Confinement of a particle for the *quasirelativistic* Schrödinger equation with a very singular potential

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MS ME-1-G 7 **Trends in nonlocal PDEs**
(J.L. Vázquez and J.I. D. organizers)

Plan:

1. Introduction: on the confinement for the classical Schrödinger equation
2. The ambiguity for the quasi-relativistic Schrödinger equation: modeling.
3. Treatment of the quasi-relativistic case: critically singular potentials.

$$\begin{cases} \mathbf{i}\frac{\partial \boldsymbol{\psi}}{\partial t} = (-\Delta u + m^2)^{1/2}\boldsymbol{\psi} + V(x)\boldsymbol{\psi} & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \boldsymbol{\psi}(0,x) = \boldsymbol{\psi}_0(x) & \text{on } \mathbb{R}^N, \end{cases}$$

$$\begin{cases} (-\Delta)^s u + V(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$P(f,V,\Omega)\begin{cases} (-\Delta)^s u + V(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\frac{\underline{C}}{d(x,\partial\Omega)^{2s}} \le V(x) \le \frac{\overline{C}}{d(x,\partial\Omega)^{2s}} \quad a.e.\ x\in\Omega,$$

$$\begin{cases} (-\Delta + m^2)^{1/2}_{\mathrm{R}} u + V(x)u = f_m(x) & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

1. Introduction: on the confinement for the classical Schrödinger equation

We recall that in Quantuum Mechanics,

$\psi : [0,\infty) \times \mathbb{R}^N \to \mathbb{C}$ the matter wave function (L. de Broglie 1924: wave-particle duality)

$\hbar > 0$ renormalized Plank constant, m mass of the elementary particle,

$V(x) \in \mathbb{R}$ the external potential

Crucial fact: $|\psi(x,t)|^2$ represents the probability density (Max Born 1926) to find the particle at point x and time t :

Question: how to "confine" (or "localize") the particle (and how to measure its linear momentum **p**) ??

Famous **negative** answers: 1927 W.Heisenberg "uncertainty principle" [1932 Nobel Prize]

Unique continuation results:

* Reed-Simon, Methods of modern Mathematical Physics (1975), Vol. 4, Theorem XIII.57

$V(x)$ is bonded on any compact interval of $\mathbb{R} - \mathcal{S}$ with $|\mathcal{S}| = 0$

* Escauriaza L, Kenig C. E., Ponce G., Vega, L. Hardy's uncertainty principle, convexity and Schrödinger evolutions J. Eur. Math. Soc. (JEMS) 10 (2008), no. 4, 883–907 ($V(x) \sim$ bounded on compact intervals of $\mathbb{R}$))

A pioneering **partially positive** answer: *the infinite well potential*

Simplifications for the linear Schrödinger equation (attributed by him, in 1935, to George Gamow (1904-1968) and repeated in any text book in Quantum Mechanics):

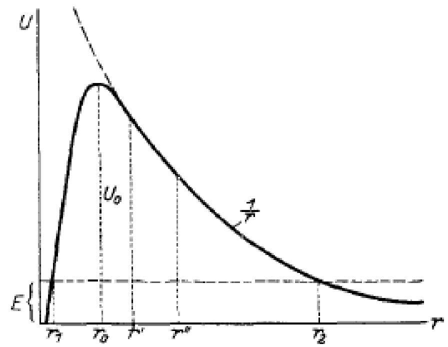


Fig. 1.

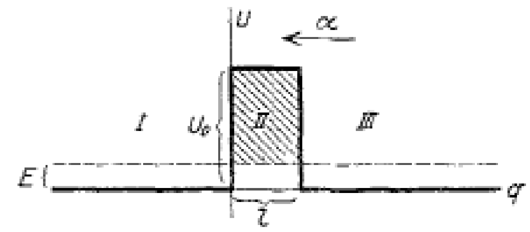


Fig. 2.

$$\psi(x,t) = e^{-\mathbf{i}Et}u(x)$$

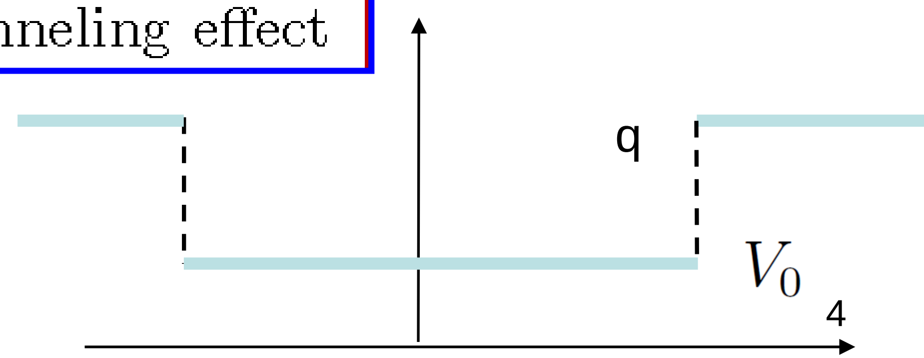
$$\mathbf{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \text{ in } (0,\infty)\times\mathbb{R}^N,$$

For simplicity $m = 1$, $\hbar = 1$ and $E = \lambda$

$$-\Delta u + V(x)u = \lambda u \quad \text{in } \mathbb{R}^N,$$

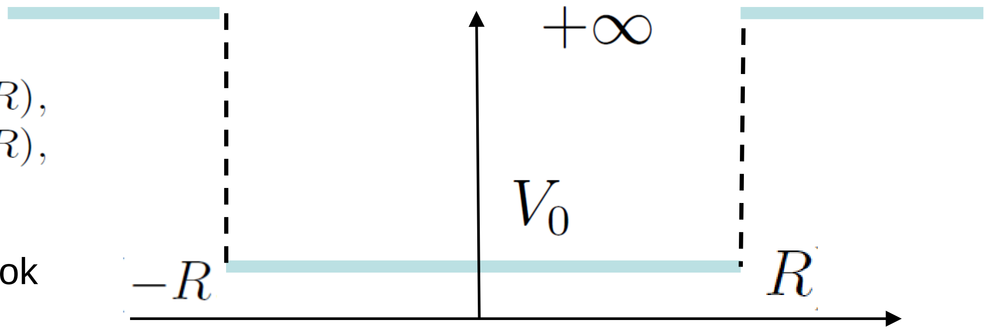
"the square well potential": tunneling effect

$$V_q(x : R, V_0) = \begin{cases} V_0 & \text{if } x \in (-R,R), \\ q & \text{if } x \notin (-R,R). \end{cases}$$



The infinite well potential: ambiguity in the standard presentation.

$$V_\infty(x:R,V_0)=\begin{cases} V_0 & \text{if } x\in(-R,R),\\ +\infty & \text{if } x\notin(-R,R),\end{cases}$$



Mandatory as one of the first examples in any book of Quantuum Mechanics.

Survey:
Belloni-Robinett, The infinite well and Dirac delta function potentials as pedagogical, mathematical and physical models in quantum mechanics, **Physics Reports** (2014)

Curiously, in this important survey the first work dealing with the infinite square well is not attributed to Gamow but **to N.F. Mott**
[book of **1930**] [1977 Nobel Prize].

Teaching Mechanics, on 2012 (and papers Bégout-Díaz on the Nonlinear Schrödinger Eq.), **I** realized some ambiguities which were the starting point of an important part of my research in the last 6 years.

In many textbooks this case is presented as a limit case of the associate *finite well potential*. In fact, there is an abuse of the notation in the above terminology.

We can introduce as definition of solution u of the *infinite well potential problem* to any function $u = \lim_{q\to\infty} u_q$ with u_q solution associated to the potential $V_q(x : R, V_0)$.

It is usually claimed that $u = \lim_{q\to\infty} u_q$ satisfies (at least in a weak sense) equation for the *infinite well potential* but, as we shall explain this is not correct since some other terms appear in the limit equation (which, in fact must be understood in distributional sense).

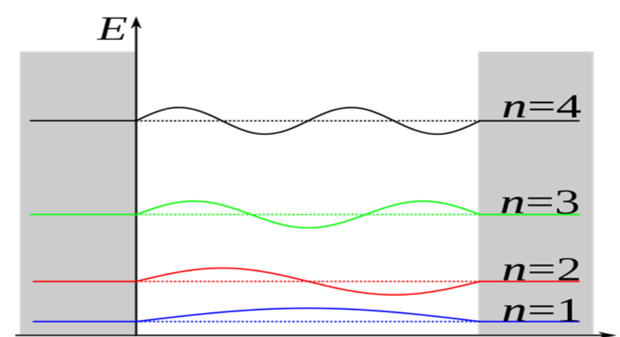
LEMMA 2.1 Given $q > 0$ and $V_q(x : R, V_0)$ defined by (1.2) problem (1.3), with $N = 1$, has a numerable sequence of eigenvalues $\lambda_n(q)$ and eigenfunctions $u_{q,n}(x)$ (renormalized such that $\|u_{q,n}\|_{L^2(\mathbb{R})} = 1$). Moreover, as $q \to +\infty$,

$$\lambda_n(q) \to \left(\frac{\pi}{2R}\right)^2 n^2, \qquad \text{with } n \in \mathbb{N},$$

and $u_{q,n} \to u_n$ weakly in $H^1(\mathbb{R})$, with u_n given by (1.5) and u_n extended by zero on $\mathbb{R} - (-R, R)$. Finally, $(u_n)_{xx}$ generate two family of Dirac deltas (depending on $n \in \mathbb{N}$): one at $x = R$ and the other at $x = -R$.

$$E_n := \frac{\hbar^2}{2m}\lambda_n$$

$$\begin{cases} u_n(x) = C \sin \dfrac{n\pi}{2R}(x+R), \\ \lambda_n - V_0 = \left(\dfrac{\pi}{2R}\right)^2 n^2, \quad n = 1, 2, \dots \end{cases}$$



The ambiguity in this mathematical treatment arises because the derivatives of such u_n are discontinuous at the points $x = \pm R$, and thus such u_n are not solutions of the equation in the whole domain $\mathbb{R}$ in the sense of distributions

$$-\frac{\hbar^2}{2m}\frac{d^2 u_n}{dx^2} + V(x)u_n = E_n u_n, \quad \text{in } \mathbb{R},$$

but they satisfy a different equation

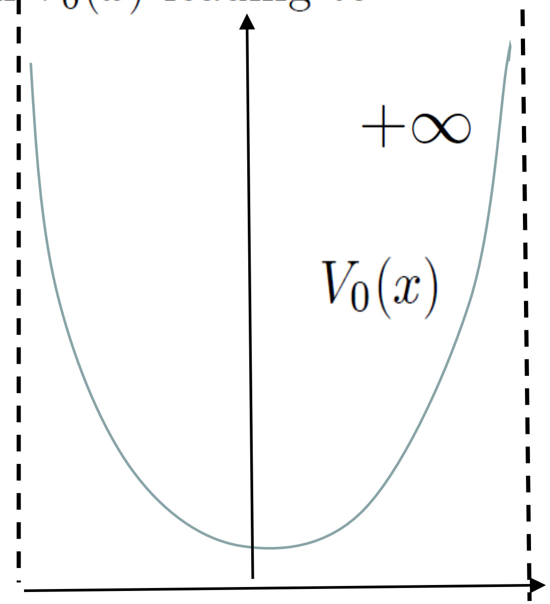
$$-\frac{\hbar^2}{2m}\frac{d^2 u_n}{dx^2} + V(x)u_n = E_n u_n + \underline{k_n(R)\delta_{\{R\}} - k_n(-R)\delta_{\{-R\}}}, \quad \text{in } \mathbb{R}, \tag{1}$$

since the second derivative develops two Dirac deltas (see Lemma above). Here

$$k_n(-R) = \frac{\hbar^2}{2m}\frac{\sqrt{2}}{R^{3/2}} n\pi \quad \text{and} \quad k_n(R) = \frac{\hbar^2}{2m}\frac{\sqrt{2}}{R^{3/2}} n\pi(-1)^n.$$

As a matter of fact, after the work by Gamow, several authors considered many generalizations of the *infinite well potential* corresponding to the case in which the constant value V_0 is replaced by a general function $V_0(x)$ leading to the potential

$$V_\infty\big(x : R, V_0(\cdot)\big) = \begin{cases} V_0(x) & \text{if } x \in (-R, R), \\ +\infty & \text{if } x \notin (-R, R). \end{cases}$$

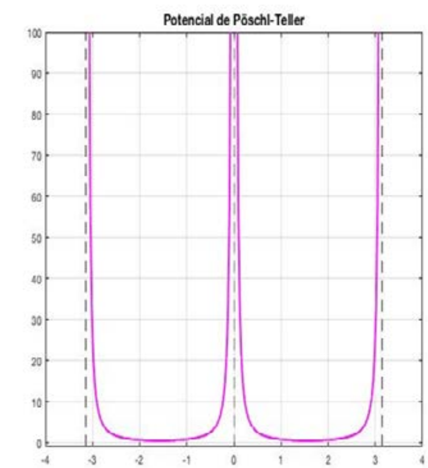


Some singular concrete potentials in the Quantum Mechanics literature:

* **Pösch-Teller potential**

$$V(x) = \frac{1}{2}V_0\{\frac{k(k-1)}{\sin^2 \alpha x} + \frac{\mu(\mu-1)}{\cos^2 \alpha x}\}, \;\; x \in [0, \frac{\pi}{2\alpha}]$$

Pöschl, G.; Teller, E. (1933), Zeitschrift für Physik, 83, 143-



***Supersymmetric potentials (SUSY)**

$$V(x) = \frac{k(k-1)}{\sin^2 x}, \;\; x \in [0, \pi]$$

(F. Cooper *et al.* (1995) Phys. Rep. 251 , 267-385)

Main program (started on 2013): consider a class of nonnegative potentials $V(x)$, being suitably singular on $\partial\Omega$, prove the existence of flat solutions $u(x)$ for some *energy* values λ, and then, extend them by zero on the rest of $\mathbb{R}^N$.

[u **flat** (on the boundary) if u and its normal derivative vanish on the boundary]

The key assumption

$$\frac{\underline{C}}{\delta(x)^\alpha} \leq V(x) \quad \left[\leq \frac{\overline{C}}{\delta(x)^\alpha} \text{ for simplicity!!}\right] \quad \text{a.e. } x \in \Omega,$$

for some $\overline{C} > \underline{C} \geq 0$ and $\alpha > 0$. Here, $\delta(x) = d(x, \partial\Omega)$.

Contrast with the case of negative singular potentials

Roughly speaking: **Theorem.** *The solutions are flat if and only if* $\alpha \geq 2$.

J. I. Díaz, On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via flat solutions: the one-dimensional case. **Interfaces and Free Boundaries**, **17** (2015),

J. I. Díaz, On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via singular potentials: the multi-dimensional case, **SeMA-Journal** (2017)

J.I. Díaz, Correction **SeMA-Journal** (2018)

J. I. Díaz, D. Gómez – Castro, J.M. Rakotoson and R. Temam,**Discrete and Continuous Dynamical Systems**(2018),

J. I. Díaz, D. Gómez-Castro, and J.-M. Rakotoson. **Differential Equations and Applications**, (2018), .

J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez (fractional Schrödinger eq.) **Nonlinear Analysis**, (2018)

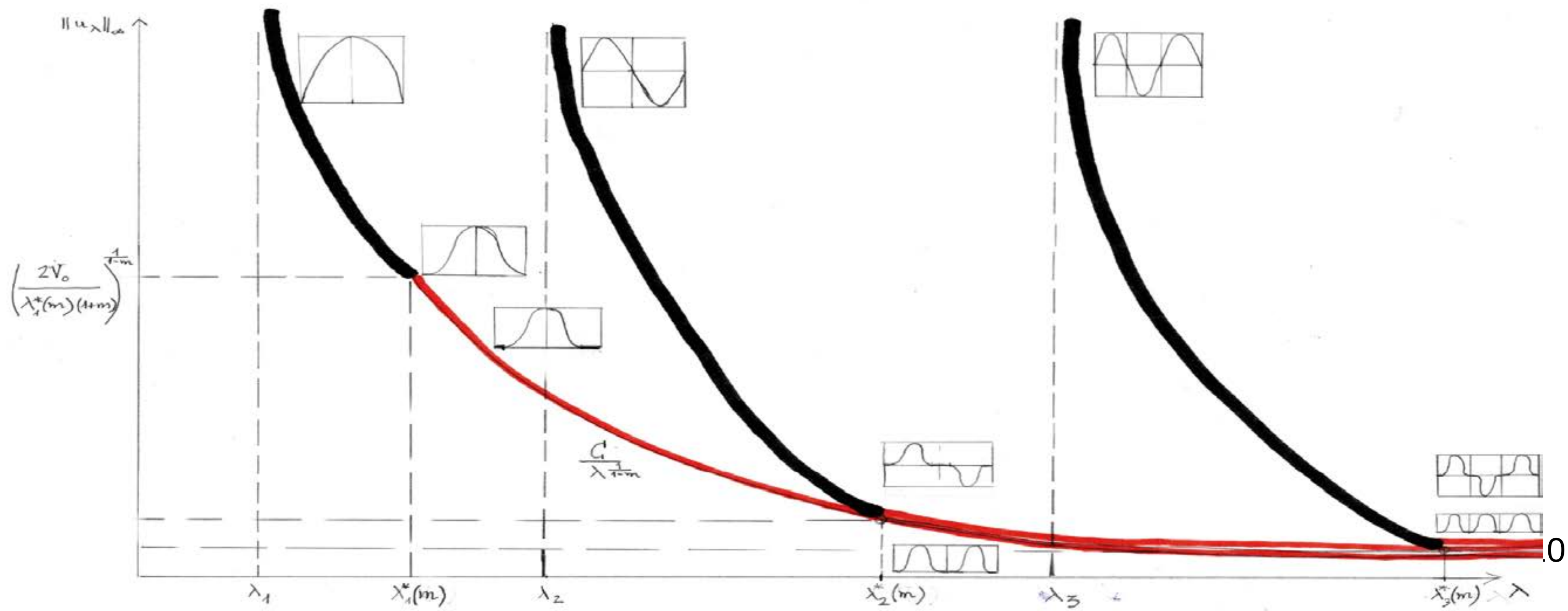
J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez (Infinite order on the boundary). **In preparation** 2019

Case N=1 and Hardy potentials (first method of proof)

A **nonlinear** problem as a *basic tool* to build super and subsolutions !!

$$P(R, m, V_0, \lambda) \equiv \begin{cases} -\dfrac{d^2 v}{dx^2} + V_0 v^m = \lambda v, \quad v \geq 0 \text{ in } (-R, R), \\ v(\pm R) = 0, \end{cases}$$

for a given $V_0 > 0$ and $m \in (0, 1)$ J. I. D. and Hernández, **Portugaliae Math.**(2015)



Proposition 3 *For any* $\lambda \geq \left(\frac{\pi}{2R}\right)^2$ *there exists a unique nonnegative solution* v_m *of* $P(R, m, V_0, \lambda)$. *Moreover, there exists a* $\lambda^*(m) > \left(\frac{\pi}{2R}\right)^2$ *such that: a) If* $\lambda \geq \lambda^*(m)$ *then*

$$v_m(x) \leq \overline{K} d(x, \partial\Omega)^{2/(1-m)} \quad \text{for any } x \in \overline{\Omega} = [-R, R], \tag{5}$$

for some constant $\overline{K}$. *In particular* $\frac{dv_m}{dx}(\pm R) = 0$. *b) If* $\lambda \leq \lambda^*(m)$ *then*

$$\underline{K} d(x, \partial\Omega)^{2/(1-m)} \leq v_m(x) \quad \text{for any } x \in \overline{\Omega} = [-R, R], \tag{6}$$

for some constant $\underline{K}$. *In particular* $v_m > 0$ *in* Ω. *c) If* $\lambda = \lambda^*(m)$ *inequalities (5) and (6) hold for some* $\overline{K} > \underline{K} > 0$.

Main idea of the application to the **linear** Schrödinger equation:

$$\frac{\overline{V}_0}{\left|v_{\lambda^*(m^\#)}(x)\right|^{1-m^\#}} \leqslant \frac{\overline{V}_0}{(\underline{K}^\#)^{1-m^\#}} \frac{1}{d(x, \partial\Omega)^2} \leqslant V(x),$$

then

$$\lambda^*(m^\#) v_{\lambda^*(m^\#)} = -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + \overline{V}_0 v_{\lambda^*(m^\#)}^{m^\#} = -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + \frac{\overline{V}_0}{\left|v_{\lambda^*(m^\#)}(x)\right|^{1-m^\#}} v_{\lambda^*(m^\#)}$$

$$\leqslant -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + \frac{\overline{V}_0}{(\underline{K}^\#)^{1-m^\#}} \frac{v_{\lambda^*(m^\#)}}{d(x, \partial\Omega)^2} \leqslant -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + V(x) v_{\lambda^*(m^\#)},$$

which proves that $v_{\lambda^*(m^\#)}(x : \overline{V}_0)$ is a supersolution (notice that for the moment $\overline{V}_0$ is arbitrary).

Case N>1 and Hardy potentials (second method of proof)

Hardy inequality + compactness $H_0^1(\Omega) \subset L^2(\Omega)$

Proposition 2.1 *Assume (10), then there exists a sequence of eigenvalues $\lambda_n \to +\infty$, $\lambda_1 > \lambda_{1,\Omega}$ (the first eigenvalue for the Dirichlet problem for the $-\Delta$ operator on Ω), λ_1 is isolated and $u_1 > 0$ on Ω.*

Flat solutions through the local comparison with a *direct local barrier* function

$$U(x) = C_U |x - x_0|^\beta \qquad -\Delta U + VU = (-\beta(\beta + N - 2) + \underline{C})C_U |x - x_0|^{\beta-2}$$

Theorem 2.1* *Assume (2) and let u_n be an eigenfunction of $DP(V, \lambda_n, \Omega)$ associated to the eigenvalue λ_n.*
(a) *There exists $\varepsilon \in [0, 2)$, $\varepsilon = \varepsilon(\underline{C}, N, n)$ and $\overline{K}_n = \overline{K}_n(\underline{C}, N, n, \Omega) > 0$ such that*

$$|u_n(x)| \leq \overline{K}_n d(x, \partial\Omega)^{2-\epsilon} \quad \text{a.e. } x \in \Omega. \tag{3}$$

(b) *If*

$$\underline{C} > N - 1, \tag{4}$$

then (3) holds for some $\varepsilon \in [0, 1)$. In particular, u_n is a flat solution.
(c) *If*

$$\underline{C} > 2N, \tag{5}$$

then (3) holds for $\varepsilon = 0$.

The flatness exponent grows with the constant $\underline{C}$!!

J. I. Díaz, SeMA (2017) 74:255–278, (2018) 75:563–568.
Easely adaptable (after a redefinition of the notion of very weak solution) to the case $\alpha > 2$:
J. I. Díaz, D. Gómez – Castro, J.M. Rakotoson (2017).

As a particular consequence of Theorem 2.1 it is possible to offer a correct alternative to the "localizing" process suggested by Gamow in his paper [40].

Corollary 2.1 *Let Ω be an open regular bounded set of $\mathbb{R}^N$, $N \geq 1$. For any $q \in [0, +\infty)$ consider the potential*

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ q & \text{if } x \in \mathbb{R}^N - \Omega. \end{cases}$$

Assume (10). Then there exists a countable set of eigenvalues λ_n and eigenfunctions $\widetilde{u}_{n,q}$ of the Schrö dinger equation

$$-\Delta u + V_{q,\Omega}(x)u = \lambda_n u \quad \text{in } \mathbb{R}^N, \tag{18}$$

such that

$$\widetilde{u}_n(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N - \Omega, \end{cases}$$

where λ_n and $u_n(x)$ are the eigenvalues and eigenfunctions of the Dirichlet problem $DP(V, \lambda, \Omega)$. Moreover the same conclusion holds for $q = +\infty$ if we define the corresponding solution as $\widetilde{u}_{n,\infty}(x) = \lim_{q \nearrow +\infty} \widetilde{u}_{n,q}(x)$.

When V is super-singular, $V(x) \geq c_V d(x, \partial\Omega)^{-(2+\varepsilon)}$ then u is flat near $\partial\Omega$ to the *infinite order*, i.e.

$$|u(x)| \leq C d(x, \partial\Omega)^{\beta}, \qquad \forall \beta > 0.$$

In fact, we can prove an exponential decay:

Theorem (D-Gómez-Vázquez (2019)). *Assume* $V(x) \geq c_V d(x, \partial\Omega)^{-(2+\varepsilon)}$, *then*

$$|u(x)| \leq C \exp\left(-d(x, \partial\Omega)^{-\varepsilon/4}\right)$$

in a neighbourhood of $\partial\Omega$.

Construction of a barrier function. $U(x) = \exp(g(x))$ where $g(x) = -\delta(x)^{-\varepsilon/4}$

$\Delta U + VU = \exp(-g(x))(|\nabla g|^2 + \Delta g(x) + V)$ Delicate (and technical) estimates...

The above results on flat solutions prove a lack of the weak and strong versions of the UCP for singular Schrödinger equations.

Corollaries for the associated **complex evolution problem** (for initial data *with compact support*) in the mentioned papers

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi & \text{in } (0,\infty) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x) & \text{on } \mathbb{R}^N. \end{cases}$$

Partial confinement (support$\psi(t,.) \subset \overline{\Omega}$ for any $t > 0$) for (nonnegative) potentials $V(x)$, being sufficiently singular near $\partial\Omega$, for initial wave functions $\psi_0 \in H^1(\mathbb{R} : \mathbb{C})$ with support $\psi_0 \subset \overline{\Omega}$.

No ambiguity, in contrast to the (Gamow-Mott) infinite well potential case !!

2. The ambiguity for the quasi-relativistic Schrödinger equation: *modeling*.

The (mean) speed of the electrons in most of the simple atoms is around c/100 **but it is much higher (near c) for very weighted elements** (mercury, indium,…)

Heisenberg principle (1927) and the identification between the momentum of the particle and an operator given by a pseudo-differential symbol (by means of the Fourier transform)

$$H = (-\hbar^2 c^2 \Delta u + m^2 c^4)^{1/2}$$

unbounded operator defined via the Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx \qquad f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$$

We shall consider here the so called quasi-relativistic particle (with a localizying potential of the type of the inifinite well). We recall that the motion of a single particle subject to an external conservative forze (e.g. an electrostatic field) can be modeled by means of the Hamiltonian

$$H = -(\frac{\hbar^2}{2m})\Delta + V(x),$$

It was observed that this model is inconsistent with special relativity, and leads to erroneous results at high energies (see, e.g. references in the series of works by H. Lieb and collaborators (since 1980)).

A better model for particles at relativistic energies was proposed by Dirac on 1928 but Dirac's equations are much more complicated, and therefore their mathematical treatment is often problematic.

Dirac derived his operator in 1928 starting from the usual classical expression of the energy of a free relativistic particle of momentum, $\mathbf{p} \in \mathbb{R}^3$, and mass m,

$$E^2 = c^2|\mathbf{p}|^2 + m^2c^4,$$

where c is the speed of light, and imposing the necessary relativistic invariances. By means of the usual identification,

$$\mathbf{p} \backsim -i\hbar\nabla,$$

where $\hbar$ is Planck's constant, he arrived to the definition of his vectorial operator

At least in some applications (see papers by Lieb) many authors considered an intermediate model, with the Hamiltonian H given by

$$H = H_0 + V(x) = (-\hbar^2c^2\Delta u + m^2c^4)^{1/2} + V(x).$$

H_0 is often called the *Klein-Gordon square root operator*, or the *quasi-relativistic Hamiltonian*. We shall define operator H_0 by starting with the definition of its domain on $L^2(\mathbb{R}^n)$: $f \in D(H)$ if $(1+\xi^2)^{1/2}\mathcal{F}f(\xi)$ is square integrable and

$$\mathcal{F}H_0f(\xi) = \sqrt{\hbar^2c^2\xi^2 + m^2c^4}\,\mathcal{F}f(\xi),$$

i.e. H_0 is a *Fourier multiplier* with symbol $\sqrt{\hbar^2c^2\xi^2 + m^2c^4}$.

For our purposes it is simpler to replace $\hbar$ and c by 1. So, at least formally, the case $m = 0$ corresponds to the usual nonlocal integro-differential operator called as the *fractional Laplacian operator*, given by the formula

$$(-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \,, \tag{1}$$

with parameter $s \in (0,1)$ and a precise constant $c_{n,s} > 0$ that we do not need to make explicit here.

So, our operator will be simplified to

$$H_0 = (-\Delta + m^2)^{1/2}.$$

Our main interest here is in the exponent $s = 1/2$ but our results can be extended to other values of $s \in (0,1)$ which corresponds to quite relevant applications

This operator admits many different equivalent formulations:
M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fractional Calculus and Applied Analysis*, 20 (1), (2017) 7-51.

Here we are interested on the quasi-relativistic Schrödinger equation problem

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = (-\Delta + m^2)^{1/2}\psi + V(x)\psi & \text{in } (0,\infty) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x) & \text{on } \mathbb{R}^N, \end{cases}$$

for an *infinite well potential (of Gamow-Mott type)* over a regular open bounded domain Ω of $\mathbb{R}^N$

$$V_{\infty,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ +\infty & \text{if } x \notin \Omega, \end{cases}$$

for some $V \in L^1_{loc}(\Omega)$, $V \geq 0$.

Now, it is possible to get some explicit version of the operator H_0 in terms of its Green function

Fall M.M. and Felli V., Unique continuation properties for the relativistic Shrördinger operator with singular potential. Discrete Contin. Dyn. Syst. 35 (2015), no. 12, 5827-5867.

$$(-\Delta + m^2)^s u(x) = c_{n,s} m^{\frac{n+2s}{2}} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{\frac{n+2s}{2}}} K_{\frac{n+2s}{2}}(m|x-y|)dy + m^{2s}u$$

for every $x \in \mathbb{R}^n$, were P.V. indicates that the integral is meant in the principal value sense, with K_ν the modified Bessel function of the second kind with order ν and

$$c_{n,s} = 2^{-(n+2s)/s+1} \pi^{-\frac{n}{2}} 2^{2s} \frac{s(1-s)}{\Gamma(2-s)}.$$

We recall that K_ν solves the linear second order equation

$$r^2 K_\nu'' + r K_\nu' - (r^2 + \nu^2) K_\nu = 0$$

and that we know

A. Erdclyi et al. (Eds.), Higher Transcendental Functions, Vol. 11, McGraw-Hill, New York 1953-1955.

$$K_\nu(r) \sim \frac{\Gamma(2)}{2} \left(\frac{r}{2}\right)^{-\upsilon} \text{ as } r \to 0$$

and

$$K_\nu(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-1/2} e^{-r} \text{ as } r \to +\infty.$$

The case $V(x) \equiv V_0 \geq 0$ on Ω corresponds to the *infinite square-well potential* (the particle in a box). see, e.g. N. Laskin, *Phys. Lett.* A 268, 298 2000, and many other papers.

Controversy with several authors:
M. Jeng, S.-L.-Y. Xu, E. Hawkins, and J. M. Schwarz, On the nonlocality of the fractional Schrödinger equation, *Journal of Mathematical Physics* 51, 062102 (2010).

A quite rigurous mathematical study for $N = 1$: K. Kaleta, M. Kwaśnicki, and J. Małecki, One-dimensional quasi-relativistic particle in the box, *Rev. Math. Phys.* 25, 1350014 (2013).

Ambiguity in the treatment of the infinite square-well potential problem:
i) assumption $V(x)\psi(t,.) = \mathbf{0}$ on $(0,\infty) \times (\mathbb{R}^N - \Omega)$
ii) explicit solution $\psi(t,.) \notin H^1_{loc}(\mathbb{R}^N)$
iii) the domain of the operator requires, $(-\Delta + m^2)^{1/2}\psi \in L^2(\mathbb{R}^N)$ [notice that $\psi(t,.) \in H^{1/2}(\mathbb{R}^N)$ implies merely $(-\Delta + m^2)^{1/2}\psi \in H^{-1/2}(\mathbb{R}^N)$].

To avoid the above ambiguity (concerning the formation of singularities on $\partial\Omega$) we will consider here the case of potentials $V \in L^1_{loc}(\Omega)$ with a singular absorption over $\partial\Omega$, i.e.,

$$\frac{\underline{C}}{d(x,\partial\Omega)^\alpha} \leq V(x) \text{ a.e. } x \in \Omega, \qquad (42)$$

for some $\overline{C} > 0$ and for some $\alpha > 0$.
As a matter of fact, in order to have a good spectral theory we will also assume also that

$$V(x) \leq \frac{\overline{C}}{d(x,\partial\Omega)^\alpha} \text{ a.e. } x \in \Omega, \qquad (43)$$

for some $\overline{C} > \underline{C} \geq 0$.

3. Treatment of the *quasi-relativistic* case: critically singular potentials.

3a. The strategy

Two different steps:

1. Show that the solution **on a bounded domain** (and a potential suitably singular on its boundary) is **flat near the boundary**
2. Show that the extensión by zero outside the domain is, in some sense, solution of the global problem (controversial…….)

Some reasons for the controversy in step 2:

Let $u \in C_0^{2s}(\overline{\Omega})$. The following non-equivalent definitions of the fractional Laplacian in a bounded Ω are known:

1. Restricted Fractional Laplacian (RFL)

$$(-\Delta)^s_{\mathrm{RFL}} u(x) = c_{n,s}\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \qquad x \in \Omega.$$

where u is extended by 0 in Ω^c.

2. Censored Fractional Laplacian (CFL)

$$(-\Delta)^s_{\mathrm{CFL}} u(x) = c_{n,s}\,\mathrm{P.V.} \int_{\Omega} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \qquad x \in \Omega.$$

3. Spectral Fractional Laplacian (SFL) For $u \in H_0^1(\Omega)$, letting the eigenfunctions of the Laplace

$-\Delta \varphi_i = \lambda_i \varphi_i$ For a function u given by the coefficient

$$u(x) = \sum_{i=1}^{+\infty} u_i \varphi_i(x)$$

We define

$$(-\Delta)^s_{\mathrm{SFL}} u(x) = \sum_{i-1}^{+\infty} \lambda_i^s u_i \varphi_i(x)$$

Two of these operators are linked by the following formula

Theorem 1 *(D-D. Gómez-Castro and J.L. Vázquez (2019)*

$$(-\Delta)^s_{\mathrm{RFL}} u(x) = (-\Delta)^s_{\mathrm{CFL}} u(x) + k(x)u(x)$$

where

$$k(x) = c_{n,s} \int_{\Omega^c} |x-y|^{-n-2s} dy$$

satisfies the estimate

$$\frac{\underline{c}}{d(x,\partial\Omega)^{2s}} \leq k(x) \leq \frac{\overline{C}}{d(x,\partial\Omega)^{2s}}$$

for some constants $c, C > 0$.

On the step 2: assume that u is a flat solution (i.e. $u = 0$ and $(\frac{\partial u}{\partial \upsilon})^{1/2} = 0$ on $\partial\Omega$) Then, although it is true that now the extended function by zero on $\mathbb{R}^n - \Omega$

$$U(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

is such that $U \in H^1(\mathbb{R}^n)$ (and thus $(-\Delta)^s U \in L^2(\mathbb{R}^n)$, i.e. there is no singularity on $\partial\Omega$), neverthelees we have

$$(-\Delta)^s U = \begin{cases} (-\Delta)^s_{\mathrm{RFL}} u(x) & x \in \Omega \\ F_u(x) & x \notin \Omega \end{cases}$$

where

$$F_u(x) := -c_{n,s}\,\text{P.V.}\int_\Omega \frac{u(y)}{|x-y|^{n+2s}}dy, \text{ for } x \notin \Omega.$$

Notice that $F_u \neq 0$ and thus, if

$$\begin{cases} (-\Delta)^s u + V(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$(-\Delta)^s U + V(x)U = F_u(x) \neq 0 = \lambda U \text{ on } \mathbb{R}^n - \Omega.$$

3.2. Statements for the evolution problem

Our main interest is the **study of the time evolution problem for** localized initial wave packets $\psi_0 \in H^1(\mathbb{R} : \mathbb{C})$, i.e. such that

$$\text{support } \psi_0 \subset \overline{\Omega}.$$

For the existence and uniqueness of solutions apply the abstract Stone theorem [generation of a group of operator T(t): see, e.g. books by Reed-Simon (Vol. II, 1980), Brezis-Cazenave (1993), Vrabie (2003)] to the operator $A(\psi) = (-\Delta + m^2)^{1/2}\psi + V(x)\psi$ on the Hilbert space $H^{-1}(\Omega : \mathbb{C})$: the sign condition $V(x) \geq 0$ is crucial !!!).

Most of the study literature about such type of problems was concerning with the *bound states* $\psi(x,t) = e^{-\mathbf{i}Et}u(x)$ [E denotes the energy and in the following we shall denote it also by λ], i.e. with $u(x)$ solving the stationary equation

$$\begin{cases} (-\Delta + m^2)^{1/2}_{\mathrm{R}} u + V(x)u = \lambda_N u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{44}$$

In fact it can be shown the following:

Theorem 4 *Assume (42) and (43) with* $\alpha \leq 2s$. *Let* $\psi_0 \in H^1(\mathbb{R}^N : \mathbb{C})$ *such that support* $\psi_0 \subset \overline{\Omega}$. *Then*

i) For $k > 0$ *consider the the truncated potential* $V_k(x) = \min(k, V_{\infty,\Omega})$. *Then Problem (40) has a unique solution* $\psi_k \in C([0,+\infty) : L^2(\mathbb{R}^N : \mathbb{C}))$ *with* $\psi_k \in L^2(0,T : H^1(\mathbb{R}^N : \mathbb{C}))$ *and* $V_k(x)\psi_k \in L^2(0,T : L^2(\mathbb{R}^N : \mathbb{C}))$ *for any* $T > 0$. *Moreover as* $k \to +\infty$, $\psi_k \to \psi$ *in* $L^2_{loc}(0,+\infty) : L^2(\mathbb{R}^N : \mathbb{C}))$ *with* ψ *the unique solution of*

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = (-\Delta + m^2)^{1/2}_{\mathrm{R}} \psi + V(x)\psi & \text{in } (0,\infty) \times \Omega, \\ \psi = \mathbf{0} & \text{on } (0,\infty) \times \mathbb{R}^N \setminus \Omega, \\ \psi(0,x) = \psi_0(x) & \text{on } \Omega. \end{cases}$$

ii) Assume (42) and (43) with $\alpha = 2s$, i.e.

$$\frac{\underline{C}}{d(x,\partial\Omega)^{2s}} \leq V(x) \leq \frac{\overline{C}}{d(x,\partial\Omega)^{2s}} \ \text{a.e. } x \in \Omega,$$

for some $\overline{C} > \underline{C} \geq 0$. Then there exists a sequence λ_N of eigenvalues, with $\lambda_N \to +\infty$ as $N \to +\infty$, of the eigenvalue problem

$$\begin{cases} (-\Delta + m^2)^{1/2}_{\mathrm{R}} u + V(x)u = \lambda_N u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover $\lambda_1 > 0$, λ_1 is isolated, $u_1 > 0$ on Ω and all the eigenfunctions u_N are in $L^\infty(\Omega)$ and if $\underline{C}$ is large enough then u_N are $s-$flat in the sense that

$$\frac{u_N(x)}{\delta(x)} \leq C\delta^{\varepsilon}(x) \to 0 \ \text{as } \delta(x) := d(x,\partial\Omega) \to 0, \tag{45}$$

uniformly as $x \to \partial\Omega$ where $\underline{C} > -\gamma_{\frac{1}{2}+\varepsilon}$ for some $0 < \varepsilon < \frac{1}{2}$. Here, for $\beta > 0$ we define (for $s \in (0,1)$)

$$\gamma_\beta = 2^{2s} \frac{\Gamma\left(\frac{n+\beta}{2}\right)\Gamma\left(s - \frac{\beta}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(-s + \frac{\beta+n}{2}\right)} \tag{46}$$

Notice that $\gamma_{s+\varepsilon} < 0$ for $0 < \varepsilon < s$.

iii) The problem

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = (-\Delta + m^2)_{\mathrm{R}}^{1/2}\psi + V(x)\psi & \text{in } (0,\infty)\times\Omega, \\ \psi = \mathbf{0} & \text{on } (0,\infty)\times\partial\Omega, \\ \psi(0,x) = \psi_0(x) & \text{on } \Omega, \end{cases} \tag{47}$$

has a unique solution $\psi_\Omega \in C([0,+\infty) : H^1(\Omega : \mathbb{C}) \cap H_0^{1/2}(\Omega : \mathbb{C}))$*, and we have the Galerkin decomposition*

$$\psi_\Omega(t,x) = \sum_{N=1}^{\infty} \mathbf{a}_N e^{-\mathbf{i}\lambda_N t} u_N(x), \tag{48}$$

with convergence at least in $L^2(\Omega : \mathbb{C})$*, where* λ_N *and* u_N *are the eigenvalues and eigenfunctions given in ii) (renormalized by (21)) for any* N *and*

$$\mathbf{a}_N = \int_\Omega \psi_0(x) u_N(x)dx.$$

iv) Assume $\alpha = 2s$ *and that*

$$\sum_{N=1}^{\infty} |\mathbf{a}_N| \overline{K}_N < +\infty, \tag{49}$$

where $\overline{K}_N > 0$ *is the uniform flat estimate of* u_N *on* $\partial\Omega$*. Then*

$$|\psi_\Omega(t,x)| \leq Kd(x,\partial\Omega) \quad \text{for any } t > 0 \text{ and a.e. } x \in \Omega, \tag{50}$$

for some $K > 0$*. In consequence, the unique solution of (40) for the extended potential* $V_{\infty,\Omega}(x)$ *is given by*

$$\psi(t,x) = \begin{cases} \psi_\Omega(t,x) & \text{if } x \in \Omega, \\ \mathbf{0} & \text{if } x \in \mathbb{R}^N - \Omega, \end{cases} \tag{51}$$

is in $C([0,+\infty) : H^1(\mathbb{R}^N : \mathbb{C}))$ *and support* $\psi(t,.) \subset \overline{\Omega}$ *for any* $t > 0$.

3.3. On the proof of i). The role of the restricted fractional Laplacian.

The restricted fractional Laplacian as the natural limit of the Schroedinger equation in $\mathbb{R}^n$ for the infinite singular-well potential: (argument in J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez. The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach. Nonlinear Analysis, Nonlinear Analysis, 177 (2018) :325-360).

Let us consider the singular infinite well potential

$$V_{+\infty,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ +\infty & \text{if } x \in \mathbb{R}^n - \Omega. \end{cases}$$

To avoid the ambiguity of the definition of Vu in Ω^c, the solutions of the associated Schrödinger problem can be understood as the limit of the solutions of the corresponding finite-well potentials

$$V_k(x) = k \wedge V(x).$$

The stationary Schrödinger equation over its natural domain, the whole space, corresponds to finding $u_k \in H^s(\mathbb{R}^n)$ such that

$$(-\Delta)^s u_k + V_k(x) u_k = f \text{ in } \mathbb{R}^n,$$

for some function $0 \leq f \in L^\infty(\Omega)$, $f = 0$ in Ω^c. Here, all the usual formulations are equivalent. Hence $u_k \geq 0$. Furthermore, $0 \leq u_k$ is a decreasing sequence, and hence has limit in $L^1(\Omega)$, $0 \leq u \in L^1(\mathbb{R}^n)$ which is also a.e. pointwise limit, due to the Monotone Convergence Theorem.

Theorem : *As $k \to \infty$ the solutions of the approximate problems in $\mathbb{R}^n$ converge to the solution of Problem with potential $V(x)$. In particular $u = 0$ in Ω^c.*

Proof. Using the solution of

$$\begin{cases} (-\Delta)^s \varphi_0 = 1 & \mathbb{R}^n \\ \varphi_0 \to 0 & |x| \to +\infty, \end{cases}$$

we deduce that, for any k

$$(1 + k \min_{\Omega_c} \varphi_0) \int_{\Omega^c} u_k \le \int_\Omega f \varphi_0$$

Hence $u = 0$ in Ω^c. On the other hand,

$$\int_{\mathbb{R}^n} u_k (-\Delta)^s \varphi + \int_{\mathbb{R}^n} V_k u_k \varphi = \int_\Omega f \varphi.$$

As before, for $K \subset \Omega$ compact $V_k u_k \to V u$ in $L^1(K)$ by the Dominated Convergence Theorem. Finally, for any $\varphi \in C_c^\infty(\Omega)$ such that $(-\Delta)^s \varphi \in L^\infty(\mathbb{R}^n)$, we pass to the limit to obtain

$$\int_{\mathbb{R}^n} u (-\Delta)^s \varphi + \int_{\mathbb{R}^n} V u \varphi = \int_\Omega f \varphi.$$

For φ the restricted fractional Laplacian and the fractional Laplacian in $\mathbb{R}^n$ coincide. Since $u = 0$ in Ω_c and φ is supported in Ω, this is precisely

$$\int_\Omega u (-\Delta)^s \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi.$$

By density, we have the previous formulation for all $\varphi \in X_\Omega^s \cap C_c(\Omega)$, where

$$X_\Omega^s = \{\varphi \in C^s(\mathbb{R}^n) : \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega \text{ and } (-\Delta)^s \varphi \in L^\infty(\Omega)\}.$$

This shows that the natural fractional Laplacian to deal with the Schrödinger equation with the singular infinite-well potential problem is the **restricted fractional Laplacian over** Ω.

We point out that, physically, the Schrödinger equation a priori must be defined over the whole space, $\mathbb{R}^n$, and that any other constraint (as, for instance, to assume a localization to a subset Ω) must be justified.

3.4. On the proof of ii). Flat eigenfunctions for α=2s.

It will be consequence of different auxiliary results

Concerning the spectrum (associated to the restricted fractional Laplacian over a bounded domain) we have:

Lemma 1 *Under the above assumption on* $V(x)$ *there exists a sequence* λ_N *of eigenvalues, with* $\lambda_N \to +\infty$ *as* $N \to +\infty$*, of the eigenvalue problem*

$$\begin{cases} (-\Delta + m^2)_{\mathrm{R}}^{1/2} u + V(x)u = \lambda_N u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N - \Omega. \end{cases}$$

Moreover $\lambda_1 > 0$, λ_1 *is isolated,* $u_1 > 0$ *on* Ω *and all the eigenfunctions* u_N *are in* $L^\infty(\Omega)$.

(Main idea of the proof: use the *fractional Hardy inequality* on Ω (see Fall and Felli (2015) and apply the Stampacchia type regularyzing iterative arguments).

In order to prove the flatness of the eigenfunctions we previously need the following result:

Lemma 2. *Assume* $V(x)$ *as before and let* $f_m, f_0 \in L^\infty(\Omega)$. *Given* $m > 0$ *let* $u_m \in L^1(\Omega)$ *be the unique very weak solution of the problem*

$$\begin{cases} (-\Delta + m^2)_{\mathrm{R}}^{1/2} u + V(x)u = f_m(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $u_0 \in L^1(\Omega)$ *be the unique very weak solution of the problem*

$$\begin{cases} (-\Delta)_{\mathrm{R}}^{1/2} u + V(x)u = f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that

$$f_0(x) \geq f_m(x) \geq 0 \text{ on } \Omega.$$

Then

$$u_0(x) \geq u_m(x) \geq 0 \text{ on } \Omega.$$

Moreover, if $f_0(x) \leq f_m(x) \leq 0$ *on* Ω *then* $u_0(x) \leq u_m(x) \leq 0$ *on* Ω.

Main ideas of the proof of Lemma 2.

1. Use the equivalent formulation for the operators $(-\Delta)_{\mathrm{R}}^{1/2}$ and $(-\Delta + m^2)_{\mathrm{R}}^{1/2}$ given in L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations*, 32 (2007), no. 7-9, 1245–1260, and Fall and Felli (2015), respectively.
2. Truncate $V(x)$.
3. Applied the comparison principle to the extended problems,
4. Pass to the limit as Díaz, Gómez-Castro and J. L. Vázquez (2018).

The flatness of the eigenfunctions is, in some sense, the main result of this program:

Theorem 5. *Assume*

$$\frac{\underline{C}}{d(x,\partial\Omega)^{2s}} \leq V(x) \leq \frac{\overline{C}}{d(x,\partial\Omega)^{2s}} \text{ a.e. } x \in \Omega,$$

for some $\overline{C} > \underline{C} \geq 0$ with $\underline{C}$ large enough. Then the eigenfunctions u_N are $s-$flat in the sense that

$$\frac{u_N(x)}{\delta^s(x)} \leq C\delta^\varepsilon(x) \to 0 \text{ as } \delta(x) := d(x,\partial\Omega) \to 0, \tag{52}$$

uniformly as $x \to \partial\Omega$ where $\underline{C} > -\gamma_{s+\varepsilon}$ and with

$$\gamma_\beta = 2^{2s}\frac{\Gamma\left(\frac{n+\beta}{2}\right)\Gamma\left(s-\frac{\beta}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(-s+\frac{\beta+n}{2}\right)}, \tag{53}$$

for $\beta > 0$. Notice that $\gamma_{s+\varepsilon} < 0$ for $0 < \varepsilon < s$.

The proof of Theorem 5 is consequence of Lemma 2 and the construction of suitable barrier functions. We still need a technical result (J.L. Vázquez (2014) and Fall (2012) obtained by applying the Fourier transform formula of a radial function as in Stein (2016)):

Lemma 3. *Let $\nu_\beta(x) = |x|^\beta$ with $\beta > 0$. Then*

$$(-\Delta)^s\nu_\beta = \gamma_\beta|x|^{-2s}\nu_\beta, \qquad \text{in } \mathbb{R}^n, \tag{54}$$

where

$$\gamma_\beta = 2^{2s}\frac{\Gamma\left(\frac{n+\beta}{2}\right)\Gamma\left(s-\frac{\beta}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(-s+\frac{\beta+n}{2}\right)} \tag{55}$$

is a constant.

As an application of Lemma 4, we have:

Lemma 4. *Let $0 < \varepsilon < s$, $0 \le f \in L^\infty$, $V \ge C_V|x - x_0|^{-2s}$ with $C_V > 0 > -\gamma_{s+\varepsilon}$, and let $x_0 \in \partial\Omega$. Then,*

$$\frac{u(x)}{|x - x_0|^{s+\varepsilon}} \le \frac{\|f\|_{L^\infty}}{(\gamma_{s+\varepsilon} + C_V)} R(x_0)^{s-\varepsilon}, \tag{56}$$

a.e. in Ω, where $R(x_0) = \max_{x\in\bar{\Omega}} d(x, x_0)$ (i.e. such that $\Omega \subset B_R(x_0)$).

Proof. Since $0 \le f \in L^\infty$ we have that $0 \le u \in L^\infty$. Let us consider $U(x) = C_U \nu_{s+\varepsilon}(x - x_0)$ where

$$C_U = \frac{\|f\|_{L^\infty}}{(\gamma_{s+\varepsilon} + C_V)} R^{s-\varepsilon} > 0. \tag{57}$$

We compute

$$\begin{aligned}(-\Delta)^s U + VU &= \gamma_{s+\varepsilon}|x - x_0|^{-2s} U + VU \\ &\ge (\gamma_{s+\varepsilon} + C_V)|x - x_0|^{-2s} U \\ &= C_U(\gamma_{s+\varepsilon} + C_V)|x - x_0|^{-s+\varepsilon} \\ &\ge C_U(\gamma_{s+\varepsilon} + C_V) R^{-s+\varepsilon} \\ &= \|f\|_{L^\infty}\end{aligned}$$

a.e. in Ω, since $-s + \varepsilon < 0$. Since also $U \ge 0 = u$ on Ω^c we have that $U \ge u$ a.e. in Ω. Therefore,

$$\frac{u}{|x - x_0|^{s+\varepsilon}} \le \frac{U}{|x - x_0|^{s+\varepsilon}} = C_U.$$

a.e. in Ω. This completes the proof.$x, x_0)$ (i.e. such that $\Omega \subset B_R(x_0)$).

Finally we arrive to a global consequence which completes the proof of Theorem 5. :
Lemma 5. *Let* $0 < \varepsilon < s$, $0 \leq f \in L^\infty$, $V(x) \geq \underline{C}\delta(x)^{-2s} \geq 0$ *with* $\underline{C} > -\gamma_{s+\varepsilon}$. *Then,*

$$\frac{u}{\delta^{s+\varepsilon}} \in L^\infty(\Omega).$$

Proof of Lemma 5. Since $\delta(x) = \min_{x_0 \in \partial\Omega} |x - x_0|$ we have that

$$\frac{u(x)}{\delta(x)^{s+\varepsilon}} = \max_{x_0 \in \partial\Omega} \frac{u(x)}{|x - x_0|^{s+\varepsilon}} \leq \frac{\|f\|_{L^\infty}}{(\gamma_{s+\varepsilon} + C_V)} \max_{x_0 \in \bar{\Omega}} R(x_0)^{s-\varepsilon}.$$

This last maximum if finite because Ω is a bounded set. This completes the proof.

3.5. On the proof of iii) and iv)

They are easy modifications of the non-relativistic case.

J. I. Díaz, On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via singular potentials: the multi-dimensional case, SeMA-Journal (2017)

No ambiguity, in contrast to the (Gamow-Mott) infinite well potential case !!

- Main conclusion: to **faster** electrons (relativistic regime) **weaker** confinement Potentials (than in the non-relativistic case).
- Many possible applications,…, even to philosophy (decreasing uncertainty with the speed) !!!

Thanks for your attention