall t\in[0,T] \qquad \int_M |u(t,x)|^2\,dA<+\infty \qquad (*)$$

and

$$\int_0^T\int_M|\nabla u|^p\,dA\,dt<+\infty\,. \qquad (**)$$

Functional ("energy") space

$$V:=\{w=w(x):\ w\in L^2(M),\ \nabla w\in L^p(M)\}$$

$(*)$ and $(**)$ satisfied if

$$u\in C([0,T]:L^2(M))\cap L^p(0,T:V)\,.$$

(3) If $p>2$ the operator becomes degenerate

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \underbrace{|\nabla u|^{p-2}}_{\substack{\| \\ 0 \\ \text{on } \{(x,t):\, \nabla u(x,t)=0\}}} \Delta u + \underbrace{\sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\right.}$$

(on $\mathbb{R}^2$)

[redacted]

- critical points of u
- flat regions of u

(4) div. and ∇. over M (Bidimensional riemannian compact manifold without boundary).

(5) If $M = S^2$ (Unit sphere in $\mathbb{R}^3$)

- ϕ = colatitude, λ = longitude
- averaging on λ (i.e. $u(t,\cdot)$ only depends on ϕ) $\Rightarrow$ 1-D model
- Finally $x := \cos\phi$ ($\Rightarrow x \in (-1,1)$) we get

2. SOME TECHNIQUES FOR THE BASIC THEORY: EXISTENCE, UNIQUENESS AND CONTINUOUS DEPENDENC ON THE DATA.

2.1. The problem: $\Omega \subset \mathbb{R}^N$ open regular set

$$(P) \begin{cases} c(x)u_t - \operatorname{div} \vec{A}(x, \nabla u) = QS(x,t)a(u) - R_e(x,t,u) & \Omega \times (0,\infty) \\ \vec{A}(x,\nabla u)\cdot\vec{\nu} = 0 & \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \Omega \end{cases}$$

Examples: $\Omega = (0,1)$ $N=1$

$A(x,\xi) = K(1-x^2)\xi$ Budyko-Sellers

$A(x,\xi) = K^*(1-x^2)^2|\xi|\xi$ Held-Suarez

Assumptions:

- $R_e(\cdot,\cdot,u) \nearrow$
- $a(u) \nearrow$ (but not necessarily continuous)
- $Q, S, c > 0$

$$\begin{cases} |\vec{A}(x,\vec{\xi})| \leq c_1 |\vec{\xi}|^{p-1} & \text{for some } p>1 \\ \vec{A}(x,\vec{\xi})\cdot\vec{\xi} \geq c_2(x)|\vec{\xi}|^p \end{cases}$$

2.2. Existence of (weak) solutions.

ⓐ via comparison arguments

$\{u^n\}_{n\in\mathbb{N}}$ solution of

$$\begin{cases} c\,u^n_t - \operatorname{div}\vec{A}(x, \nabla u^n) = QSa(u^{n-1}) - R_e(x,t,u^n) \\ +\text{b.c and i.c.} \end{cases}$$

- Then if we start with a supersolution $\overline{u}^0$ i.e.
$$\begin{cases} \text{PDE} \geqslant \text{—} \\ \text{BC} \geqslant - \\ \text{IC} \geqslant u_0(x) \end{cases}$$
we deduce that

$$\overline{u}^n \ldots \leq \overline{u}^2 \leq \overline{u}^1 \leq \overline{u}^0$$

(use the maximum principle for the "monotone" operator $-\operatorname{div}\vec{A}(x,\nabla u) + Re(\cdot, u)$)

- If we start with a subsolution $\underline{u}^0$ we deduce

$$\underline{u}^n \ldots \geqslant \underline{u}^2 \geqslant \underline{u}^1 \geqslant \underline{u}^0$$

- Finally as in Satinger (1971) it is shown that:

Theorem

Assume $\bar{u}^\circ$, $\underline{u}^\circ$ super and sub solutions
Let $u_0(x)$ such that

$$\underline{u}_0(x,0) \leq u_0(x) \leq \bar{u}^\circ(x,0)$$

Then

$$\lim \bar{u}^n = \bar{u}$$

$$\lim \underline{u}^n = \underline{u}$$

are (not necessarily different) solutions and

$$\underline{u}^\circ \leq \underline{u}^1 \leq \ldots \leq \underline{u}^n \leq \underline{u} \leq \bar{u} \leq \ldots \bar{u}^n \leq \ldots \leq \bar{u}^1 \leq \bar{u}^\circ$$

Remark: A similar non-linear iterative algoritm in D-STAKGOLD (1990, ...)

- Book by Lakshmikantam (PITMAN 1989)

(b) via compactness methods

D-VRABIE (CRAS, 1989, ...)
I. VRABIE: Compactness methods for nonlinear Evolutions. PITMAN 1987.

Define

$$G : L^2(\Omega \times (0,T)) \longrightarrow L^2(\Omega \times (0,T))$$

Non linear Green Operator

$$f \longmapsto v$$

v solution of $\begin{cases} c v_t - \operatorname{div} A(x, \nabla v) = f \\ \text{B.C.} + v(x,0) = u_0(x) \end{cases}$ fixed

· Characterization as fixed point

$$u \text{ solution of } (P) \iff u = G\big(QSa(u) - R_e(u)\big)$$

· Technical results :

(i) G is a compact operator
(bounded subsets → relatively compact)

(ii) $QSa(u) - R_e(u)$ is a "upper semicontinuous" operator (even if a is discontinuous)

(iii) (compact) ∘ (upper semicontinuous) has at least one fixed point
(variant of the Tychonoff fixed point theorem).

Remarks ① The method applies to systems of nonlinear parabolic equations (even with righ hand side termes discontinuous).

Application to HETZER model on multi-layer models (lecture of Stakgold)

② Other methods can be also applied :
· Parabolic [or elliptic] regularization (Xu 1979) ALT-LUCKHA (198
· Semi-discretization (Semigroups theory)..
Brezis, Benilan, ...

Remarks:

- V is a reflexive Banach space (if $p>1$).
- Space of test functions:

$$v \in C([0,T]: L^2(M)) \cap L^p(0,T:V)$$ make sense t

$$\int_M u(T,x)v(T,x)\,dA \quad , \int_0^T\int_M |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dA\,dt$$

(use Young inequality $ab \le \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$

- $\int_0^T\int_M u(t,x)v_t(t,x)\,dA\,dt$ can be understood as $\int_0^t \langle u(t,\cdot), v_t(t,\cdot)\rangle_{VV'}\,dt$ assumed

$$v_t \in L^{p'}(0,T:V')$$

- $\int_0^T\int_M (Q S(x)\beta(u)+f(t,x)-g(u))v\,dA\,dt$ is finite if we assume

$$f,u,v \in L^\infty((0,T)\times M)$$

space of bounded measurable functions on $(0,T)\times M$.

- Definition of weak solution: A function $u(t,x)$ is a bounded weak solution of (P) if $\forall\, T>0$
  - i) $u \in C([0,T]:L^2(M)) \cap L^p(0,T:V) \cap L^\infty((0,T)\times M)$
  - ii) the integral identity (II) holds for any $v \in C([0,T]:L^2(M)) \cap L^p(0,T:V) \cap L^\infty((0,T)\times M)$ such that $v_t \in L^{p'}(0,T:V')$.

2.1. Existence via a compactness abstract method.

Fixed point argument:

(o) Abstract framework

$$(P^*)\begin{cases}\dfrac{du}{dt}(t) + Au(t) \in R(t, u(t)) \\ u(0) = u_0(\cdot)\end{cases}$$

Possibly multivalue

$u \in C([0,T] : X)$, X a Banach space (in our case $L^2(M)$)

(1) For a given $w \in L^2(0,T : \underbrace{L^2(M)}_{X})$ we define the operator

$$\mathcal{T} : L^2(0,T : L^2(M)) \longmapsto L^2(0,T : L^2(M))$$

$$w \longmapsto z$$

z solution of $\begin{cases}\dfrac{dz}{dt}(t) + Az(t) = w \\ z(0) = u_0(\cdot)\end{cases}$

(2) If u is a fixed point of $\mathcal{T}(R(\cdot))$ i.e. if

$$u = \mathcal{T}(R(u))$$ to be well defined if R multivalued

then u is a solution of (P^*).

(3) If the operator A generates a compact semigroup and (for instance) $R(t,\cdot)$ is a bounded maximal monotone graph of $X \times X$ then $\mathcal{T}(R(\cdot))$ has a fixed point.

Vrabie's book, Pitman 1987

D-Vrabie : MAAA 1993

D- Vrabie : TMNA 1995

(4) The conditions of (3) hold in our case with

$$Au \equiv -\Delta_p u + g(u)$$

(use Sobolev compact embedding)

<u>Remark</u>. The solution is so regular as the solution of the unperturbed problem

$$\begin{cases} \frac{dz}{dt} + Az = f \\ z(0) = u_0 \end{cases}$$

is. In particular

$$u(t) \in D(A) \ \forall t > 0 \Rightarrow \Delta_p u \in L^2(M)$$

and

$$u_t \in L^2((0,T) \times M)$$

[of interest for the numerical approach

Bermejo (1994)

Bermejo-D-Tello (1995)]

· Correct formulation via Analysis on Manifolds:

$\mathbb{W} = \{(W_\lambda, \underline{w}_\lambda)\}_{\lambda\in\Lambda}$ atlas

local chart

W_λ open set of M

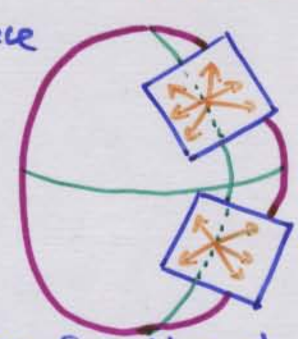
$\underline{w}_\lambda : W_\lambda \to \mathbb{R}^2$

homeomorphism of class C^∞

(for simplicity!)

$T(M) = \bigcup_{p\in M} T_p(M)$ tangent space

$T_p(M)$ Tangent space at $p \in M$



$\left\{\frac{\partial}{\partial\theta_\lambda}, \frac{\partial}{\partial\varphi_\lambda}\right\}$ is a basis of $T_p(M)$

if $\{\theta_\lambda, \varphi_\lambda\}$ is a coordinates system for the chart $(W_\lambda, \underline{w}_\lambda)$

g C^∞-Riemannian metric:

$i,j = 1,2, \quad g_{ij}^\lambda(\theta,\varphi) := \langle D_i P_\lambda(\theta,\varphi) \cdot D_j P_\lambda(\theta,\varphi) \rangle$

$$D_1 = \frac{\partial}{\partial\theta}, \quad D_2 = \frac{\partial}{\partial\varphi}$$

$$P_\lambda : \underline{w}_\lambda(W_\lambda) \longrightarrow M$$

parametrization $(\theta,\varphi) \longmapsto \underline{P} = (x,y,z)$

$(\alpha_\lambda)_{\lambda\in\Lambda} \equiv$ partition of unity subordinate to the covering $\{W_\lambda\}_{\lambda\in\Lambda}$

$\operatorname{supp} \alpha_\lambda \subset W_\lambda$

$\alpha_\lambda \geq 0, \quad \sum_{\lambda\in\Lambda} \alpha_\lambda(\underline{P}) = 1 \quad \forall \underline{P} \in M$

$$g = \sum_{\lambda\in\Lambda} \alpha_\lambda g^\lambda, \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

- Application to our case:
  - $u_t - (\rho(x)|u_x|^{p-2}u_x)_x = R_a(x,t,u) - R_e(x,t,u)$
  - Let $v(x,t)$ the "test function" (satisfyin[g] suitable conditions to be stated).
  - Multiplying and integrating by parts on $I \times (0,T)$

$I = (-1,1)$

$$(*)\left\{\begin{aligned} &\int_I u(x,T)v(x,T)\,dx - \int_0^T\int_I u(x,t)v_t(x,t)\,dx\,dt \\ &+ \int_0^T\int_I \rho(x)|u_x|^{p-2}u_x v_x\,dx\,dt \\ &= \int_0^T\int_I (R_a - R_e)v\,dx\,dt \\ &+ \int_I u_0(x)v(x,0)\,dx \quad + \int_0^T\int_{\partial I} \underbrace{\rho(x)|u_x|^{p-2}u_x}_{\substack{\shortparallel \\ 0}} v\,dx\,dt \end{aligned}\right.$$

Boundary condition

- Key idea: The own solution u "must" be a test function $\Rightarrow$

$$\forall t \in [0,T] \qquad \int_I |u(x,t)|^2\,dx < +\infty$$

$$\Uparrow$$

$$u \in C([0,T] : L^2(I))$$

$$\int_0^T \int_I \rho(x)\,|u_x|^p \,dx\,dt < +\infty$$

$$\Uparrow$$

$$u \in L^p(0,T:V)$$

$$V := \left\{ w = w(x) \;;\; w \in L^2(I) \;,\; w_x \in L^p(I:\rho) \right\}$$

ENERGY SPACE

Weighted Lebesgue Space

$$L^p(I:\rho) = \left\{ v = v(x) : \int_I \rho(x)\,|v(x)|^p \,dx < +\infty \right\}$$

- We can measure how big (or small) are the distances between two functions satisfying those conditions ($L^2(I)$, $L^p(I:\rho)$. $L^p(0,T:V)$, $C([0,T]:L^2(I))$ are normed spaces)
- These functional spaces are complete (they do not present the difficulties of $\mathbb{Q}$ rational nombers w.r.t. $\mathbb{R}$ real nombers). THEY ARE BANACH SPACES

• Coming back to the list of conditions on the set of test functions:

$$v \in C([0,T]:L^2(I)) \cap L^p(0,T:V)$$

(if u is in these spaces the integrals

$$\int_I u(x,T)\,v(x,T)\,dx$$

$$\int_0^T \int_I \rho(x)|u_x|^{p-2} u_x v_x$$

are "justified", i.e. finite

[use Young's inequality $ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$
$\frac{1}{p}+\frac{1}{p'}=1$]

• To "justify"

$$\int_0^T \int_I u(x,t)\,v_t(x,t)\,dx\,dt$$

and

$$\int_0^T \int_I (R_a - R_e)\,v\,dx\,dt \quad \text{assumed } u \in \underbrace{L^\infty}_{\text{SET OF BOUNDED FUNCTIONS}}(I\times(0,T))$$

we add the conditions

$$v \in L^\infty(I\times(0,T)) \text{ and } v_t \in L^{p'}(0,T:\underbrace{V'}_{\text{DUAL SPACE}})$$

Definition: u is a bounded weak solution of (P) if

$$u \in C([0,T]:L^2(I)) \cap L^\infty(I\times(0,T)) \cap L^p(0,T:V)$$

and satisfies the integral equality (*) for any $v \in C([0,T]:L^2(I)) \cap L^\infty(I\times(0,T)) \cap L^p(0,T:V)$ such that $v_t \in L^{p'}(0,T:V')$.

THEOREM.

For any $u_0 \in L^\infty(I)$ there exist, at least one bounded weak solution of (P).

Different methods of proof:

(a) via Volterra integral equations
(see Hetzer's lecture notes page 6)
(fails for the Budyko case)

apply to the Budyko case

(b) via a compactness method
(gives additional information on the regularity of the solution)
(applies to 2-D Models D-Tello (1983))

(c) via a regularization method
(useful for the study of the free boundary)

(3) If operator A generates a compact semigroup and (for instance) $R(t, \cdot)$ is a bounded maximal momotone graph of $\cancel{X} \times X$ then $\mathcal{T}(R(\cdot))$ has a fixed point (D- Vrabie : 1987
Vrabie's book : 1987
D- Vrabie : 1995)

(4) The conditions assumed in (3) holds in the case of problem (P)

[the proof uses some compact embedding detailed in Lemma 1 of the Lecture notes.]

Remark. The solution is a fixed point so it is so regular as the solution of the problem

$$\begin{cases} \frac{dz}{dt}(t) + Az(t) = w \\ z(0) = u_0 \end{cases}$$

is. In our case

$$u(t) \in D(A) \Rightarrow \left(\rho |u_x|^{p-2} u_x\right)_x \in L^2(I)$$

$$u_t \in L^2(0,T: L^2(I)) \cap L^{p'}(0,T: V')$$

[Used in the numerical approach: Bermejo (1994)] ■

2.2. Existence via a regularizat method.

(1) Approximation of the data:

β maximal monotone graphe $\sim \begin{cases} \beta_\varepsilon \in C^\infty(\mathbb{R}) \\ \beta'_\varepsilon > 0 \\ \beta_\varepsilon \text{ bounded} \end{cases}$

$\rho(x) = (1 - x^2) \quad \sim \quad \rho_\varepsilon(x) = \rho(x) + \varepsilon \ (>0$

$Q \quad \sim \quad Q_\varepsilon \in C^\infty$

$R_e \quad \sim \quad \begin{cases} R_{e,\varepsilon} \in C^\infty \\ R_{e,\varepsilon}(t, x, \cdot) \nearrow \end{cases}$

$u_0 \quad \sim \quad u_{0,\varepsilon}$

we assume that

$\beta_\varepsilon \longrightarrow \beta$
$Q_\varepsilon \longrightarrow Q$
$R_{e,\varepsilon} \to R_e$
$u_{0,\varepsilon} \to u_0$

in suitable senses.

(2) Approximate problem

$$(P^\varepsilon) \begin{cases} u_t - \left(\rho_\varepsilon(x) |u_x|^{p-2} u_x\right)_x \underbrace{- \varepsilon u_{xx}}_{\text{CRUCIAL}} = Q_\varepsilon \beta_\varepsilon(u) - R_{e,\varepsilon}(u) \\ \rho_\varepsilon(x)\left(|u_x|^{p-2} u_x + \varepsilon u_x\right) = 0 \quad \text{on } \partial I \times (0,T) \\ u(x,0) = u_{0,\varepsilon}(x) \end{cases}$$

NON DEGENERATE PROBLEM

(now is a true boundary condition

(3) A priori estimates:

Bounds on

$\|U\|_{L^\infty(I\times(0,T))}$, $\|\rho_\varepsilon U_x\|_{L^p(0,T:L^p(I))}$

independent of ε, assumed U solution of (P^ε)

(4) Passing to the limit

- Weak convergence in the energy space $\Rightarrow$ strong in some subespace
- difficulties:
  - $p \neq 2$ (identification of the limit of $\left(\rho_\varepsilon |U_x|^{p-2} U_x\right)_x$ Minty-Browder type argument)
  - β multivalued (strong $\times$ weak closedness of maximal monotone graphs: see Brezis' book).

· Normalization:

$$\underline{e}_\theta := \frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial\theta} \quad , \underline{e}_\varphi := \frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial\varphi}$$

orthonormal basis of $T_p(M)$.

· $\mathrm{grad}_M u : M \longrightarrow T(M)$

$$\mathrm{grad}_M u = \left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\tilde{u}}{\partial\theta}, \frac{1}{\sqrt{g_{22}}}\frac{\partial\tilde{u}}{\partial\varphi}\right) \quad , \tilde{u} = u \circ \underline{\psi}_\lambda^{-1}$$

· $\forall X \in T(M) \quad , X = h_1\underline{e}_\theta + h_2\underline{e}_\varphi$

$$\mathrm{div}_M X := \langle D_{\underline{e}_\theta}(h_1\underline{e}_\theta + h_2\underline{e}_\varphi), \underline{e}_\theta\rangle + \langle D_{\underline{e}_\varphi}(h_1\underline{e}_\theta + h_2\underline{e}_\varphi), \underline{e}_\varphi\rangle$$

· If $M = S^2_R$ $\quad \begin{cases} x = R\,\mathrm{sen}\,\varphi\cos\theta \\ y = R\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta \\ z = R\cos\varphi \end{cases}$

$\Delta_p u =$

$$= \frac{u_\theta}{R^2\mathrm{sen}^2\varphi}\frac{(p-2)}{2}\left(\frac{u_\theta^2}{R^2\mathrm{sen}^2\varphi} + \frac{u_\varphi^2}{R^2}\right)^{\frac{p-4}{2}}\left(\frac{2u_\theta u_{\theta\theta}}{R^2\mathrm{sen}^2\varphi} + \frac{2u_\varphi u_{\varphi\theta}}{R^2\mathrm{sen}^2\varphi}\right)$$

$$+ \frac{u_\varphi}{R^2}\frac{(p-2)}{2}\left(\frac{u_\theta^2}{R^2\mathrm{sen}^2\varphi} + \frac{u_\varphi^2}{R^2}\right)^{\frac{p-4}{2}}\left(\frac{2u_\theta u_{\theta\varphi}}{R^2\mathrm{sen}^2\varphi} - \frac{2u_\theta^2\cot\varphi}{R^2\mathrm{sen}^2\varphi} + \frac{2u_\varphi u_{\varphi\varphi}}{R^2}\right)$$

$$+ \frac{1}{R\,\mathrm{sen}\,\varphi}\left(\frac{u_\theta^2}{R^2\mathrm{sen}^2\varphi} + \frac{u_\varphi^2}{R^2}\right)^{\frac{p-2}{2}}\left(\frac{u_{\theta\theta}}{R\,\mathrm{sen}\,\varphi} + \frac{u_\varphi\cos\varphi}{r}\right)$$

$$+ \frac{1}{R}\left(\frac{u_\theta^2}{R^2\mathrm{sen}^2\varphi} + \frac{u_\varphi^2}{R^2}\right)^{\frac{p-2}{2}}\left(\frac{u_{\varphi\varphi}}{R}\right)$$

if $p=2$ the Laplace-Beltrami operator

$$\Delta u = \frac{1}{R\,\mathrm{sen}\,\varphi}\left(\frac{\partial}{\partial\varphi}\left(\frac{\mathrm{sen}\,\varphi\, u_\varphi}{R}\right) + \frac{1}{R\,\mathrm{sen}\,\varphi}u_{\theta\theta}\right)$$

2. THE TRANSIENT MODEL.

2.1. EXISTENCE OF SOLUTIONS

- Different methods
- "Energy space"

$$\mathbb{V} := \{u \in L^2(\mathcal{M}) : \nabla u \in L^p(T\mathcal{M})\}$$

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}$$ tangent bundle

- Bounded weak solution $(\forall T > 0)$
  - $u \in C([0,T] : L^2(\mathcal{M})) \cap L^\infty((0,T)\times\mathcal{M}) \cap L^p(0,T : \mathbb{V})$
  - $u_t \in L^{p'}(0,T : \mathbb{V}')$
  - $\exists\, z \in L^\infty((0,T)\times\mathcal{M})$, $z(t,x) \in \beta(u(t,x))$ a.e. (t,x)

$$\int_{\mathcal{M}} c(x)\, u(T,x)\, v(T,x)\, dA - \int_0^T \langle c(x)\, v_t(t,x), u(t,x)\rangle_{\mathbb{V}',\mathbb{V}}\, dt$$

$$+ \int_0^T\int_{\mathcal{M}} (k(x)|\nabla u|^{p-2}\nabla u) \cdot \nabla v \, dA\,dt + \int_0^T\int_{\mathcal{M}} G(u)\, v\, dA\,dt$$

$$= \int_0^T\int_{\mathcal{M}} QS(x)\, z(t,x)\, v(t,x)\, dA\,dt + \int_0^T\int_{\mathcal{M}} f\, v\, dA\,dt$$

$$+ \int_{\mathcal{M}} c(x)\, u_0(x)\, v(0,x)\, dA$$

$$\forall\, v \in L^p(0,T : \mathbb{V}),\ v_t \in L^{p'}(0,T : \mathbb{V}')$$

Theorem 1 (D[1993]), (D-TELLO[1994]) (BERMEJO-D-TELLO[1996])

For any $u_0 \in L^\infty(\mathcal{M})$ there exists, at least, a bounded weak solution

Idea of the proof. Define

$$w(t,x) := c(x)\,u(t,x) \,,$$

$$\varphi : \mathcal{M} \times \mathbb{R} \to \mathbb{R} \,, \quad \varphi(x,w) := \frac{w}{c(x)}$$

Then

$$\begin{cases} w_t - \operatorname{div}\left(k(x)\,|\nabla\varphi(w)|^{p-2}\nabla\varphi(w)\right) + \mathcal{G}\left(\frac{w}{c}\right) \in \\ \qquad \in Q\,S(x)\,\beta\left(\frac{w}{c}\right) + f(t,x) & \text{in } (0,T)\times\mathcal{M} \\ w(0,x) = \dfrac{u_0(x)}{c(x)} & \text{on } \mathcal{M}. \end{cases}$$

Let

$$X := L^q(\mathcal{M}) \quad 1 \le q \quad \text{(to be chosen)}$$

$$\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$$

$$w \longmapsto -\operatorname{div}\left(k\,|\nabla\varphi(w)|^{p-2}\nabla\varphi(w)\right) + \mathcal{G}\left(\frac{w}{c}\right)$$

Taking

$$X = L^2(\mathcal{M}) \quad \text{if} \quad c(x) \equiv c_0$$

$$X = L^1(\mathcal{M}) \quad \text{if} \quad c(x) \not\equiv c_0$$

we prove

- $\mathcal{A}$ is a m-accretive operator in $L^q(\mathcal{M})$ and $\overline{D(\mathcal{A})} = L^q(\mathcal{M})$
- $\mathcal{A}$ generates a compact semigroup of contractions in $L^q(\mathcal{M})$

Now, let

$$\mathcal{K} := \{ z \in L^\infty(0,T : L^\infty(\mathcal{M})) \; : \; \| z \|_{L^\infty} \leq C_0 \}$$

$$\mathcal{S} : \mathcal{K} \longrightarrow C([0,T] : L^q(\mathcal{M}))$$

$$z \longmapsto v \quad \text{mild solution of} \quad \begin{cases} \dfrac{dv}{dt} + \mathcal{A}(v) = z \\ v(0) = \dfrac{u_0}{c} \end{cases}$$

$$\mathcal{F} : L^p(0,T : L^q(\mathcal{M})) \longrightarrow 2^{L^p(0,T:L^q(\mathcal{M}))}$$

selection operator

$$v \longmapsto \{ h : h \in QS\beta(v) + f \text{ a.e.} \}.$$

Finally:

$$\mathcal{L} : \mathcal{K} \longrightarrow 2^{L^p(0,T:L^q(\mathcal{M}))}$$

$$z \longmapsto \{ h : \; h \in \mathcal{F}(\mathcal{S}(z)) \}$$

Then:

"w solution $\Leftrightarrow$ z is a fixed point of $\mathcal{L}$" $(z \in QS\beta(w) + f)$

But:

- $\mathcal{F}(v)$ nonempty convex closed set of $L^p((0,T) : L^q(\mathcal{M}))$ $\forall v$
- graph $\mathcal{F}$ (strongly) $\times$ (weakly) sequentially closed in $L^p((0,T) \times L^q(\mathcal{M})) \times L^p((0,T) \times L^q(\mathcal{M}))$
- $\mathcal{S}$ is sequentially continuous from $L^p((0,T) : L^q(\mathcal{M}))$ (weak) to $C([0,T] : L^q(\mathcal{M}))$ strong

D-Vrabie (89)

$\Longrightarrow$

Arino, Gautier, Penot (1984) $\mathcal{L}$ has a fixed point.

- local solutions can be extended to $[0,T]$ $\forall T$
- mild solutions are bounded weak solutions

■

Remarks.

1. Similar approach for the existence of periodical solutions when

   $QS(x)$ replaced by $Q(x,t) \geqslant 0$
   Q time-periodic

   (M. Badii - D, 1996)
   (G. Hetzer, 1987)
   (G. R. North, J. Coakley (1979) study of "ice ages"
   Snowcover over the summer is a necessary condition for the growth of continental glaciers)

2. Existence via approximation by classical solutions

$$\begin{cases} C(x)u_t - \operatorname{div}\left(K(x)\left[\varepsilon + |\nabla u|^2\right]^{\frac{p-2}{2}} \nabla u\right) + \mathcal{G}_\varepsilon(u) = \\ \qquad\qquad = QS(x)\beta_\varepsilon(u) + f_\varepsilon(t,x) \\ u(0,x) = u_{0,\varepsilon}(x) \end{cases}$$

(X. XU (1991) $p=2$), (P. TOLKSDORFF (1983) $p \neq 2$, $M \equiv \Omega$)

(Regularizing merely β_ε by β_ε also leads to existence results : D-(1993)).

2.2. ON THE UNIQUENESS OF SOLUTIONS

- If the coalbedo function $\beta(u)$ is Lipschitz continuous the uniqueness of solutions is standar

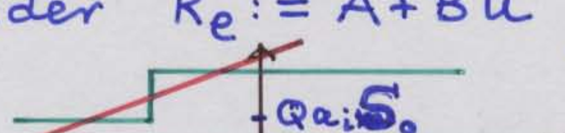
2.2a. NON UNIQUENESS OF SOLUTIONS FOR BUDYKO TYPE MODELS (β DISCONTINUOUS)

- β discontinuous (or multivalued) at the right hand side of the equation (source term)

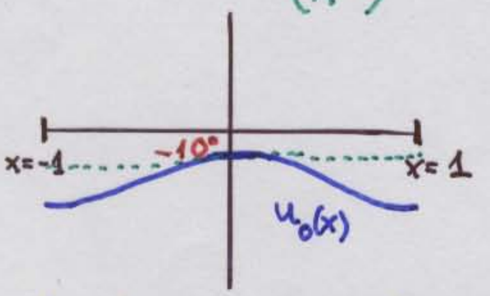
- <u>Counterexample for 1-d model</u>.
  - D (1991) in D-J.L. LIONS (eds): Mathematics, Climat and Environment, Masson 1993 (curso de Verano, EL ESCORIAL) Agosto 1991)

To fix ideas, consider $R_e := A + Bu$ with

$$\boxed{Q a_i \leq A - 10B} \quad (H^*) \quad \left[\Rightarrow \right.$$



$\left. \right]$



$$(H^*_{u_0}) \begin{cases} u_0 \in C^\infty(-1,1),\ u_0(x) = u_0(-x) \\ u_0(0) = -10 \quad u_0'(0) = u''(0) = 0 \\ u_0'(x) < 0 \quad x \in (0,1),\ u_0'(1) = 0 \end{cases}$$

Let us construct two different solutions:

2.3. On the uniqueness of solutions.

The main difficulty comes from the fact that a may be discontinuous:

Easy case a Lipschitz (e.g. $a \in C^1$)

Idea of the proof.

We assume a stronger assumption on the diffusion operator:

$$\left(\vec{A}(x,\vec{\xi}) - \vec{A}(x,\vec{\eta})\right)\cdot\left(\vec{\xi}-\vec{\eta}\right) \geq 0 \quad \forall \vec{\xi},\vec{\eta} \in \mathbb{R}^N$$

[o.k. for the Held-Suarez model]

If u and $\hat{u}$ are two solutions

$$c(u_t - \hat{u}_t) - \operatorname{div}\left(A(x,\nabla u) - A(x,\nabla \hat{u})\right) =$$

$$= QS\left[a(u) - a(\hat{u})\right] - R_e(u) + R_e(\hat{u})$$

Easiest case:
[if $a(u) = a(\hat{u}) \equiv 0$]

$$\frac{d}{dt}\int_\Omega c(u-\hat{u})^2 dx + \underbrace{\int_\Omega \left(A(x,\nabla u) - A(x,\nabla\hat{u})\right)\cdot(\nabla u - \nabla \hat{u})\, dx}_{\geq 0} =$$

$$= - \underbrace{\int_\Omega \left(R_e(u) - R_e(\hat{u})\right)(u-\hat{u})}_{\leq 0} \qquad R_e(u) \nearrow$$

$\Rightarrow$

$$\|c(u(\cdot,t) - \hat{u}(\cdot,t))\|_{L^2(\Omega)} \leq \underbrace{\|c(u(\cdot,0) - \hat{u}(\cdot,0))\|_{L^2(\Omega)}}_{= 0} \Rightarrow u = \hat{u}$$

If a Lipschitz

Change of unknowns

$u = e^{\lambda t} w \qquad \lambda > 0$ large enough

$u_t = e^{\lambda t} w_t + \lambda e^{\lambda t} w = e^{\lambda t}(w_t + \lambda w)$

Multiply the equation of $u - \hat{u}$ by $e^{-\lambda t}$

$$c(w_t - \hat{w}_t) - \operatorname{div} e^{-\lambda t}\left(A(x, \nabla w e^{\lambda t}) - A(x, \nabla \hat{w} e^{\lambda t})\right)$$

$$= \underbrace{-\lambda(w - \hat{w}) + Qs e^{-\lambda t}\left[a(e^{\lambda t} w) - a(e^{\lambda t}\hat{w})\right]}_{\text{DECREASING IF } \lambda \to +\infty}$$

$$- R_e(e^{\lambda t} w) + R_e(e^{\lambda t} w)$$

$(t \in (0,T))$.

$\Downarrow$

Thm (!, continuous dependence, comparison).

- REMARK.
  - The uniqueness for $a(u)$ ↗ and discontin (and u_0 in some suitable class) seems to be an OPEN PROBLEM.
  - A non-uniqueness result for some special initial data can be built using the following simple formulations

ODE $\begin{cases} \dfrac{dz}{dt} = H(z) \\ z(0) = 0 \end{cases}$ $\qquad z_1(t) = 0 \ \forall t$ is a solutio

$\qquad\qquad\qquad\qquad z_2(t) = t$ is a different solution

- A recent partial result (uniqueness) E. FEIRESEL Non Linear Analysis 199

A first solution represents a "completly ice covered Earth"

Proposition 1

Under the above conditions there exists, at least, one solution u^* such that

$$u^*(t,x) < -10 \quad \forall t \in (0,T], \; x \in (-1,1)$$

Proof. Take as u^* the (unique) solution of the simpler problem

$$P^* \begin{cases} u^*_t - (\rho(x)|u^*_x|^{p-2}u^*_x)_x + Bu^* = QS(x)a_i - A & \text{in } (0,T)\times(-1,1) \\ \rho(x)|u^*_x|^{p-2}u^*_x = 0 & \text{on } x = \pm 1, \; t \in (0,T) \\ u^*(0,x) = u_0(x) & \text{on } (-1,1) \end{cases}$$

$(\rho(x) := (1-x^2)^{p-1})$.

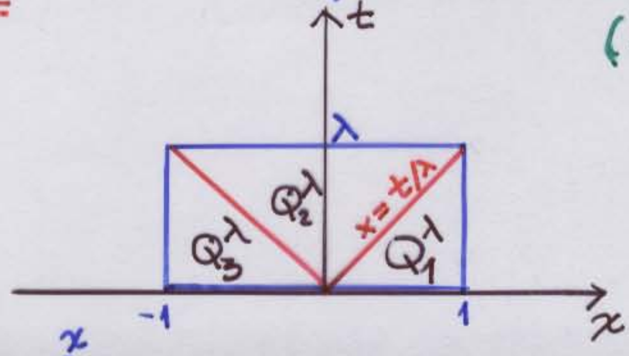
$u^*(t,x) < -10 \quad \forall t \in (0,T], x \in (-1,1)$ (by (H^*) the strong maximum principle applies to $-10 - u^*(t,x)$) $\Rightarrow u^*$ is a solution with discontinuous coalbedo $\beta(u^*)$ ∎

- A second solution (in fact a family of solutions) can be buit having a "free-ice zone"

Proposition 2.

There exists, at least, one solution u such that the "free-ice zone" $\{(t,x) \in (0,T]\times(-1,1) : u(t,x) > -10\}$ is not empty

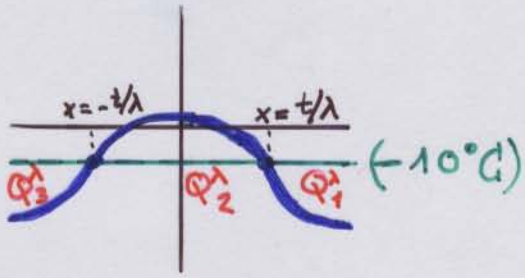
Proof. Given a parameter $\lambda > 0$ we define $(\lambda < T)$



$$\overset{t}{(0,\lambda)} \times \overset{x}{(-1,1)} = Q_1^\lambda \cup Q_2^\lambda \cup Q_3^\lambda$$

$$Q_1^\lambda := \{(t,x) \in (0,\lambda) \times (0,1),\ \ x > t/\lambda\}$$
$$Q_3^\lambda := -Q_1^\lambda = \{(t,x) \in (0,\lambda) \times (-1,0),\ \ x < -t/\lambda\}$$
$$Q_2^\lambda := \{(t,x) \in (0,\lambda) \times (-1,1) :\ \ -t/\lambda \le x \le t/\lambda\}$$

Define $v^\lambda(t,x)$ by:



Q_1^λ: v^λ is the (unique) solution of the auxiliary problem

$$\begin{cases} v_t - (\rho(x)|v_x|^{p-2}v_x)_x + Bv = QS(x)a_i - A & \text{in } Q_1^\lambda \\ v_x(t,1) = 0\ ,\ \ v(t,t/\lambda) = -10 & t \in (0,\lambda) \\ v(x,0) = u_0(x) & x \in (0,1) \end{cases}$$

Q_3^λ: $\quad v^\lambda(x,t) := v^\lambda(-x,t)$

Q_2^λ: $\quad v^\lambda(x,t) := -10 + C^\lambda(t)\,(x - t/\lambda)(x + t/\lambda)$

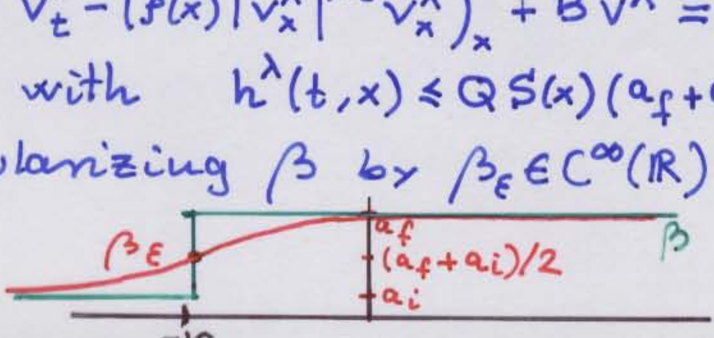
- It is possible to choose $C^\lambda(t)$ such that

  $v^\lambda \in C^0\ ,\ \ v_x^\lambda \in C^0$

  $v_t^\lambda - (\rho(x)|v_x^\lambda|^{p-2}v_x^\lambda)_x + Bv^\lambda = -A + h^\lambda(t,x)$

  with $\quad h^\lambda(t,x) \le QS(x)(a_f + a_i)/2$

- Regularizing β by $\beta_\varepsilon \in C^\infty(\mathbb{R})$



and taking $u^\varepsilon(t,x)$ solution of the approxima
problem

$$\begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x + Bu = Q\,S(x)\beta_\varepsilon(u) - A & \text{in } (0,T)\times(-1,1) \\ \rho(x)|u_x|^{p-2}u_x = 0 & \text{on } x=\pm 1,\ t\in(0,T) \\ u(x,0)=u_0(x) & x\in(-1,1) \end{cases}$$

it is easy to see that

$$u^\varepsilon(t,x) \geq V^\lambda(t,x) \quad \text{on } [0,\lambda]\times(-1,1)$$

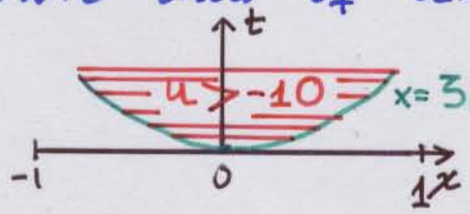
and that

$$u^\varepsilon \xrightarrow{L^\infty((0,T)\times(-1,1))} u\ , \text{ when } \varepsilon\downarrow 0\ , \ u \text{ solution}\ ,\ u(t,\cdot)\in C($$

$(\Rightarrow \ u(x,t) > -10 \quad \forall x\in(-t/\lambda\ ,\ t/\lambda)\ ,\ t\in(0,\lambda))$. ■

Remarks.

1. By the Implicit Function Theorem it can be shown that if $u = \lim u_\varepsilon$ then



free boundary
$\zeta(0)=0\ ,\ \zeta'(0)=+\infty$

2. Other non-uniqueness results can be obtained by using other subsolutions:
   - self-similar solutions of $u_t - (\rho(x)u_x)_x + Bu = Q S_{q_i} -$ in $(0,T)\times\mathbb{R}$

     $u(t,x) = t f(\eta) \quad \eta: \frac{x}{\sqrt{t}}$
   - travelling wave solutions

     $u(t,x) = f(\xi) \quad \xi = x \pm ct$

2.2b. A UNIQUENESS CRITERION.

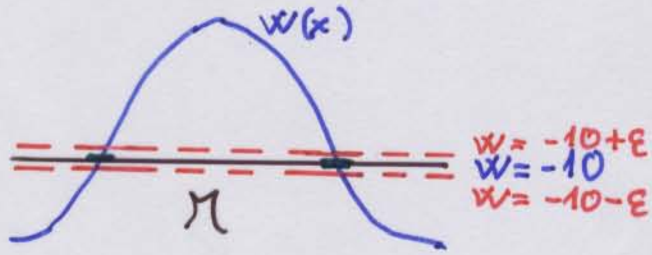
- The crucial point in the counterexample (Proposition 2) is that $u(0,x)\ (\equiv u_0(x))$ is "very flat" ($u_0'(0)=u_0''(0)=0$) at the point where $u_0(x)=-10$ (i.e. $x=0$).
- Solutions which are not flat (nondegenerate solutions) are the unique solutions (when they exist) D (1993), D-Tello (1994), Bermejo-D-Tello (199
- M Riemannian manifold

Definition. Let $w\in L^\infty(M)$. Function w satisfies the strong (resp. weak) p-nondegeneracy property if $\exists\, C>0$ and $\varepsilon_0>0$ such that

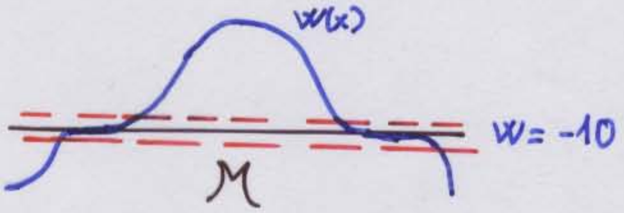
$$|\{x\in M : |w(x)+10|\leq\varepsilon\}|\leq C\varepsilon^{p-1}$$

(resp. $|\{x\in M: 0<|w(x)+10|\leq\varepsilon\}|\leq C\varepsilon^{p-1}$) $\forall\varepsilon\in(0,\varepsilon_0)$.

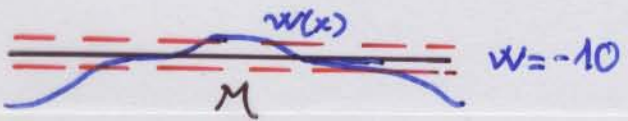
Examples.



(strong nondegeneracy $\Rightarrow |\underbrace{\{x\in M: w=-10\}}_{\text{THE MUSHY REGION OF } w}|=0$)



(weak nondegeracy allows $|\{x\in M: w=-10\}|>0$)



(very flat functions w does not satisfy nondegeneracy properties)

Theorem.

Assume that there exists a solution u such that $u(t,\cdot)$ satisfies the weak p-nondegeneracy property $\forall t \in [0,T]$. Then u is the unique bounded weak solution of P.

Idea of the proof.

The multivalued "selection operator"

$$\mathfrak{F} : w \longmapsto \{ h : h(x) \in \beta(w(x)) \text{ a.e. } x \in \mathcal{M} \}$$

is continuous from $L^\infty(\mathcal{M})$ into $L^q(\mathcal{M})$ $(\forall q \in [1,\infty))$ at any "point" w satisfyiug the strong p-degeneracy property

Lemma.

i) Let $w, \hat{w} \in L^\infty(\mathcal{M})$ and assume that w satisfies the strong p-nondegeneracy property. Then $\forall q \in [1,\infty)$ $\exists \hat{C} > 0$ such that $\forall z, \hat{z} \in L^\infty(\mathcal{M})$ with $z(x) \in \beta(w(x))$, $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in \mathcal{M}$ we have

$$\| z - \hat{z} \|_{L^q(\mathcal{M})} \leq (a_f - a_i) \min \{ \hat{C} \| w - \hat{w} \|_{L^\infty(\mathcal{M})}^{(p-1)/q}, |\mathcal{M}|^{1/q} \}$$

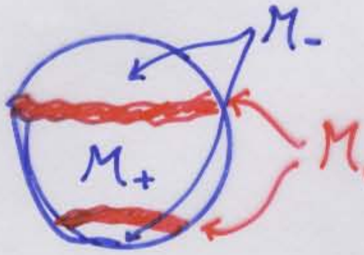
ii) if w satisfies the weak p-nondegeneracy property

$$\int_{\mathcal{M}} (z(x) - \hat{z}(x))(w(x) - \hat{w}(x))\, dA \leq (a_f - a_i) \hat{C} \| w - \hat{w} \|_{L^\infty(\mathcal{M})}^{p}$$

Proof of the Lemma.

i) Assume $\|w-\hat{w}\|_{L^\infty(M)} \le \varepsilon_0$ (otherwise trivial)

Define the decomposition



$M = M_0 \cup M_+ \cup M_-$ (resp. $M = \hat{M}_0 \cup \hat{M}_+ \cup \hat{M}_-$)

$M_0 := \{x \in M : w(x) = -10\}$

$M_+ := \{x \in M : w(x) > -10\}$

$M_- := \{x \in M : w(x) < -10\}$

Then

$$|z(x) - \hat{z}(x)| \le (a_f - a_i) \quad \text{on} \quad \underbrace{M_0 \cup \hat{M}_0 \cup (M_+ \cap \hat{M}_-) \cup (M_- \cap \hat{M}_+)}_{:= B}$$

$$\Rightarrow \quad \|z - \hat{z}\|_{L^q(M)} \le (a_f - a_i) \min\{|B|^{1/q}, |M|^{1/q}\}$$

but strong p-nondegeneracy $\Rightarrow |B| \le C_1 \varepsilon_0^{p-1}$

ii) Analogous. ■

Proof of the Theorem (contin.)

[Assume strong p-nondegeneracy]

Case(a): $p > 2$. Let $u, \hat{u}$ be bounded weak solutions

Applying the Lemma and the imbedding $V \subset L^\infty(M)$

$$\frac{1}{2}\frac{d}{dt}\|u(t) - \hat{u}(t)\|^2_{L^2(M)} \le (Q M (a_f - a_i) - C_1)\|u(t) - \hat{u}(t)\|^p_{L^\infty(M)} + C_2\|u(t) - \hat{u}(t)\|^2_{L^2(M)}$$

(we have assumed the heat capacity $c(x) \equiv 1$, for simplicity)

If $QM(a_f - a_i) - C_1 \le 0$ the conclusion comes from Gronwall' inequality

If $QM(a_f - a_i) - C_1 > 0$ we introduce a rescaling in the manifold $M_\delta := \delta M$ (and in u) and making δ small enough we reduce the situation to the above case.

(carefull study of the constant in the inclusion $V \hookrightarrow L^\infty(M)$

Case (b) : $p=2$. Use that $V \hookrightarrow L^q(M)$ for any $q \in [1,\infty)$ and that $\hat{u}(t)$ and $u(t)$ are bounded.

Remarks.

1. The class of solutions satisfying the strong (resp. weak) p-nondegeneracy is not empty under additional conditions on u_0

Example : $\left.\begin{array}{l} M = S^1 \\ S(x) = \hat{S}(\phi) \\ c(x) = c(\phi) \end{array}\right\}$

- $u_0(\lambda, \phi) = \widetilde{u}_0(\phi)$
- the set $M_0 := \{\phi \in (-\pi,\pi) : \widetilde{u}_0(\phi) = -10\}$ has a finite number of connected components

- $\boxed{|\widetilde{u}_0(\phi) + 10| \geq K |\phi - \phi_i|^{1/(p-1)} \quad \forall \phi_i \in \partial M_0}$

Then there exists a (unique) bounded weak solution satisfying the weak p-nondegeneracy property.
If in addition

$$\frac{\partial}{\partial \phi} \widetilde{u}_0(\phi_i) \neq 0 \quad \forall \phi_i \in \partial M_0$$

then u satisfies the strong p-nondegeneracy property

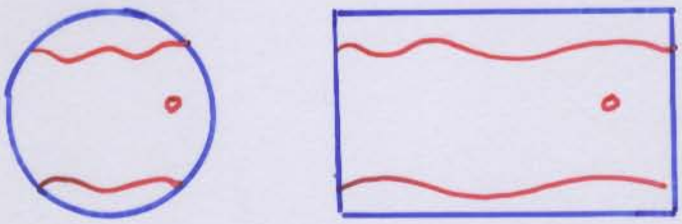
2. Relate model arises in Combustion Theory ($p=2$, $\Omega \subset \mathbb{R}^N$, ...). Uniqueness criterion via a different approach by
R. Gianni - J. Hulshof (1992)
R. Gianni - P. Mannuci (1993, 94)

2.3. (On the free boundaries).

- For the Budyko type model (β discont.) and $p=2$ (linear diffusion) the level set

$$M_0(t) := \{x \in M : u(t,x) = -10\}$$

is formed by smooth curves



(see Xu (1991) for the 1-d model)
(works by R. Gianni, R. Ricci, ... for the Combustion problem)

- Satelite pictures $\Rightarrow$ $M_0(t)$ is far to be a set of lignes but a narrow zone (400 km

- If $p>2$ it can be shown (D-(1993)) (N=1) that

$$|M_0(0)| > 0 \Rightarrow |M_0(t)| > 0 \quad \text{at least for } \underset{\text{small}}{t>0}$$

(proof by a local energy method: S. N. Antontsev (1981), D-Veron (1983), Antontsev-D (book in preparation)) + Shmarev

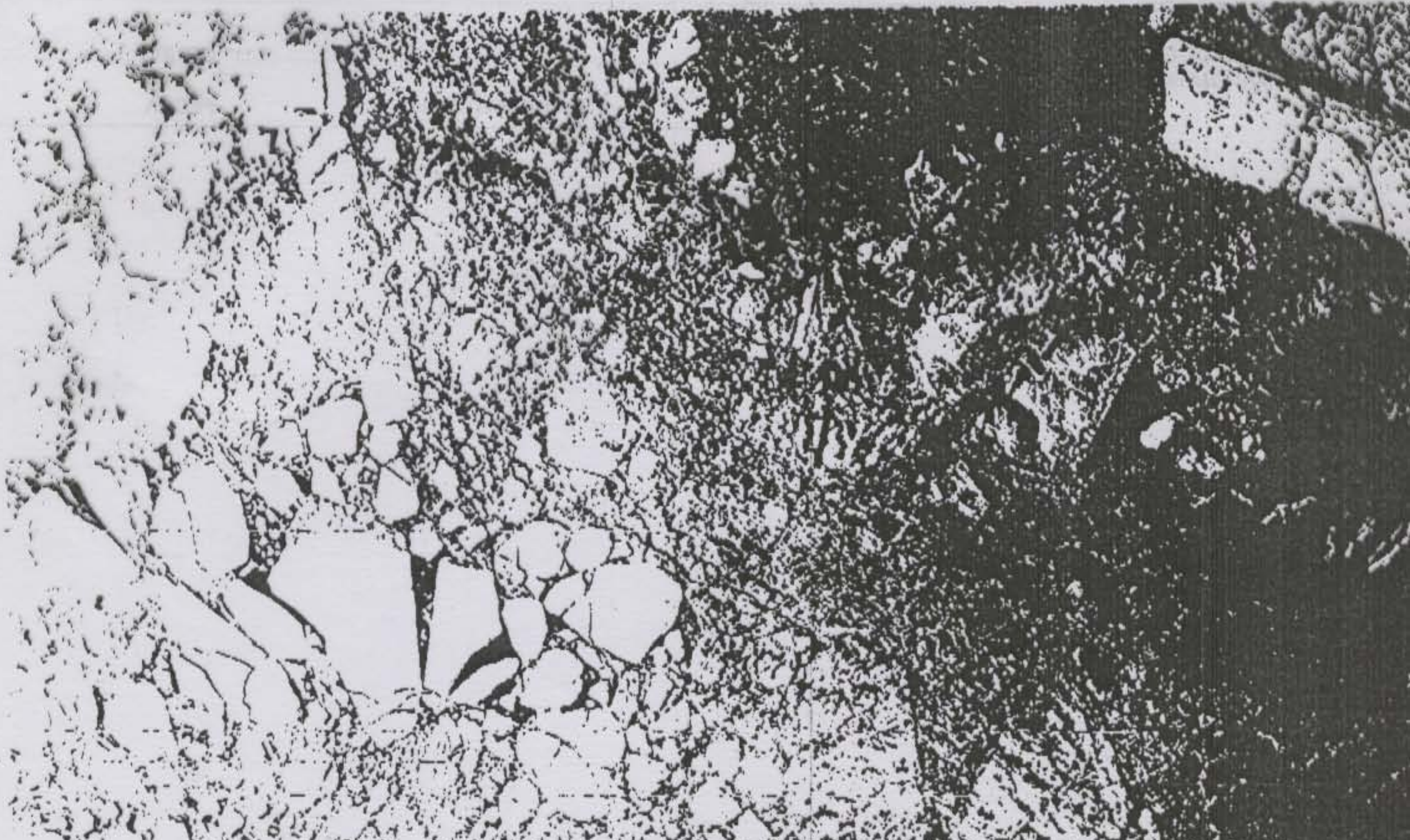
- For the general model ($M \neq I$) use local charts


Hielo perenne
Groenlandia
Hielo estacional
3
Agua dulce del hielo
que se deshace
Europa
Norteamérica
2
Agua fría y salada
Agua templada desde los trópicos
CINTA
TRANSPORTADORA
OCEÁNICA
Océano Atlántico norte

EL PAÍS / NYT

MER DE WEDDELL

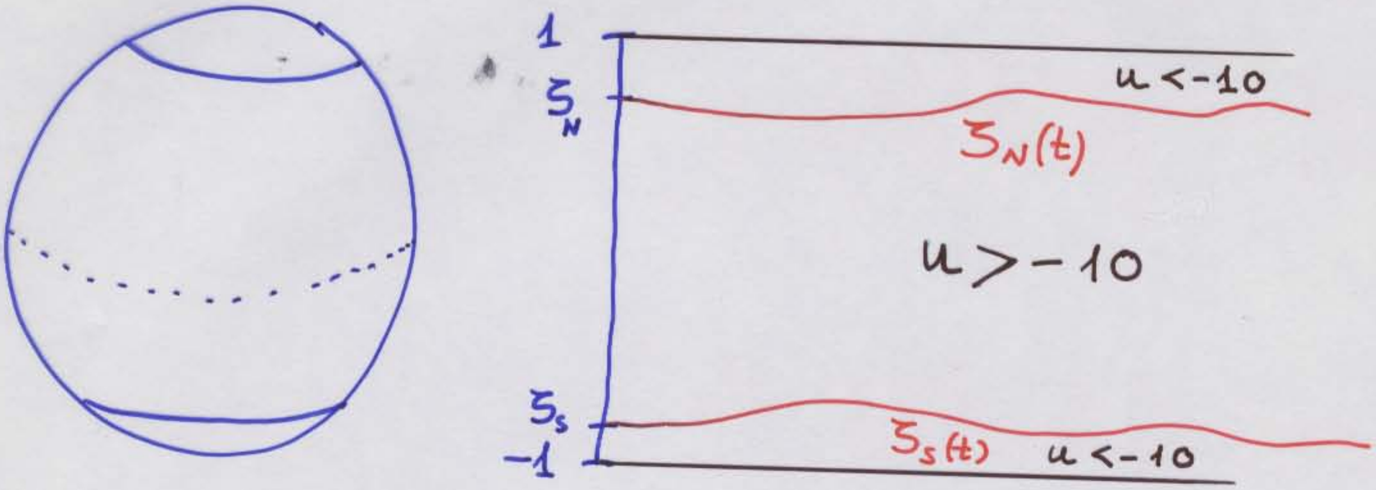
Image acquise par le satellite Spot le 10 décembre 1987



SPOT IMAGE

cnes

4. The free boundaries in the Budyko type EBMs.



If the set $\{x\in I : u(x,t)>-10\}$ has a unique connexe component $\Rightarrow$

$$\forall t \qquad \{x\in I : u(x,t)>-10\} = (\xi_s(t), \xi_N(t))$$

FREE BOUNDARIES

- Unknown "a priori".
- If $p=2$ (linear diffusion) u is very regular except at the free boundaries where $u_{xx}(x,t)$ is discontinuous.

· Mathematical results

· $p=2$, X. Xu (1991)

[E. Fereisl - J. Norbury (1991)]
[R. Gianni - J. Hulshof (1992)]

$\zeta_N, \zeta_S \in C^\infty([0,T^*))$ with $T^*>0$ defined as the first time $\zeta_N(T^*)=\zeta_S(T^*)$

- (T^* may be not finite).
- assumptions on the regularity of u_0 and ecuator - symmetry.
- proofs by studying the level sets of regularized problem
- (implicit) formula for

$\frac{d}{dt}\zeta_N(t)$, $\frac{d}{dt}\zeta_S(t)$ (speeds of the fronts)

· Similar results for $p\neq 2$ remain an open question.

· Conjecture (· if $p\in(1,2)$ same type of results)

· new behaviour for $p>2$

- Separation between the ice-free and ice-covered zones:
    - Satellite pictures $\Rightarrow$ is not a ligne but a narrow zon

$$M(t) = \{x \in I : u(x,t) = -10\}$$ positively measured ?

mushy region

- for $p = 2$ $\quad M(t) = \{\xi_s(t)\} \cup \{\xi_N(t)\}$
  $\Rightarrow |M(t)| = 0$

- Very different answer for the Stone type model

<u>Theorem</u>. Assume $p > 2$ and let u_0 such that

$$M(0) := \{x \in I : u_0(x) = -10\} \supset B(x_0, R_0)$$

Then there exists $T^* \in [0, +\infty]$ and a nonincreasing function $R(t)$, $R(0) = R_0$ such that

$$M(t) := \{x \in I : u(x,t) = -10\} \supset B(x_0, R(t))$$

- Proof by energy method (Antontsev-D)
- Behaviour of $M(t)$?? (D-Fasano-Meirmanov(91))

3. THE STATIONARY MODEL : S-SHAPED BIFURCATION CURVE

· The asymptotic behaviour for $t \to \infty$

Theorem (D-Hernandez-Tello, 1996)

i) If $u_0 \in V \cap L^\infty(M)$, $f \in L^\infty((0,\infty)\times M) \cap W^{1,1}_{loc}((0,\infty) : L^1(M))$ and

$$\int_t^{t+1} \|f_t(s,\cdot)\|_{L^1(M)} ds \le C_0 \quad (\text{indep. on } t) \quad \forall t \ge 0$$

then there exists a weak solution u such that

(*) $\quad u \in L^\infty(0,\infty : V)$ and $u_t \in L^2(0,\infty : L^2(M))$

(ii) If u is any weak solution satisfying (*) then $\omega(u) := \{u_\infty \in V \cap L^\infty(M) : \exists t_n \to \infty$ such that $u(t_n,\cdot) \to u_\infty$ in $L^2(M)\} \ne \emptyset$. In fact, if $u_\infty \in \omega(u)$ then $\exists \hat{t}_n \to \infty$, $u(\hat{t}_n,\cdot) \to u_\infty$ (strongly) in V and u_∞ is a solution of the stationary problem

$$P_Q \left\{ \; -\operatorname{div}(\kappa(x)|\nabla u_\infty|^{p-2}\nabla u) + G(u_\infty) \in Q S(x) \beta(u_\infty) + f_\infty \quad \text{in } M \right.$$

assumed

$$\int_{t-1}^{t+1} \|f(\tau,\cdot) - f_\infty(\cdot)\|_{V'} d\tau \to 0 \quad \text{as } t \to \infty$$

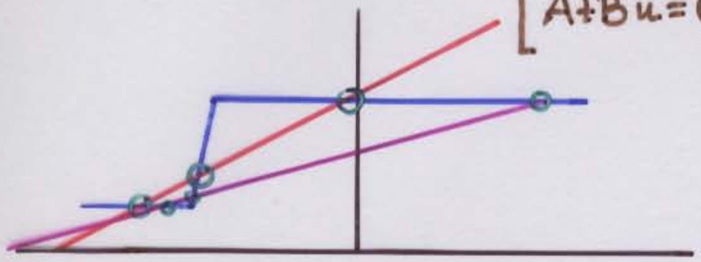
(Langlais-Phillips (1985))
(D-de Thélin, (1994))

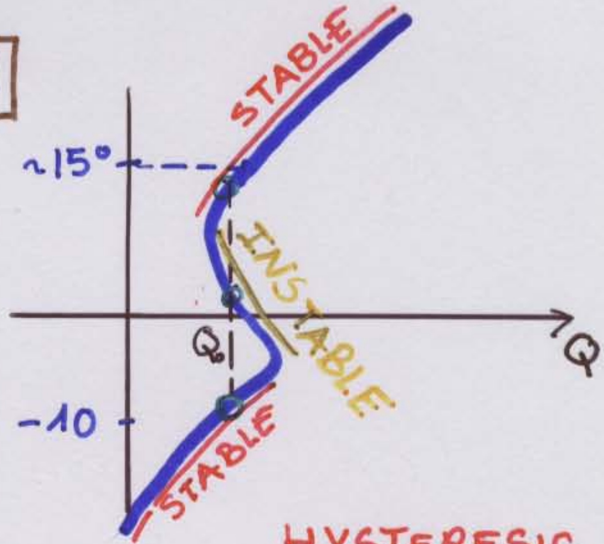
- Convergence of the free boundaries under nondegeneracy property

- EBM used for feedback analysis and stability problems : The simpler model (0-d stationary EBM) exhibits a phenomenon of multiple equilibria

$$0 = R_a - R_e$$

$$\left[A + Bu = Q\beta(u)\right]$$





HYSTERESIS LOOPS

QUESTION:

- Diffusive (stationary) Energy Balance Models ?

$$P_Q \quad \begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + \mathcal{G}(u) \in Q S(x)\beta(u) + f(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{M} \end{cases}$$

Multiplicity results:

Theorem (D-Hernandez-Tello, 1997)

Assume

(H_S) $\quad S : \mathcal{M} \to \mathbb{R}$, $S \in L^\infty(\mathcal{M})$, $S_1 \geq S(x) \geq S_0 > 0$

$(H_{\mathcal{G}}^*)$ $\begin{cases} \mathcal{G} : \mathbb{R} \to \mathbb{R} \text{ continuous, strictly increasing}, \ \mathcal{G}(0) = 0 \\ \lim\limits_{|s| \to \infty} |\mathcal{G}(s)| = +\infty \end{cases}$

(H_f^*) $f \in L^\infty(\mathcal{M})$ and $f(x) \leq -C_f$ for some $C_f > 0$

(H_β^*) $\begin{cases} \beta \text{ is a maximal monotone graph of } \mathbb{R}^2 \text{ with} \\ \beta(r) = \{m\} \ \forall r \in (-\infty, -10-\varepsilon), \ \beta(r) = \{M\} \ \forall r \in (-10+\varepsilon, +\infty) \end{cases}$

$0 < m < M$

Then:

i) for any $Q > 0$ there is a minimal solution $\underline{u}$ (resp. maximal solution $\overline{u}$) of (P_Q). Moreover, any other solution u must satisfy

$$\underline{u} \leq u \leq \overline{u} \qquad \text{in } \mathcal{M}$$

ii) for any $Q > 0$ there is, at least, a solution u of (P_Q) which is a global minimum of

$$J(w) := \frac{1}{p}\int_{\mathcal{M}} |\nabla w|^p dA + \int_{\mathcal{M}} \mathcal{G}(w)\, dA - \int_{\mathcal{M}} f w\, dA - \int_{\mathcal{M}} Q S(x) j(w)$$

$$\left(\mathcal{G}(s) := \int_0^s \mathcal{g}(\tau)\, d\tau \ , \ \beta = \partial j \right)$$

on the set $K := \{ w \in V, \ \mathcal{G}(w) \in L^1(\mathcal{M}) \}$

Moreover, if in addition

(H_{C_f}) $\left\{ \mathcal{g}(-10-\varepsilon) + C_f > 0 \ , \ \dfrac{\mathcal{g}(-10-\varepsilon) + \|f\|_{L^\infty}}{\mathcal{g}(-10-\varepsilon) + C_f} \leq \dfrac{S_0 M}{S_1 m} \right.$

holds then defining

$$Q_1 := \frac{\mathcal{g}(-10-\varepsilon) + C_f}{S_1 M} \qquad Q_2 := \frac{\mathcal{g}(-10+\varepsilon) + \|f\|_{L^\infty}}{S_0 M}$$

$$Q_3 := \frac{\mathcal{g}(-10-\varepsilon) + C_f}{S_1 m} \qquad Q_4 := \frac{\mathcal{g}(-10+\varepsilon) + \|f\|_{L^\infty}}{S_0 m}$$

we have

a) if $0 < Q < Q_1 \Rightarrow (P_Q)$ has a unique solution

b) if $Q_2 < Q < Q_3 \Rightarrow (P_Q)$ has at least three solutions

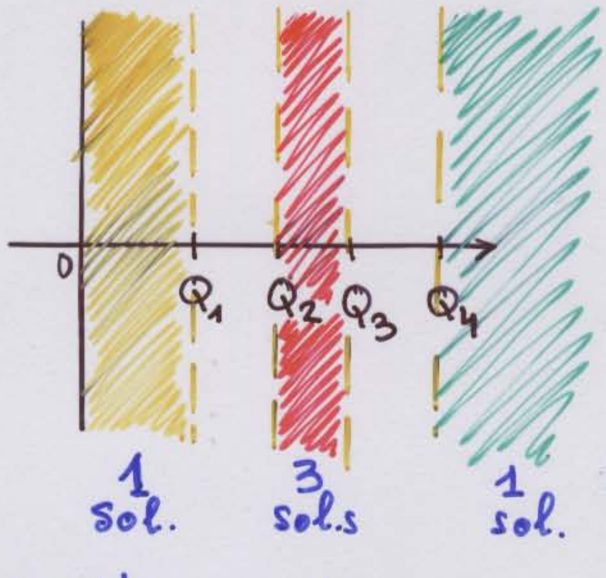
c) if $Q_4 < Q \Rightarrow (P_Q)$ has a unique solution. ∎

(• Method of super and subsolutions,

• in case (b): construction of two subsolutions (for an approximate equation) V_1, V_2 and two supersolutions U_1, U_2 such that

$$V_2 < U_2 < -10-\varepsilon < -10+\varepsilon < V_1 < U_1,$$

Amann theorem (1976) + passing to the limit)



• Sharp results under

(H_f^{**}) $\quad f(x) \equiv C_f$ on $\mathcal{M}$ (Budyko's choice)

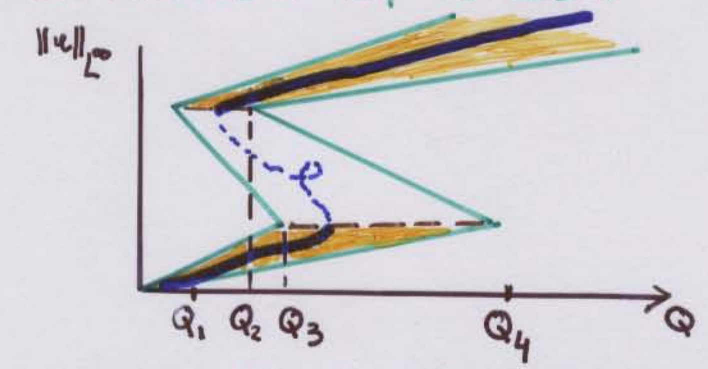
Bifurcation diagram: Define

$$\Sigma := \{(Q,u) : Q \geq 0 \text{ and } u \text{ is solution of } (P_Q)\}$$

Theorem (Arcoya-D-Tello, 1998)

Σ has an unbounded S-shaped component "starting" in $(0, G^{-1}(-G_f))$ with, at least, two "turning points" contained in the region $(Q_1, Q_2) \times L^\infty(\Omega)$ and $(Q_3, Q_4) \times L^\infty(\Omega)$, respectively.

- Approximate model $\beta \to \beta_\varepsilon$
- Rabinowitz bifurcation theorem (1971)
- Construction of a box



- Passing to the limit by a topological argument (Whyburn 1955)

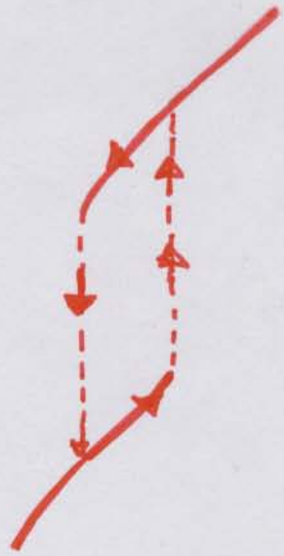
Remarks

1. Previous results ($p=2$, β regular)

   1d - North (1975), Ghil (1976), ...
   Drazin-Griffel (1976), ...
   B.E. Schmidt (1996) S

   2d Hetzer (1990), Hetzer-P.L. Schmidt(1990...
   North, Mengel, Short (1983), ...

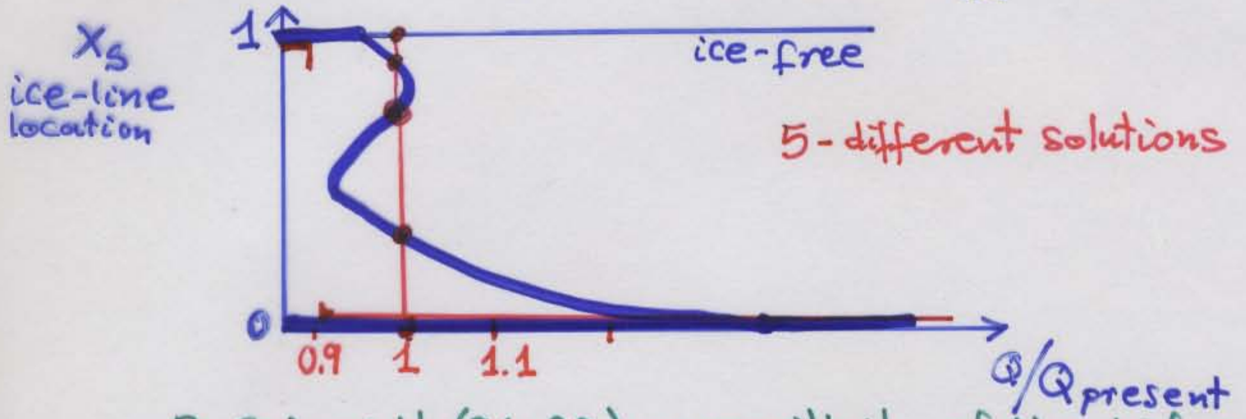
2. Numerical approximation of Σ
   Hetzer, Jarausch-Mackens (1989)



· Drazin-Griffel (77, 82, ...) [computations]

$x\in(0,1)$, symmetric solutions

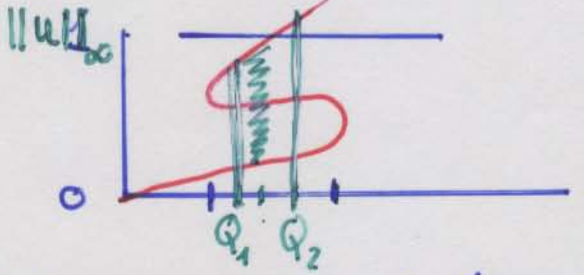
$$\begin{cases} -\kappa \dfrac{d}{dx}\left((1-x^2)\dfrac{du}{dx}\right)+A+Bu = QS(x)\beta(u) \\ (1-x^2)\dfrac{du}{dx}=0 \ \text{ at } x=\pm 1, \ \dfrac{du}{dx}(0)=0. \end{cases} \qquad (E)$$



- B.Schmidt (94, 99): sensitivity of the bifurcation diagram in Sellers type w.r.to slope of β(u
- D-Tello (99) $p \geq 2$ ("shooting method")

$$\begin{cases} -\left(|u_x|^{p-2}u_x\right)_x + g(u) \in Q\beta(u), \quad x\in(0,1) \\ u_x(0)=u_x(1)=0 \end{cases} \qquad \begin{cases} g(0)=A, \ g(-10)\in\beta(-10) \end{cases}$$

"$\exists$ $0<Q_1<Q_2$ such that if $Q\in(Q_1,Q_2)$ $\forall N\in\mathbb{N}$ there exists at least one solution u_N which crosses the level $u_N=-10$ N times"



- G. HETZER (1999): "Climatic EBM Budyko's type": $\left[\begin{array}{c} \cdot(1-x^2) \\ S(x)\not\equiv 1 \\ (p=2) \end{array}\right]$ Topological arguments + Rabinowitz + Whyburn

OPEN PROBLEM : The asymptotic behaviour

conjecture : Very few solutions are stable (climates)