# *Control of the turbulence in oscillatory reaction-diffusion systems*

***J.I. Díaz***

Universidad Complutense de Madrid

**COMPLEJIDAD 2008:
III Encuentro sobre
Modelización de Sistemas Complejos**

MÓSTOLES, 20-21 DE NOVIEMBRE DE 2008
Universidad Rey Juan Carlos

## 1. Introduction

We consider the *non local delayed complex Ginzburg-Landau equation*

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - (1+i\epsilon)\Delta \mathbf{u} = (1-i\omega)\mathbf{u} - (1+i\beta)\left|\mathbf{u}\right|^2 \mathbf{u} \\ \qquad +\mu e^{i\chi_0}\mathbf{F}(\mathbf{u},t,\tau) & \Omega \times (0,+\infty), \\ \left.\begin{array}{l} \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+2}}, \\ \left(-\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\Big|_{\Gamma_j} = \right)\frac{\partial \mathbf{u}}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial \mathbf{u}}{\partial x_j}\Big|_{\Gamma_{j+2}}\left(=\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\Big|_{\Gamma_{j+2}}\right) \end{array}\right\} & \partial\Omega \times (0,+\infty), \\ \mathbf{u}(x,s) = \mathbf{u}_0(x,s) & \Omega \times [-\tau, 0], \end{cases}$$

domain $\Omega = (0, L_1) \times (0, L_2)$, periodic boundary conditions,

$\Gamma_j$ the faces of the boundary

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}, \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}, j = 1, 2.$$

$\mathbf{n}$ is the normal vector

$$\mathbf{F}(\mathbf{u}, t, \tau) = [m_1\mathbf{u}(t) + m_2\overline{\mathbf{u}}(t) + m_3\mathbf{u}(t-\tau, x) + m_4\overline{\mathbf{u}}(t-\tau)],$$

$\overline{\mathbf{u}}(s) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}(s,x)dx,$ $\epsilon, \beta, \omega, \mu, \chi_0, m_i$ and $\tau$ are real numbers,

but $\mathbf{u}(x,t) = u_1(x,t) + iu_2(x,t)$.

This type of equations (called as of Stuart-Landau in absence of the diffusion term) arise in the study of the stability of reaction diffusion equations

$\dfrac{\partial \mathbf{X}}{\partial t} - \mathbf{D}(\varepsilon)\Delta\mathbf{X} = \mathbf{f}(\mathbf{X} : \eta), \quad \mathbf{X} : \Omega \times (0, +\infty) \to \mathbb{R}^n, \quad \eta$ real scalar parameter,

the deviation $\mathbf{v}$ from the uniform state solution $\mathbf{X}_\infty$ is developed asymptotically in terms of some multiple scales (Kuramoto, (1984)).

Our special motivation:

Mathematical modeling of control of pattern formation and spatiotemporal chaos (chemical turbulence) in spatially extended nonlinear systems, far from equilibrium.

Many examples: photosynthetic Belousov-Zhabotinsky reaction (Kuramoto 1984)

In particular, the catalytic oxidation of carbon monoxide on a platinum (110) single crystal surface (idealized setting of the reaction that proceeds in the **catalytic converter of a car exhaust**).

# Krischer-Eiswirth-Ertl model

CO coverage

$$\frac{\partial u}{\partial t} = k_1\, s_{\mathrm{CO}}\, p_{\mathrm{CO}}(1-u^3) - k_2 u - k_3 uv + D\nabla^2 u$$

Oxygen coverage

$$\frac{\partial v}{\partial t} = k_4\, p_{\mathrm{O}_2}[s_{\mathrm{O},1\times 1} w + s_{\mathrm{O},1\times 2}(1-w)](1-u-v)^2 - k_3 uv$$

Local ratio of 1 x 1 phase of the Pt(110) surface

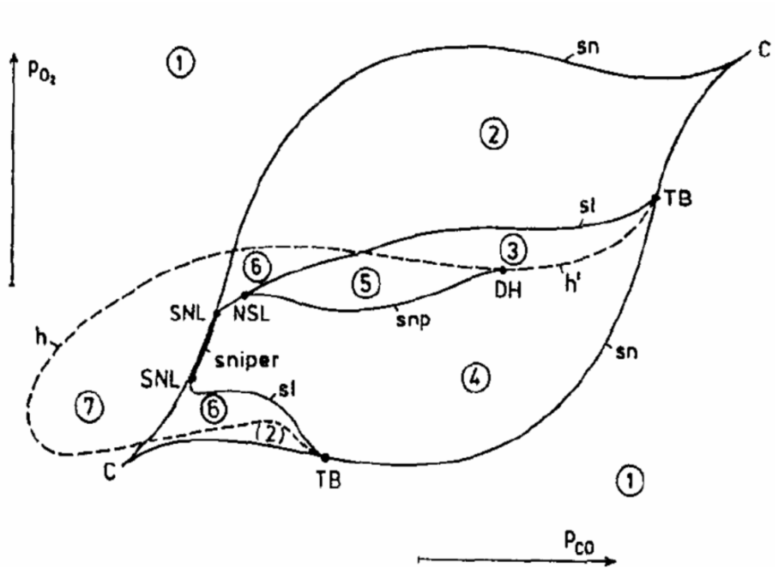
$$\frac{\partial w}{\partial t} = k_5 \left( \frac{1}{(1+\exp\left((u_0-u)/\delta u\right))} - w \right)$$

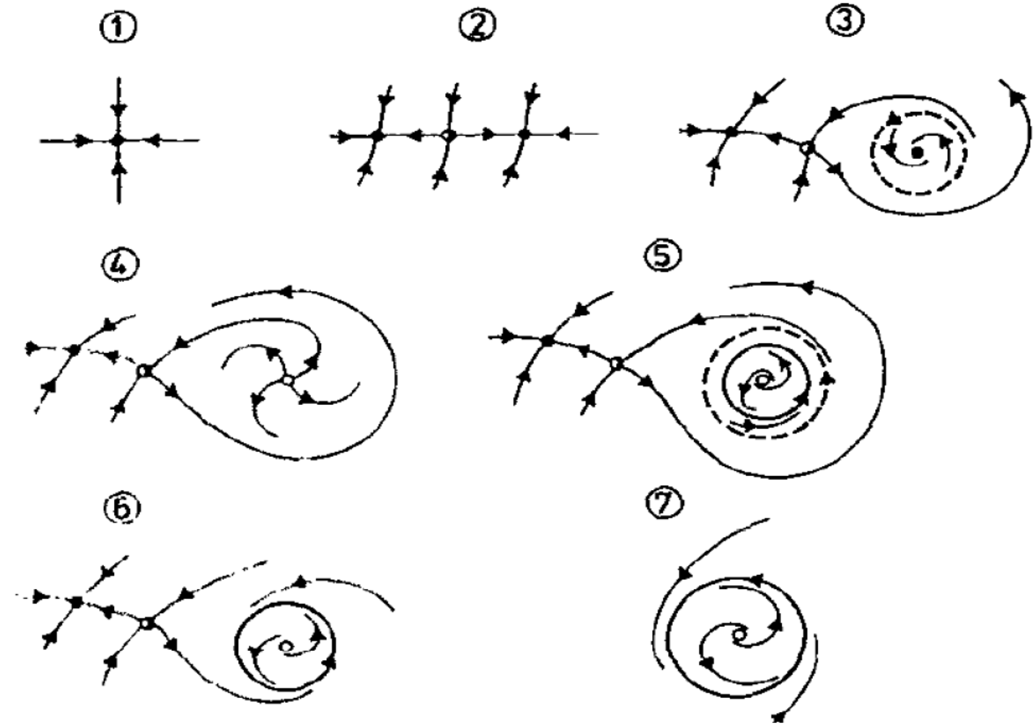
$$p_{CO}(u,t,\tau) := p^0_{CO} + \mu(\overline{u}(t) - \overline{u}(t-\tau))$$

or, more in general,

$$p_{CO}(u,t,\tau) := p^0_{CO} + \mu\,[m_1 u(t) + m_2 \overline{u}(t) + m_3 u(t-\tau,x) + m_4 \overline{u}(t-\tau)]\,,$$

$$\overline{u}(s) = \frac{1}{|\Omega|}\int_\Omega u(s,x)\,dx.$$




Any uniformly spatial solution (i.e. solution of the system without diffusion term) is also solution of the system. Then, bifurcations and oscillations appear.

The **Complex GL Equation** represents a **normal form** of a dynamical system in the vicinity of a supercritical **Hopf bifurcation**.

**Individual elements** perform harmonical **limit-cycle oscillations**, but the local **diffusion coupling** between them can produce (chemical) **turbulence**.

Depending on the **phase shift** of the control signal, the feedback can be positive or negative, **stabilizing or destabilizing** uniform oscillations.

The principal **role of delays** is to **modify phase shifts** between the control signal and the oscillating pattern.

In this case, the **derivation of the GL equation** from the Krischer, Eiswirth & Ertl Model, can be seen in

**Ipsen, Kramer and Sorensen**, Physics Reports, 337(2000)

We focus our attention on the so called *slowly varying complex amplitudes* defined by $\mathbf{u}(x,t) = \mathbf{v}(x,t)e^{-i\omega t}$. Thus, $\mathbf{v}$ satisfy $(P_2)$ :

$$
(P_2)\begin{cases}
\dfrac{\partial \mathbf{v}}{\partial t} - (1+i\epsilon)\Delta\mathbf{v} = \mathbf{v} - (1+i\beta)|\mathbf{v}|^2\,\mathbf{v} + \\
+\mu e^{i\chi_0}\left[m_1\mathbf{v} + m_2\overline{\mathbf{v}} + \right. & \Omega\times(0,+\infty), \\
\qquad\qquad \left. e^{i\omega\tau}(m_3\mathbf{v}(t-\tau,x) + m_4\overline{\mathbf{v}}(t-\tau))\right] \\
\left.\begin{array}{l}
\mathbf{v}|_{\Gamma_j} = \mathbf{v}|_{\Gamma_{j+2}}, \\
\left(-\dfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\Big|_{\Gamma_j} = \right)\dfrac{\partial\mathbf{v}}{\partial x_j}\Big|_{\Gamma_j} = \dfrac{\partial\mathbf{v}}{\partial x_j}\Big|_{\Gamma_{j+2}}\left(= \dfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\Big|_{\Gamma_{j+2}}\right),
\end{array}\right\} & \partial\Omega\times(0,+\infty), \\
\mathbf{v}(x,s) = \mathbf{u}_0(x,s)e^{i\omega s} & \Omega\times[-\tau,0].
\end{cases}
$$

We study the stability of *uniform oscillations*, i.e., special solutions of $(P_2)$ of the form $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ which determines completely $\rho_0$ and $\theta$.

In absence of delay ($\tau = 0$), and for $|\Omega| = +\infty$ and $\mu = 0$, it is known (see Kuramoto[23] and Mertens[25]) that the *Benjamin-Feir condition* $\beta < -\frac{1}{\epsilon}$ implies the instability of such uniform oscillations. Here we shall assume merely that

$$\beta \leq 0 \text{ and } \varepsilon \geq 0 \qquad (18)$$

and we shall prove that this instability holds, in absence of delay, for $L < +\infty$ once $\chi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\mu > \frac{1}{|\cos \chi_0|}$. Moreover, we shall also prove that when $\tau > 0$ is suitably chosen then the uniform oscillation becomes linearly stable. We point out that the above stabilization phenomenon requires a non zero complex component perturbation (notice that $\chi_0$ cannot be zero) and that it applies to the case of $\mu > 0$ and $\epsilon = \beta = \omega = 0$.

**A possible pedagogical explanation:**

K. Pyragas, Continuous control of chaos by self-controlling feedback, Physics Letters A 170, 1992, 421-428.

N.Y.Hu and Z.H. Wang: *Dynamics of controlled Mechanical Systems with Delayed Feedback*, Springer, 2002.

$$\dot{z}(t) = f(z(t), t), \qquad z \in R^n, \tag{8.3.1}$$

where $f: R^{n+1} \to R^n$ is of period $T$ with respect to $t$. Suppose that the system has a chaotic attractor, where an infinite number of unstable periodic orbits are embedded. Among them, an unstable motion $z_p(t)$ of period $\tau = mT$ is selected as the target of control, where $m$ is a positive integer. To direct a chaotic motion $z(t)$ near $z_p(t)$ to $z_p(t)$, a control force $g(t)$ is introduced into the system. More specifically, Eq. (8.3.1) with the control force $g(t)$ is partitioned as

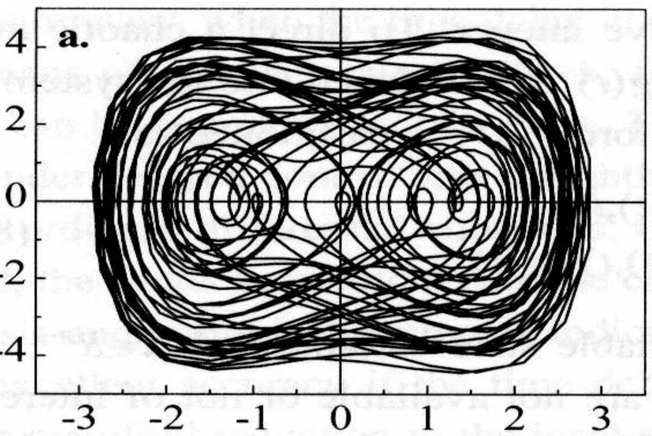
$$\begin{cases} \dot{x}(t) = p(x(t), y(t), t), \\ \dot{y}(t) = q(x(t), y(t), t) + g(t), \end{cases} \tag{8.3.2}$$

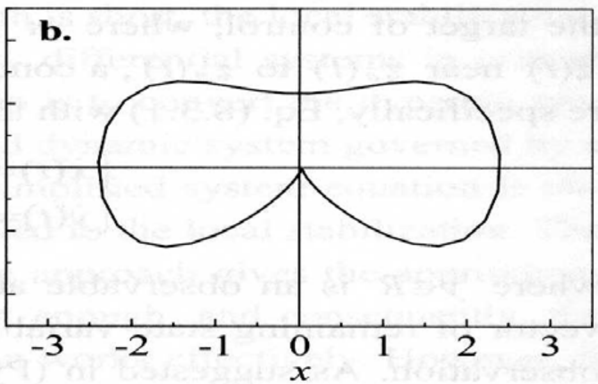
where $y \in R$ is an observable and controllable state variable and $x \in R^{n-1}$ is the vector of remaining state variables which are not available or not of interest for observation. As suggested in (Pyragas 1992), the control force can be a delayed linear feedback as following

$$g(t) = v[y(t) - y(t - \tau)], \tag{8.3.3}$$

**Example 8.3.1** To demonstrate the efficacy of delayed linear feedback, we consider a harmonically forced Duffing oscillator as follows

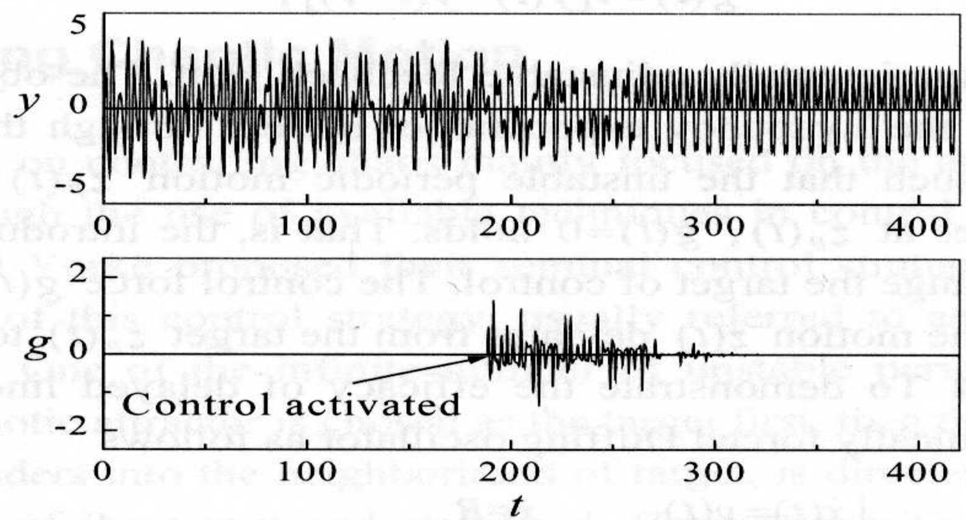
$$\begin{cases} \dot{x}(t)=y(t), \qquad x\in R, \\ \dot{y}(t)=x(t)-x^3(t)-0.02y(t)+2.5\cos t. \end{cases} \tag{8.3.4}$$




A periodic solution

$$g(t)=v[y(t)-y(t-2\pi)],$$ Delayed linear feedback



$$x_{n+1} = 4x_n(1 - x_n) + F_n \quad (6)$$

has the unstable fixed point $x_n = \frac{3}{4}$ with the eigenvalue $\lambda = -2$. The perturbation in the form of a delay $F_n = K(x_{n-1} - x_n)$ does not change the $x$-coordinate of this fixed point, but increases the dimension of the map to two. The analysis of this two-dimensional map shows that the absolute values of both eigenvalues of the fixed point are less than 1 in the interval of the parameter $K = [-1, -0.5]$. Therefore, for these values of $K$ a "one-dimensional" unstable fixed point turns into a "two-dimensional" stable fixed point. A more detailed theory of this stabilization is in progress and will be reported elsewhere.

Plan of the rest of the lecture:

## 2. An abstract pseudo-linearization principle

**A. C. Casal, J. I. Díaz, On the principle of pseudo-linearized stability: application to some delayed parabolic equations, *Nonlinear Analysis* 63 (2005) e997 – e1007.**

## 3. Aplications: stabilization by delay terms of unstable uniform oscilations in GL and to a degenerate system with possible nondifferentiable terms

**A. C. Casal, J. I. Díaz, J. F. Padial, L. Tello, On the stabilization of uniform oscillations for the complex Ginzburg-Landau equation by means of a global delayed mechanism. 2003**

**A. C. Casal, J. I. Díaz, On the complex Ginzburg-Landau equation with a delayed feedback. *Mathematical Models and Methods in Applied Sciences*, 16, 1 (2006) 1-17.**

## 4. Numerical experiences (some remarks).

**A. C. Casal, J. I. Díaz and M. Stich, On some delayed nonlinear parabolic equations modeling CO oxidation, *Dynamics of Continuous, Discrete and Impulsive Systems*, serie A, Vol 13b (Supplementary Volume), 2006, 413-426**

**A. C. Casal, J. I. Díaz and M. Stich: Control of Turbulence in Oscillatory Reaction-Diffusion Systems through a Combination of Global and Local Feedback. *Physical Review E*. E 76, 036209 1-9, 2007**

## 2. An abstract pseudo-linearization principle

### 2.1. Motivation.

We are interested in the stability analysis of the time-periodical function $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$. In order to avoid the application of techniques for the study of the stability of periodic solutions we can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t) = \mathbf{v}(x,t)e^{i\theta t}$ where $\mathbf{v}(x,t)$ is a solution of $(P_2)$. Thus $\mathbf{z}(x,t)$ satisfies

$$(P_3)\begin{cases} \frac{\partial \mathbf{z}}{\partial t} - (1+i\epsilon)\Delta\mathbf{z} = (1+i\theta)\mathbf{z} - (1+i\beta)\left|\mathbf{z}\right|^2\mathbf{z} + \\ \qquad +\mu e^{i\chi_0}\left[m_1\mathbf{z} + m_2\overline{\mathbf{z}} + \right. & \Omega\times(0,+\infty), \\ \qquad \left. e^{i(\omega+\theta)\tau}\left(m_3\mathbf{z}(t-\tau,x) + m_4\overline{\mathbf{z}}(t-\tau)\right)\right], & \\ \mathbf{z}|_{\Gamma_j} = \mathbf{z}|_{\Gamma_{j+2}}, & \\ \left(-\frac{\partial \mathbf{z}}{\partial \vec{n}}\Big|_{\Gamma_j} =\right)\frac{\partial \mathbf{z}}{\partial x_j}\Big|_{\Gamma_j} = \frac{\partial \mathbf{z}}{\partial x_j}\Big|_{\Gamma_{j+2}}\left(=\frac{\partial \mathbf{z}}{\partial \vec{n}}\Big|_{\Gamma_{j+2}}\right), & \partial\Omega\times(0,+\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x,s)e^{i(\omega-\theta)s}, & \Omega\times[-\tau,0]. \end{cases}$$

Now, $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_{uosc}(x,t)e^{i\theta t} = \mathbf{z}_\infty = \rho_0$ is an stationary solution of $(P_3)$: i.e.

$$\mathbf{0} = (1+i\theta)\mathbf{z}_\infty - (1+i\beta)\left|\mathbf{z}_\infty\right|^2 \mathbf{z}_\infty + \mu e^{i\chi_0}\left[m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3+m_4)\right]\mathbf{z}_\infty.$$

This delayed system can be also (formally) linearized (Battogtokh and Mikhailov (1996), and Mertens, Imbihl, Mikhailov, (1994)), but losing the meaning of the original operator.

## Motivation for an abstract formulation:

Relevant examples of nonlinear functional differential equations arise in the most different contexts (Díaz, Hetzer (1998), in Climatology, Chukwu (2001), on the wealth of nations, and the general exposition in Wu, (1996) and.in Hale (1977)).

## 2.2. The abstract pseudo-linearization principle

We study the stabilization, as $t \to \infty$, of the solutions of

$$\begin{cases} \frac{du}{dt}(t) + Au(t) + Bu(t) \ni F(u_t(.)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0] \end{cases}$$

on a Banach space $X$, where

$$u_t(\theta) = u(t+\theta), \theta \in [-\tau, 0],$$

to the associated equilibria: $w \in D(A) \subset D(B) \subset X$ such that

$$Aw + Bw \ni F(\widehat{w}(.)),$$

$\widehat{w} \in C := C([-\tau, 0] : X)$ is the function which takes constant values equal to $w$.

**The term** ***pseudo-linearization***

If in $(P_3)$ we make $\mathbf{z}(x,t) = \rho(x,t)e^{i\phi(x,t)}$, use the representation $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$,

$\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$, and make $\mathbf{q} = (\rho,\phi)$,

$\mathbf{P}$ is nonlinear, $\mathbf{z}(x,t) = \mathbf{P}(\mathbf{q}(x,t))$ and $(P_3)$ is

$$\frac{d\mathbf{P}(\mathbf{q}(\cdot,t))}{dt} + A\mathbf{P}(\mathbf{q}(\cdot,t)) + B\mathbf{P}(\mathbf{q}(\cdot,t)) = F(\mathbf{P}(\mathbf{q}(\cdot))_t).$$

$\mathbf{C}(\mathbf{q}(\cdot,t)) = grad\mathbf{P}(\mathbf{q}(\cdot,t))$ is not singular and, in terms of $\mathbf{q}$,

$$\frac{d\mathbf{q}}{dt}(\cdot,t) + \mathbf{C}(\mathbf{q}(\cdot,t))^{-1}[A\mathbf{P}(\mathbf{q}(\cdot,t)) + B\mathbf{P}(q(\cdot,t))] = \mathbf{C}(\mathbf{q}(\cdot,t))^{-1}F(\mathbf{P}(\mathbf{q}(\cdot))_t).$$

Notice that the above diffusion operator $\mathbf{C}(\mathbf{q}(\cdot,t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot,t))$ becomes quasilinear in $\mathbf{q}$ (we would still need a mathematical justification of the stability under perturbations).

We may (and, sometimes, need to) keep $A$ nonlinear after the process of linearizing the rest of the terms of the equation. Example 2, justify also this philosophy of keeping $A$ non-linear ($A$ multivalued, or nondifferentiable or a degenerate quasilinear operator).

## Structural assumptions for the abstract formulation:

**(H1):** $A \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, with $\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathcal{P}(X)$ such that $A + \omega I$ is a m-accretive operator$\}$,

(Brezis (1973), $X = H$, Hilbert space, and Benilan, Crandall, Pazy, Vrabie (1995), $X$, general Banach space),

**(H2):** the operators semigroup $T(t) : \overline{D_x(A)}^X \to X$, $t \geq 0$, generated by $A$, is compact (Vrabie(1995))

**(H3):** $B \in \mathcal{A}(0 : X)$, $B$ is single valued, Fréchet differentiable, and $B$ is dominated by $A$:

$$D_X(A) \subset D_X(B) \text{ and } |Bu| \leq k \left|A^0 u\right| + \sigma(|u|)$$

$$\left|A^0 u\right| := \inf\{|\xi| : \xi \in Au\},$$

$$u \in D_X(A), \text{ some } k < 1, \ \sigma(\text{continuous}) : \mathbb{R} \to \mathbb{R},$$

Here and in what follows, $|.|$ denotes the norm in the space $X$ (in contrast with the norm in space $C$ which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces $X$ and $Y$ the corresponding norms will be indicated).

**(H4):** $F : C \to X$ satisfies a local Lipschitz condition.

**(H5):** there exists $\delta^F > 0$ such that $F : B^X_{\delta^F}(\widehat{w}) \to X$ is Fréchet differentiable with the Fréchet derivative $\mathrm{D}F(\widehat{w})$ given by $\mathrm{D}(F(\widehat{w}))\phi = \int_{-\tau}^{0} d\eta(\theta)\phi(\theta), \phi \in C$, for $\eta : [-\tau, 0] \to B(X, X)$ of bounded variation and the Fréchet derivative is locally Lipschitz continuous, where $B^X_{\delta^F}(\widehat{w}) = \left\{\phi \in C; \|\phi - \widehat{x}\| < \delta^F\right\}$,

**(H6):** the operator $y \to Ay + By - \mathrm{D}F(\widehat{w})(e^{\omega\cdot}y)$ belongs to $\mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ with $\mathrm{Re}\,\omega = \gamma < 0$ where $e^{\omega\cdot}v \in C$ is defined by

$$(e^{\omega\cdot}v)(s) = e^{\omega s}\widehat{v}(s), \text{ with } \widehat{v}(s) = v, \text{ for any } s \in [-\tau, 0] \text{ for } v \in X.$$

**Theorem 1** *Assume (H1)-(H6). Then there exists $\alpha > 0$, $\epsilon > 0$ and $M \geq 1$ such that if $u_0 \in B^X_\epsilon(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ then the solution $u(\cdot : u_0)$ of (1) exists on $[-\tau, +\infty)$ and*

$$|u(t : u_0) - w| \leq Me^{-\alpha t}\|u_0 - \widehat{w}\|, \text{ for any } t > 0.$$

## Outline of the Proof

- $F$ and $\mathrm{D}F$ being locally Lipschitz continuous (H4 and H6)
- $A + B + \mathrm{D}F$ being m-accretive (H6)
- $B$ dominated by $A$ (H3)

give estimates and bounds on some intervals

- (H2), compactness of $T(t)$, allows to apply some maximal continuation results, allowing to use Gronwall's inequality ■

Structural assumptions when B is differentiable:

**(H7):** there exists a Banach space $Y$ and there exists $\delta^B > 0$ such that $B$ is Fréchet differentiable as function from $B_{\delta^B}(w) = \left\{ z \in D(B);\ |w - z| < \delta^B \right\}$ into $Y$, with the Fréchet derivative $\mathrm{D}B(w)$ locally Lipschitz continuous,

**(H8)** the operator $y \to Ay + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega^* \cdot} y\right)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\mathrm{Re}\,\omega^* = \gamma^* < 0$.

**Remark 1** *It can be found in Ambrosetti and Prodi (1993), results on the Frechet differentiability of Nemitsky operators, Th. 2.6 (with $p = 4$)) that, in Example 1,(H7) holds, with $\mathrm{D}B(\mathbf{y})\mathbf{v} = 3(1+i\beta)\,|\mathbf{y}|^2\,\mathbf{v}$, if we take $Y = \mathbf{L}^{4/3}(\Omega)$. However, assumption (H7) does not hold here if we take $X = Y = \mathbf{L}^2(\Omega)$.*

**Theorem 2** *If we also assume (H7), that (H1)-(H5) also holds on the space $Y$ and (H8) then there exists $\alpha^* > 0$, $\epsilon^* \in (0, \epsilon]$ and $M^* \geq 1$ such that if $u_0 \in B_{\epsilon^*}^{X \cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ then, for any $t > 0$,*

$$|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \leq M^* e^{-\alpha^* t}(\|u_0 - \widehat{w}\|_X + \|u_0 - \widehat{w}\|_Y),.$$

**Outline of the Proof**: Repeat the same arguments as for Theorem1, but for the space Y, and sum up. ■

It is not difficult to show that the assumption (H8) is implied (when $A$ is linear) by the condition: "if $\lambda \in \mathbb{C}$ is given so that there exists $y \in D(B) \setminus \{0\}$ such that $Ay + \mathrm{D}B(w)y - \lambda y \ni \mathrm{D}F(\widehat{w})\left(e^{\lambda \cdot}y\right)$ then $\mathrm{Re}\,\lambda > 0$". This allow to see Theorem 4.1 of Wu (1996) (see also Parrot (1992) and its references) as an special case of our abstract result with $B = 0$. In that case the "variation of the constants formula" can be used to get a different proof of the theorem since $A$ is linear.

**Remark 2** *When $A$ is linear, as in the case without delay, assumption (H7) implies that the zero solution of the linearized problem $\frac{dU}{dt}(t) + AU(t) + \mathrm{D}B(w)U(t)$-$\mathrm{D}F(\widehat{w})U_t(.) = 0$ in $X$, is locally asymptotically stable (Wu (1996)).*

**Remark 3** *It is possible to prove the existence of global solutions for a general class of initial data (not necessarily near* $\widehat{w}$*) by using that* $A + B \in \mathcal{A}(\omega : X)$*, for some* $\omega \in \mathbb{C}$*, some truncation of the nonlocal term* $F(u_t)$ *and passing to the limit by the compactness of the semigroup generated by* $A$ *(Vrabie (1995) for some related results).*

Díaz, J.I., Padial, J.F., Tello, J.I.and Tello, L., To appear,

J. GINIBRE, G. VELO, The Cauchy problem in local spaces for the complex Ginzburg – Landau equation. I. Compactness methods, *Physica D.* **95** 191-228 (1996).

J. GINIBRE, G. VELO, The Cauchy problem in local spaces for the complex Ginzburg – Landau equation. II. Contraction methods, *Comm. Math. Phys.* **187** 45-79 (1997).

B. GUO, R. YUAN, The time-periodic solution to a 2D generalized Ginzburg – Landau equation, *J. Math. An. Appl.* **266** 186-199 (2002).

An easy adaptation of the above proof leads to the following linearization result (now on a possibly smaller neighborhood of $w$) when $A$ is differentiable

**Theorem 3** *The conclusion of the above result remains true if we assume, additionally, that condition (H7) also holds for $A$ and we replace condition (H8) by*

**(H9):** *the operator* $y \to \mathrm{D}A(w)y + \mathrm{D}B(w)y - \mathrm{D}F(\widehat{w})\left(e^{\omega \cdot} y\right)$ *belongs to* $\mathcal{A}(\omega)$, *for some* $\omega \in \mathbb{C}$ *with* $\mathrm{Re}\,\omega = \gamma < 0$ ■

## 3. Aplications: stabilization by delay terms of unstable uniform oscilations in GL and to a degenerate system with possible nondifferentiable terms

For the special form of the nonlinear term of the equation in $(P_3)$ we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$

In contrast with the case of scalar equations (see Parrot (1992)) the space $\mathbf{L}^\infty(\Omega)$ is not suitable space to check assumption (H1). (Auscher, Bathélemy, Bénilan, (2000)).

Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} \Delta & -\epsilon\Delta \\ \epsilon\Delta & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

For the relevant properties of the associated diffusion operator, some previous results can be used (Amann (1990)).

If $\epsilon \neq 0$ the diffusion matrix has a nonzero antisymmetric part, $A$ is the generator of a semigroup of contractions $\{T(t)\}_{t \geq 0}$ on $X$. The compactness of the semigroup is consequence of the compactness of the inclusion $D(A) \subset X$ (since $N = 2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems.

Cubic terms in $(P_3)$. Define $B\mathbf{u} = (1 + i\beta) |\mathbf{u}|^2 \mathbf{u}$ with $D(B) = \mathbf{L}^{12}(\Omega)$. By using the characterizarion of the semi inner-bracket $[,]$ for the spaces $L^p(\Omega)$ (i.e., Benilan, Crandall and Pazy) it is easy to see that $\mathbf{B}$ verifies (H3).

Nonlocal term.
$$F(\mathbf{u}_t) = (1 + i\theta)\mathbf{u}(t) + \mu e^{i\chi_0} \Big[ m_1 \mathbf{u}(t) + m_2 \overline{\mathbf{u}}(t) + e^{i(\omega+\theta)\tau} (m_3 \mathbf{u}(t-\tau) + m_4 \overline{\mathbf{u}}(t-\tau)) \Big],$$

is locally Lipschitz continuous and its Frechet derivative is given by

$$\mathrm{D}F(\widehat{\mathbf{y}})\mathbf{v}(t) = -(1+i\theta)\mathbf{v}(t) - \mu e^{i\chi_0}\left[m_1\mathbf{v}(t)+m_2\overline{\mathbf{v}}(t)- e^{i(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau)-m_4\overline{\mathbf{v}}(t-\tau))\right]$$

since for any $\phi \in C$, the non-local operator $\phi \to \frac{1}{|\Omega|}\int_\Omega \phi(s)dx$ is linear.
We can write, then,

$$\mathrm{D}F(\widehat{\mathbf{y}})\phi = \int_{-\tau}^{0} d\eta(s)\phi(s),$$

with

$$d\eta(s)v(s) = \delta_0(s)(1+i\theta)v(s) + \mu e^{i\chi_0}\left[\delta_0(s)(m_1\mathbf{v}(s)+m_2\overline{\mathbf{v}}(s))+ e^{i(\omega+\theta)\tau}\delta_{-\tau}(s)(m_3\mathbf{v}(s)+m_4\overline{\mathbf{v}}(s))\right]$$

for any $\mathbf{v} \in C([-\tau,\infty)\colon \mathbf{L}^4(\Omega))$ and any $s \in [-\tau,\infty)$, where $\delta_0(s)$, $\delta_{-\tau}(s)$ denote the Dirac delta at the points $s=0$ and $s=-\tau$ respectively. By well-known results, we have that $\eta : [-\tau, 0] \to B(X,X)$ has a bounded variation and so, conditions (H4) and (H5) hold (and analougously replacing $X$ by $Y$).

**Remark 5** *By introducing the representation operator* $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$ *it is clear that the quasilinear operator* $A\mathbf{P}(\mathbf{q})$ *obtained from the operator* $A\mathbf{u}$=-$(1+i\epsilon)\Delta\mathbf{u}$ *satisfies also condition* $A \in \mathcal{A}(\omega)$ *(since* $\mathbf{P}$ *is merely a change of variables). We point out that,*

$$A\mathbf{P}(\mathbf{q}) = -(1+i\epsilon)[\Delta\rho - \rho\,|\nabla\phi|^2 + i(2\nabla\rho\cdot\nabla\phi + \rho\Delta\phi)]e^{i\phi}.$$

Then, the "formal linearization" of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ at $\mathbf{q}^*(x,y) := \mathbf{y} \equiv \rho_0$ becomes

$$D\mathbf{E}(\mathbf{q}^*)(\rho e^{i\phi}) = -(1+i\epsilon)[\Delta\rho + i\rho_0\Delta\phi]e^{i\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ needs a slight modification of the above linear expression. ■

Finally, assumption (H6) can be read as a condition on the stationary state $\mathbf{y}$ (a study of the eigenvalue of operator $A$ can be found in Temam (1988)).

We are interested in the stability analysis of the time-periodical function $\mathbf{v}_{uosc}(x,t)$ $\rho_0 e^{-i\theta t}$ $\quad \mathbf{z}(x,t) = \mathbf{v}(x,t)e^{i\theta t}$ Thus $\mathbf{z}(x,t)$ satisfies $(P_3)$

Now, $\mathbf{v}_{uosc}(x,t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_{uosc}(x,t)e^{i\theta t} = \mathbf{z}_\infty = \rho_0$ is an stationary solution of $(P_3)$: i.e.

$$\mathbf{0} = (1+i\theta)\mathbf{z}_\infty - (1+i\beta)\left|\mathbf{z}_\infty\right|^2 \mathbf{z}_\infty + \mu e^{i\chi_0}\left[m_1+m_2+e^{i(\omega+\theta)\tau}(m_3+m_4)\right]\mathbf{z}_\infty.$$

In order to keep some resemblance with Battogtokh[6] we shall assume that

$$m_1+m_2 = 0 \; and \; m_3+m_4 = 1 \qquad (21)$$

Then we get the expressions $\rho_0(\tau) = (1 + \mu \cos \chi(\tau))^{1/2}$, where $\chi(\tau) = \chi_0 + (\omega + \theta(\tau))\tau$ and with $\theta(\tau)$ given as the solution of the implicit equation

$$\theta = \beta - \mu(\sin(\chi_0 + (\omega + \theta)\tau) - \beta \cos(\chi_0 + (\omega + \theta)\tau)).$$

Notice that if $\mu = 0$ we deduce that $\rho_0(\tau) = 1$ and that $\theta(\tau) = \beta$ for any $\tau$ and that $\rho_0(0) = (1 + \mu \cos \chi_0)^{1/2}, \theta(0) = \beta - \mu(\sin \chi_0 - \beta \cos \chi_0)$.

Our main stabilization result is the following

**Theorem 4** *Assume (18), (21),* $\chi_0 \in (\pi, \frac{3\pi}{2})$,

$$3 - m_1 - 2m_3 \geq 0,\ m_1 + m_3 \geq 0,\ 3 + 2m_3 > 0, \tag{23}$$

$$\mu > \max\{\frac{1}{|\cos \chi_0|}, \frac{3\beta - \omega + 3(\omega + \beta)\sin \chi_0 + \cos \chi_0}{5(-\beta)\sin \chi_0 \cos \chi_0 + 1},$$
$$\frac{m_3(3\beta - \omega - \varepsilon \frac{\pi^2}{L^2}) + 3(\omega + \beta)\sin \chi_0 + (m_1 + m_3)\cos \chi_0}{(3 - m_1 - 2m_3)\sin^2 \chi_0 + (m_1 + m_3)\cos^2 \chi_0 + (-\beta)(3 + 2m_3)\sin \chi_0 \cos \chi_0}\}.$$

*Then there exists some* $\tau_0 \in (0, 1)$ *such that if we assume* $\tau \in (\tau_0, 1)$ *we get that*

$$|\mathbf{v}(x,t) - \rho_0| \leq M e^{-\alpha t} \left\| \mathbf{u}_0(\cdot,\cdot)e^{i\omega \cdot} - \rho_0 \right\|.$$

## Example 2. Nonlinear and nondifferentiable operator $A$

It is not difficult to adapt the results of the first example to the case in which the vectorial operator is given by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} A_1 & -\epsilon\Delta \\ \epsilon\Delta & A_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with $A_i : D(A_i) \to \mathcal{P}(L^4(\Omega))$ two (possibly different) m-accretive operators in $L^4(\Omega)$, as for instance,

$$\begin{cases} A_i u = -div(|\nabla u|^{p_i-2}\nabla u) + \beta_i(u) \\ D(A_i) = \{u \in W^{1,1}(\Omega) \cap L^4(\Omega),\ u(x) \in D(\beta) \text{ a.e. } x \in \Omega,\ A_i u \in L^4(\Omega) \\ \text{and } -\left|\frac{\partial u}{\partial n}\right|^{p_i-2} \frac{\partial u}{\partial n} \in \gamma_i(u) \text{ on } \partial\Omega\} \end{cases}$$

where $p_i \in (1, +\infty)$ and $\beta_i, \gamma_i$ are maximal monotone graphs of $\mathbb{R}^2$ (not necessarily associated to differentiable functions).

We refer the reader to Vrabie (1995) (and its references) for the study of the assumptions (H1) and (H2) for each of the nonlinear operators $A_i$. We point out that the structure of the nonlinear diffusion operator (24) allows to guarantee that the diffusion operator is m-accretive in $\mathrm{L}^4(\Omega)$. The same holds also on $\mathrm{L}^{4/3}(\Omega)$. ■

**Remark 6** *For some (partial) results on the linearization of the p-Laplacian operator see Aftalion and Pacella (Preprints). See also Bermejo and Infante (2000) for the application to numerical analysis of some associated stationary equations. Finally, we mention the linearization results by Hernández, Mancebo and Vega de Prada (2002) for some semilinear equations with a singular zeroth order term.*

## 5. Numerical experiences

We have studied the system

$$\frac{\partial \mathbf{u}}{\partial t} = (1 - \mathrm{i}\omega)\mathbf{u} - (1 + \mathrm{i}\alpha)|\mathbf{u}|^2\mathbf{u} + (1 + \mathrm{i}\beta)\frac{\partial^2 \mathbf{u}}{\partial x^2} + F(\mathbf{u}, t, \tau)$$

with a time-delayed feedback term $F$

$$F(\mathbf{u}, t, \tau) = \mu \mathrm{e}^{\mathrm{i}\xi}\left[m_1\mathbf{u}(x,t) + m_2\overline{\mathbf{u}}(t) + m_3\mathbf{u}(x, t-\tau) + m_4\overline{\mathbf{u}}(t-\tau)\right],$$

which has global and local contributions since

$$\overline{\mathbf{u}}(t) = \frac{1}{L}\int_L \mathbf{u}(x,t)\,\mathrm{d}x$$

denotes the spatial average of the complex variable $\mathbf{u}(x,t)$. In particular, we investigated the cases
(a): $m_1 = -1, m_2 = 0, m_3 = 1, m_4 = 0$,
(b): $m_1 = -1, m_2 = -1, m_3 = 1, m_4 = 1$,
(c): $m_1 = -0.7, m_2 = -0.3, m_3 = 0.7, m_4 = 0.3$,
(d): $m_1 = -1.8, m_2 = -0.2, m_3 = 1.8, m_4 = 0.2$.

For time integration, we use an explicit Euler scheme with $\delta t = 0.002$. The Laplacian operator is discretized using a next-neighbor representation. The system size of the one-dimensional medium is $L = 128$ with a spatial resolution of $\delta x = 0.32$. We apply periodic boundary conditions and the initial conditions consist either of slightly perturbed uniform oscillations or developed spatio-temporal chaos. The overall simulation time is $\Delta t = 700$ (usually the systems reaches the stable asymptotic state before $\Delta t = 200$). To obtain the phase diagram, $\tau$ was changed in steps of $0.05$ from $0.05$ to $2.0$ and $\mu$ in steps of $0.05$ from $0.05$ to $1.0$.

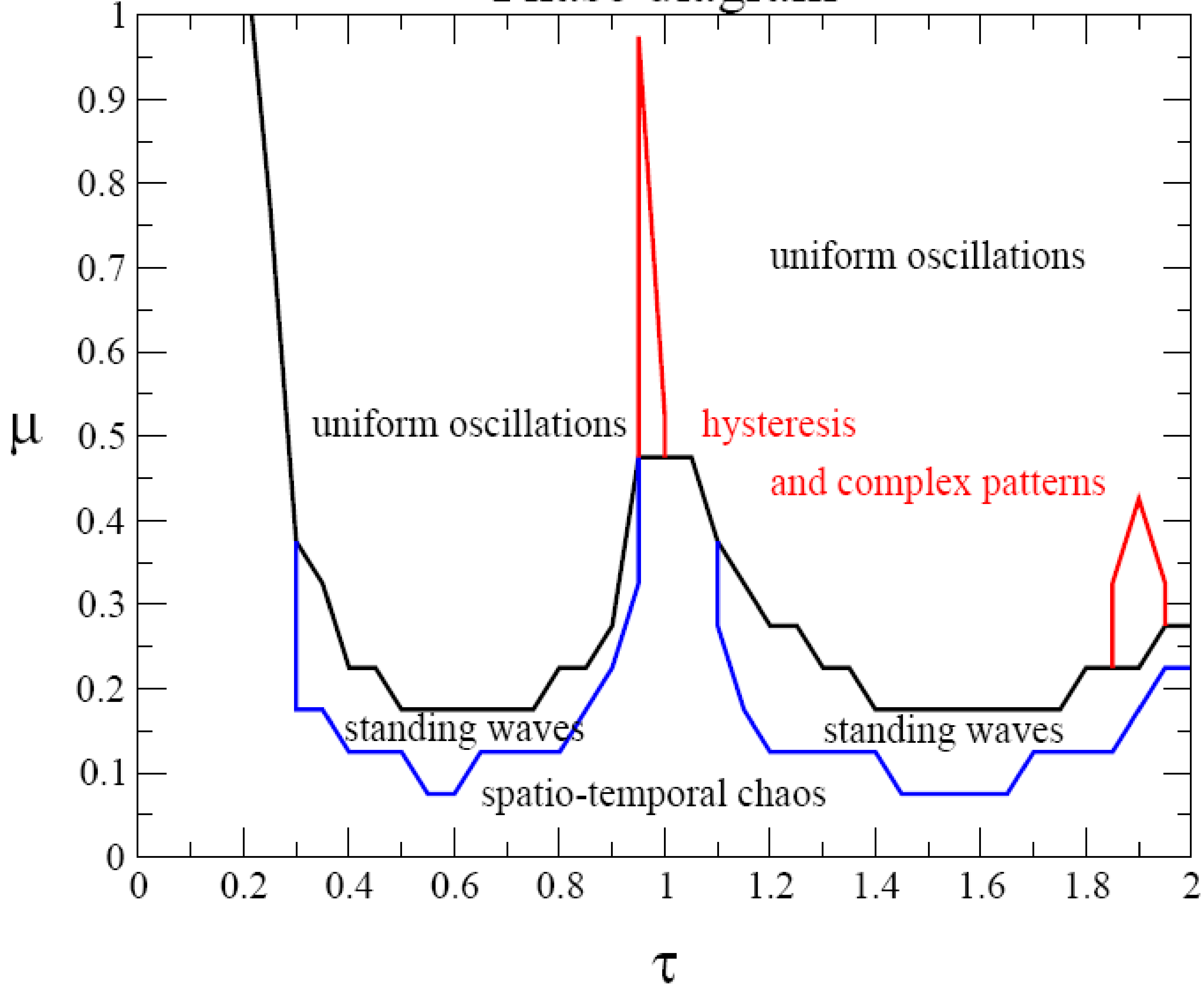
Among the local and global terms within the feedback, the idea of time-delayed autosynchronization is applied, i.e., $m_1 = -m_3$ and $m_2 = -m_4$. We are interested in the stabilization of uniform oscillations in a parameter range where such oscillations are unstable without any feedback $\mu = 0$. To be precise, the parameters $\alpha$ and $\beta$ fulfill the Benjamin-Feir criterion for instability with respect to phase perturbations $1 + \alpha\beta < 0$. In fact, we are in the regime of amplitude turbulence (spatio-temporal chaos).

In the case of purely local feedback (a), we find no stabilization of uniform oscillations, but the formation of traveling waves. This has been discussed before for a slightly different setting [11].

If we apply local and global feedbacks with the same strength (b), global feedback is clearly dominant and the phase diagram, shown in the figure, is very similar to the one for the strictly global case discussed in [10].

Even if the contribution of the local feedback terms is larger than the contribution of the global terms (c), we qualitatively observe a very similar picture that we do not present here through a figure.

At the moment, simulations for the case (d) are taken out in order to find both traveling waves and stabilization of uniform oscillations for the same values of $m_i$.




**Thanks for your attention**