Explosiones controladas: dinámica después de la explosión para problemas semilineales con condiciones dinámicas en el borde.

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Modelos No Lineales
en Ingeniería Matemática

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Plan:

1. Explosión en tiempo finito para EDO con retardo (Athenas 2013, 3-47)

Casal, A.C., Díaz, J.I. and Vegas, J.M.: Blow-up in some ordinary and partial differential equations with time-delay, Dynam. Systems Appl., 18 1, 29-46 (2009).

1.1. Introducción
1.2. Preliminares
1.3. La EDO (lineal) con retardo "básica"
1.4. La ecuación (funcional) neutra equivalente
1.5. Soluciones con singularidades "evitables"
1.6. Prolongación más allá del retardo
1.7. La ecuación neutra con una perturbación lineal
1.8. La ecuación neutra no lineal
1.9. La ecuación neutra no lineal general: fórmula de variación de las constantes de Alekseev
1.10. Aplicación a una EDP lineal con retardo
1.11. Aplicación a una EDP no lineal con retardo y con explosión en tiempo finito
1.12 Continuación más allá de la explosión

2. Control de la explosión en EDO no lineales sin retardo (Post-Athenas I: Orlando 2014, 48-62)

- Casal, A.C., Díaz, J.I. and Vegas, J.M.: Controlled explosions of blowing-up trajectories in semilinear problems and a nonlinear variation of constant formula, XXIII Congreso de Ecuaciones Diferenciales y Aplicaciones, XIII Congreso de Matemática Aplicada, Castellón, 9-13 septiembre 2013. e-Proccedings.
- Casal, A.C., Díaz, J.I. and Vegas, J.M.: Complete recuperation after the blow up time for semilinear problems, AIMS Procceding 2015, 223-229. (2015).

2.1 Introducción

2.2 Explosiones controladas para EDO con términos superlineales

2.3. Explosiones controladas para f(u) localmente Lipschitz continua y superlineal

3. Aplicación a EDPs estacionarias con condiciones dinámicas (Post-Athenas II: Rakotoson 60, 2017, 63-86)

Controlled explosions: dynamics after blow-up time for semilinear problems with a dynamic boundary condition

A.C. Casal, G. Díaz, J.I. Díaz, J.M. Vegas

3.1. Introducción

3.2. Perfiles explosivos y resultados previos

3.3. El problema dinámico sin control

3.4. Explosiones controladas

1. Explosión en tiempo finito para EDO con retardo (Athenas 2013, 3-47)

There is a very extensive bibliography on blow-up phenomena. The relation of blow-up and time delay has not been studied in so much detail.

The authors have done a similar analysis for the "opposite" situation, namely, finite-time extinction, in which some (nonzero) solutions vanish identically after some finite "extinction time".

Some of the techniques used are similar in both cases, but there are important differences.

The first of these comes from the very nature of blow-up and how "infinity" is involved, which requires analyzing some technical aspects of the regularity properties of the solution.

The second difference is that the structure of the equation enables us to apply delay-PDE comparison techniques which are not usually available in the extinction phenomenon.

Our goal will be to analyze the delay PDE (where f is C^2 and Ω an open bounded set of $\mathbb{R}^N$)

$$(NLP_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(t, u(t,x), u(t-\tau, x)), & (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t,x) = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(\theta, x) = \xi(\theta, x), & (\theta, x) \in (-\tau, 0) \times \Omega, \end{cases}$$

going through the study of

$$(P_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = B'(t) u(t-\tau, \mathbf{x}), & (t,\mathbf{x}) \in (0,+\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t,\mathbf{x}) = 0, & (t,\mathbf{x}) \in (0,+\infty) \times \partial\Omega, \\ u(\theta, \mathbf{x}) = \xi(\theta, \mathbf{x}), & (\theta, \mathbf{x}) \in (-\tau, 0) \times \Omega, \end{cases}$$

based (for g in C^1 and B defined below) in that of

$$(NLDDE)\begin{cases} u'(t) = f(t, u(t)) + B'(t)g(t, u(t-\tau)), & 0 < t \\ u(\theta) = \xi(\theta), & -\tau \le \theta \le 0, \end{cases}$$

after considering carefully properties of

$$(DDE)\begin{cases} u'(t) = B'(t)(t, u(t-\tau)), & 0 < t \\ u(\theta) = \xi(\theta), & -\tau \le \theta \le 0, \end{cases}$$

$B : [0, \tau] \to \mathbb{R}$ is a positive L^1 function which behaves like $1/|t - t^*|^\alpha$, for some $\alpha \in (0, 1)$ and $t^* \in (0, \tau)$,

B' represents its distributional derivative. (The product with u will be justified later on)

We analize the (delicate) possibility of extending blow-up solutions beyond the explosion time t^* (in a certain sense).

For this, we will come to define a generalized solution by means of the following integral identity in a suitable space of functions on Ω

$$u(t) = e^{At}\xi(0) + B(t)\xi(t-\tau) + \int_0^t e^{A(t-s)}B(s)\left[-A\xi(s-\tau) + \xi'(s-\tau)\right] ds,$$

where $\xi(\theta, \mathbf{x})$ is the initial historyfunction, A is the abstract operator associated to $-\Delta$

with Neumann boundary conditions and e^{At} is the associated semigroup, and give sufficient conditions for the integral to exist beyond $t = t^*$(for instance on $[0, \tau]$).

Preliminary analysis

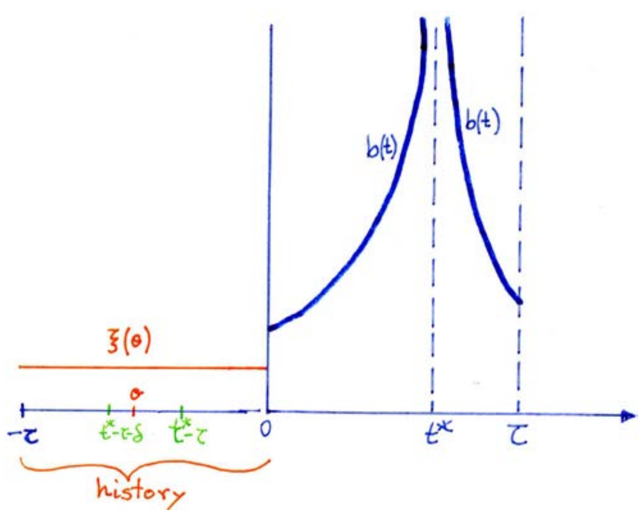
Let $t^* > 0$, let $b : [0, t^*) \to \mathbb{R}$ be a continuous function such that $b(t) \geq 0$ on $[0, t^*)$

Assume that b *"blows up"* at t^*, that is, $b(t) \to \infty$ as $t \nearrow t^*$.

Consider the delay differential equation

$$(DDE^*) \begin{cases} u'(t) = b(t)u(t-\tau), & \text{for } 0 \leq t < t^*, \\ u(\theta) = \xi(\theta), & \text{for } -\tau \leq \theta \leq 0, \end{cases}$$

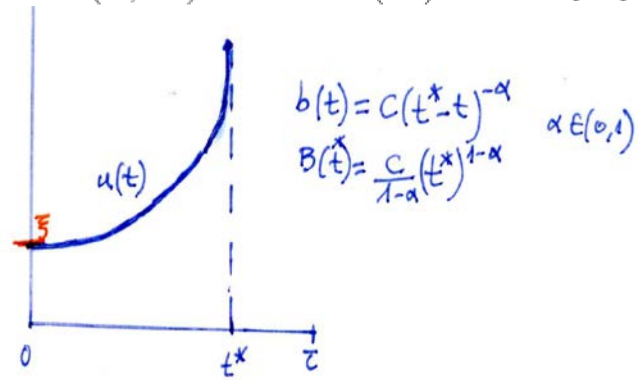
where $\tau > t^*$ is a given delay and ξ represents the "history" or "initial function", which is usually assumed to be continuous on $[-\tau, 0]$, (other function spaces can also be considered).



If $\xi(t) \equiv \xi \in \mathbb{R}$ is a nonzero constant, then direct integration of both sides of above equation gives

$$u(t) = u(t,\xi) = u(0) + \int_0^t b(s)\xi ds = \xi(1 + B(t)), \quad 0 \leq t < t^*, \tag{1}$$

where $B(t) = \int_0^t b(s)ds$. If b is integrable on $(0, t^*)$ then $B(t^*) = \lim_{t \to t^*} B(t)$ exists.



If b(t) is not integrable !!

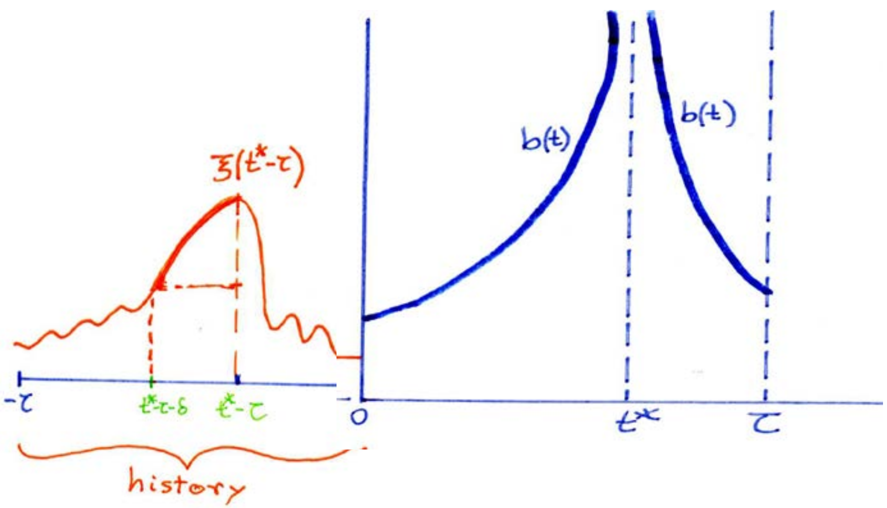
Otherwise, u blows up at t^* but the singularity of the solution is weaker than that of b, a fact that reminds the "smoothing effect" usually found on delay equations.

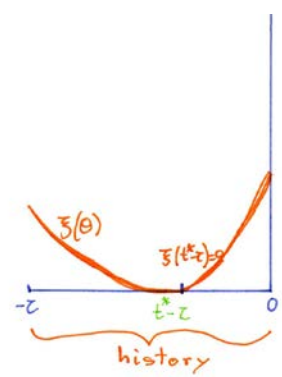
If the initial condition ξ is not constant and if $\xi(t^* - \tau) > 0$, then $\xi(t) \geq \xi(t^* - \tau)/2$ on some interval $[t^* - \tau - \delta, t^* - \tau]$ (where $0 < \delta < t^*$) and we may write for $t \in [0, t^*)$:

$$u(t) = u(t,\xi) = u(t^* - \delta) + \int_{t^*-\delta}^{t} b(s)\xi(s-\tau)ds \geq \tag{2}$$

$$\geq u(t^* - \delta) + \frac{\xi(t^* - \tau)}{2}\left[B(t) - B(t^* - \delta)\right]$$

which implies that, $u(t,\xi)$ blows up at t^* like $B(t)$ as before.





If $\xi(t^* - \tau) = 0$, the product $b(t)\xi(t - \tau)$ may be integrable or not on $(0, t^*)$, depending on the (fractional) order of $t^* - \tau$ as a zero of ξ. If ξ is C^1, for instance, the order will be an integer and the product will certainly be integrable.

If the function b is also defined and is continuous for $t > t^*$, it is natural to ask whether the solution itself can be continued beyond t^* in some sense. In other words, can the formal integral expression

$$u(t, \xi) = \xi(0) + \int_0^t b(s)\xi(s-\tau)ds, \quad 0 \leq t \leq \tau,$$

be considered a as an "integral solution" of some kind, defined on the whole interval $[0, \tau]$?

This is the real difficulty, since continuation beyond τ is always possible as long as b remains continuous.

Let us start again with constant initial functions $\xi(t) \equiv \xi$. If $B \in L^p(0,\tau)$ for some $p \in [1,\infty]$, the function

$$u(t,\xi) = \xi(1 + B(t)),$$

is a well-defined L^p function. For a general continuous initial ξ, the function

$$u(t,\xi) = \xi(0) + \int_0^t b(s)\xi(s-\tau)ds, \quad 0 \le t \le \tau,$$

is also well defined and belongs to the same L^p class as B does, it is also C^1 except a $t = t^*$ and satisfies the differential equation for all $t \in [0,\tau]$ except for t^*.

Of course, one could *define* an integral solution to be just that, but it is clear that further analysis is necessary in order to justify such a procedure. This is the purpose of the next section, which deals with primitives $B(t)$ only assumed to be in $L^p(0,\tau)$, thus allowing for infinitely many singularities and other more complicated situations.

The basic equation

Let $B \in L^p(0,\tau)$ such that $B' \notin L^1_{\rm loc}(0,\tau)$, where B' is to be understood in the sense of distributions. Without loss of generality we will assume that $B(0) = 0$.

Example: $B'(t) = C(t^* - t)^{-\beta}$ with $\beta \in (1,2)$ [so that $B(t) \sim K(t^* - t)^{-\beta+1} \in L^p(0,t^*)$ for some $p \geq 1$]

We consider the retarded functional differential equation

$$DDE\tau \begin{cases} u'(t) = B'(t)u(t-\tau), & 0 < t < \tau, \\ u(\theta) = \xi(\theta), & -\tau \leq \theta \leq 0, \end{cases}$$

where ξ is a given initial function whose smoothness properties will be discussed below. For the time being we will concentrate on the initial “basic interval” $[0,\tau]$.

As discussed previously, if B is C^1 except for a singularity $t^* \in (0, \tau)$, for instance

$$B(t) = 1/|t - t^*|^\alpha, \quad \text{where} \quad 0 < \alpha < 1, \tag{3}$$

we can integrate both sides, thus obtaining

$$u(t) = \xi(0) + \int_0^t B'(s)\xi(s - \tau)ds, \tag{4}$$

but, in general, this formula will make sense only for $t \in [0, t^*)$ because the product $B'(t)\xi(t - \tau)$ need not be integrable. In fact, it will *never* be integrable for nonzero constants ξ. As mentioned above, in order to get a better understanding of the problem and check whether the solution can be continued "beyond" the singular point t^* in a meaningful way we need to give a more precise meaning to the right-hand side of $(DDE\tau)$

A strategy in the theory of differential equations with discontinuous right-hand sides (Filippov) is to transform the equation into another with integrable discontinuities, that is, a "Carathéodory form", considering an

Equivalent neutral equation

1.4. La ecuación (funcional) neutra equivalente

By writing

$$B'(t)u(t-\tau) = [B(t)u(t-\tau)]' - B(t)u'(t-\tau),$$

equation $(DDE\tau)$ becomes

$$NTL \begin{cases} \dfrac{d}{dt}[u(t) - B(t)u(t-\tau)] = -B(t)u'(t-\tau), & t > 0, \\ u(\theta) = \xi(\theta), & -\tau \leq \theta \leq 0, \end{cases}$$

which is a **neutral** differential-delay equation.

J. K. Hale, Theory of functional differential equations, Springer, New York, 1977.

Integrating (formally) both sides of (NTL) on $[0,\tau]$ and since we assume $B(0)=0$ (nonessential)

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)\xi'(s-\tau)ds, \quad 0 \le t \le \tau$$

which gives an explicit representation of the solution in terms of the initial function.

This is just the standard "method of steps" as long as the integral in the right-hand side is defined. As is usual in neutral FDE's, more smoothness in the initial function is required than in the retarded case. Since $B \in L^p(0,\tau)$, the hypothesis $\xi' \in L^q(-\tau,0)$ $(1/p + 1/q = 1)$ will be enough.

We have just proved the following result:

Theorem 1 *1. Let $B \in L^p(0,\tau)$. Then, for every $\xi \in W^{1,q}(0,\tau)$ (where $1/p+1/q=1$) the Cauchy problem (NTL) has a unique solution given by the identity*

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)\xi'(s-\tau)ds, \quad 0 \le t \le \tau, \tag{5}$$

Therefore $u \in L^p(0,\tau)$ and $u(t) - B(t)\xi(t-\tau)$ is an absolutely continuous function and we may write symbolically

$$u(t) = B(t)\xi(t-\tau) + AC,$$

where "AC" means "an absolutely continuous function". As a consequence, the singularities of the solution on $[0,\tau]$ are also singularities of B

2. *In particular, let* $t^* \in (0,\tau)$, $0<\alpha<1$, *let* m *be continuous on* $[0,\tau]$ *and let*

$$B(t) = \frac{a}{|t-t^*|^\alpha} + m(t),$$

If the initial function ξ *satisfies* $\xi(t^*-\tau) \neq 0$, *then* t^* *is also a singularity of* u *and*

$$u(t) \simeq \frac{a}{|t-t^*|^\alpha}\xi(t^*-\tau), \quad \text{as } t \to t^*,$$

is an asymptotic expansion of u *near* t^*.

3. *If* $|\xi(t^*-\tau-t)| \leq C\,|t^*-\tau-t|^\alpha$ *near* t^*, *then* u *is bounded near* t^*.

1.5. Soluciones con singularidades "evitables"

Solutions with removable singularities

More in general, following on point 3 of the previous theorem, let us concentrate again on the single-singularity case as above:

$$B(t) = \frac{a}{|t - t^*|^\alpha} + m(t), \quad t \in [0, \tau],$$

with $t^* \in (-\tau, 0)$, $\alpha \in (0, 1)$ and m continuous. For any $\gamma > \alpha$ let us consider the following class of initial values:

$$E_\gamma = \{\xi \in W^{1,q}(-\tau, 0) : \text{There exists } C > 0 \text{ such that}$$
$$|\xi(t^* - \tau - \theta)| \leq C\, |t^* - \tau - \theta|^\gamma \text{ for all } \theta \in [-\tau, 0]\}.$$

Because of the Sobolev embedding $W^{1,q}(-\tau, 0) \subset C[-\tau, 0]$, E_γ is a closed subspace of $W^{1,q}(-\tau, 0)$. We have thus an immediate consequence of representation

$$u(t) = \xi(0) + B(t)\xi(t - \tau) - \int_0^t B(s)\xi'(s - \tau)ds, \quad 0 \leq t \leq \tau$$

Proposition 2 *If* $\xi \in E_\gamma$, *the solution* u *of (NTL) is absolutely continuous on* $[-\tau, 0]$.

Remark 3 *If we restrict ourselves to* C^1 *initial functions (a very standard procedure in neutral delay-differential equations), the hypothesis that the exponent* γ *be strictly larger than* α *means that the condition* $|\xi(t^* - \tau - t)| \leq C\,|t^* - \tau - t|^\gamma$ *is automatically satisfied if* $t^* - \tau$ *is simply a zero of* ξ, *and the definition of* E_γ *is much easier:*

$$E_\gamma \cap C^1([-\tau, 0]) = \{\xi \in C^1([-\tau, 0]) : \xi(t^* - \tau) = 0\}.$$

Take, for instance, $\xi \equiv$ constant, $1/2 \leq \alpha < 1$, $B(t) = 1/\left|t - \tau/2\right|^{\alpha}$ on $[0, \tau]$ and extended periodically to all of $\mathbb{R}$. Then $u(t) = \xi(1 + B(t)) = \xi$ on $[0, \tau]$, on $[\tau, 3\tau/2)$ the equality $u'(t) = B'(t)u(t - \tau)$ does hold and then

$$\begin{aligned} u(t) &= \xi(1 + B(\tau)) + \xi \int_{\tau}^{t} B(s)B'(s - \tau)ds \\ &= \xi \left(1 + B(\tau) + \frac{1}{2}\left[B(t)^2 - B(\tau)\right]\right), \quad \tau \leq t < \frac{3\tau}{2}, \end{aligned}$$

because of the periodicity of B. Since B^2 is not integrable, the solution cannot be extended

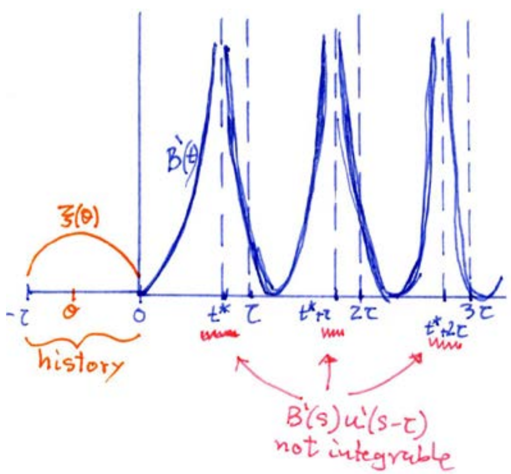
beyond $3\tau/2$ in a meaningful way.

Continuation beyond τ

Assume that B is defined on a larger interval $[0, T)$, where $T > \tau$. As can be easily seen from the explicit formula

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)u'(s-\tau)ds, \quad 0 \le t \le \tau,$$

even for very smooth ξ, if B also contains singularities on the interval $[\tau, 2\tau]$ (for instance, if B is τ-periodic, a very important case), the function $B(s)u'(s-\tau)$ may not be integrable beyond τ.



On the other hand, results of the general theory of functional differential equations imply that if B is differentiable on $[0, T)$ except at a unique singularity t^*, the solution can be extended to all $[0, T)$. The following theorem is stated in a simplified situation which enables us to give a direct proof.

Theorem 4 *Let $T > \tau$ (including $+\infty$), $0 < \alpha < 1$, let B_1 be given by $B_1(t) = \dfrac{a}{|t - t^*|^\alpha}$, and let $m : [0, T) \to \mathbb{R}$ be continuously differentiable and let*

$$B(t) = B_1(t) + m(t), \quad 0 \le t < T.$$

Let $\xi \in C^1([-\tau, 0])$. Then the initial value problem

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - B(t)u(t-\tau)\right] = -B(t)u'(t-\tau), \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0, \end{cases}$$

has a unique solution on $[0, T)$ which belongs to $L^p($ $-\tau, \tau)$, for every $p < 1/\alpha$, and continuous on $[0, T)$ except at t^* and continuously differentiable at every $t \in [0, T)$ except t^* and $\tau + t^*$.

Proof. We already know that the expression

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)\xi'(s-\tau)ds, \quad 0 \le t \le \tau$$

gives us an L^p solution $[0,\tau]$. Since $\xi \in C^1$, it is also continuously differentiable except at t^*. In order to extend it beyond τ, we go back to the original retarded presentation

$$u'(t) = B'(t)u(t-\tau),$$

which does not give any trouble for values $t \ge \tau$, since the "coefficient" $B'(t)$ is continuous on $[\tau, T)$. On $[\tau, 2\tau]$ we can write

$$u(t) = u(\tau) + \int_\tau^t B'(s)u(s-\tau)ds, \quad \tau \le t \le 2\tau,$$

which is absolutely continuous on $(\tau, 2\tau)$ and continuously differentiable except at $t = \tau + t^*$. ■

Remark 5 *Since linear retarded functional differential equations are well-posed on L^p spaces (Webb) and these equations have a "smoothing effect" (Hale), the above result can be extended in a number of ways. For instance, if $B : [0, T) \to \mathbb{R}$ is L^p on $[0, \tau]$ and continuously differentiable on $[\tau, T)$, then the solution belongs to $L^p_{loc}(0, T)$, belongs to $W^{1,p}(\tau, 2\tau)$, to $W^{2,p}(2\tau, 3\tau)$ and so on.*

G. F. Webb, Functional differential equations and nonlinear semigroups in L^p-spaces. J. Differential Equations 20 (1976), no. 1, 71–89.

J. K. Hale, Theory of functional differential equations, Springer, New York, 1977.

Linear perturbations

This aparently easy argument will be important later on.

The above analysis is easily adapted to the case

$$\begin{cases} u'(t) = \lambda u(t) + B'(t)u(t-\tau), & t > 0 \\ u(\theta) = \xi(\theta), & \tau \leq \theta \leq 0, \end{cases}$$

by first applying the Euler change of variables $v(t) = e^{-\lambda t}u(t)$, which gives

$$\begin{aligned} v'(t) &= -\lambda e^{-\lambda t}u(t) + e^{-\lambda t}\left[\lambda u(t) + B'(t)u(t-\tau)\right] \\ &= e^{-\lambda\tau}B'(t)v(t-\tau), \end{aligned}$$

and successively obtaining the *equivalent neutral formulation*

$$\begin{cases} \dfrac{d}{dt}\left[v(t) - e^{-\lambda\tau}B(t)v(t-\tau)\right] = -e^{-\lambda\tau}B(t)v'(t-\tau), & t > 0, \\ v(\theta) = e^{-\lambda\theta}\xi(\theta), & \tau \leq \theta \leq 0, \end{cases}$$

the *representation* for $v(t)$ is

$$\begin{aligned} v(t) = {} & e^{-\lambda\tau}B(t)e^{-\lambda(t-\tau)}\xi(t-\tau)+\xi(0) \\ & -\textstyle\int_0^t e^{-\lambda\tau}B(s)e^{-\lambda(s-\tau)}\left[-\lambda\xi(s-\tau)+\xi'(s-\tau)\right]ds \\ & = \xi(0)+e^{-\lambda t}B(t)\xi(t-\tau)-\textstyle\int_0^t e^{-\lambda s}B(s)\left[-\lambda\xi(s-\tau)+\xi'(s-\tau)\right]ds, \end{aligned}$$

and the *representation* for $u(t)=e^{\lambda t}v(t)$ is

$$\begin{cases} u(t)=e^{\lambda t}\xi(0)+B(t)\xi(t-\tau) \\ \qquad\qquad +\int_0^t e^{\lambda(t-s)}B(s)\left[-\lambda\xi(s-\tau)+\xi'(s-\tau)\right]ds, \end{cases}$$

which is very similar to

$$u(t)=\xi(0)+B(t)\xi(t-\tau)-\int_0^t B(s)\xi'(s-\tau)ds, \quad 0\le t\le\tau$$

. The qualitative statements of theorem 1 and the asymptotic expansion near t^* are translated to this case without change.

Similar results can be written for non-autonomous versions of the above equation

$$\begin{cases} u'(t) = \lambda(t)u(t) + B'(t)u(t-\tau), \quad t > 0, \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0, \end{cases}$$

obtaining the representation

$$\begin{cases} u(t) = B(t)\xi(t-\tau) \\ \qquad\qquad + \int_0^t e^{\Lambda(t)-\Lambda(s)} B(s)\left[-\lambda(s)\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases}$$

where $\Lambda(t)$ is a primitive of $\lambda(t)$ on $[0,\tau]$. It suffices that $\lambda \in L^1(-\tau, 0)$, thus allowing for singularities on the coefficient λ which give rise to very interesting interactions with the singularities of B.

We now generalize the results presented above to the "partially nonlinear" case, that is

$$\begin{cases} u'(t) = B'(t)g(t, u(t-\tau)), & 0 < t < \tau, \\ u(\theta) = \xi(\theta), & -\tau \leq \theta \leq 0, \end{cases}$$

where g is C^1. By formally writing

$$B'(t)g(t, u(t-\tau)) = \frac{d}{dt}\left[B(t)g(t, u(t-\tau))\right] - B(t)\frac{d}{dt}\left[g(t, u(t-\tau))\right],$$

we see that the equivalent neutral equation is completely similar to those obtained in the previous section, that is

$$\begin{cases} \frac{d}{dt}\left[u(t) - B(t)g(t, u(t-\tau))\right] = -B(t)\frac{d}{dt}\left[g(t, u(t-\tau))\right], & t > 0, \\ u(t) = \xi(t), & \tau \leq t \leq 0. \end{cases}$$

On $[0,\tau]$ we have (formally)

$$\begin{cases} u(t) = B(t)g(t,\xi(t-\tau)) + \xi(0) \\ \qquad\qquad - \int_0^t B(s)\dfrac{d}{ds}\left[g(s,\xi(s-\tau))\right]ds, \quad 0 \le t \le \tau. \end{cases}$$

But if $\xi \in W^{1,q}(-\tau,0)$ and g is C^1, $s \mapsto g(s,\xi(s-\tau))$ is also in $W^{1,q}(-\tau,0)$ and the integral actually is an absolutely continuous function. Therefore, the representation or "asymptotic expansion" $u(t) = B(t)g(t,\xi(t-\tau)) + AC$ is still valid.

If an additive term $\lambda u(t)$ appears in the right-hand side, a similar analysis can be performed by means of the change of variable $v(t) = e^{-\lambda t}u(t)$, although the nonlinearity $g(t,u(t-\tau))$ makes the integral representation much more complicated than the above expression.

The fully nonlinear case

Let us now analyze the "fully nonlinear" case, that is

$$\begin{cases} u'(t) = f(t, u(t)) + B'(t)g(t, u(t-\tau)), & 0 < t < \tau \\ u(\theta) = \xi(\theta), & -\tau \le \theta \le 0 \end{cases}$$

where f is C^2 and g is C^1. Its reduction to a "neutral form" is still possible:

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - B(t)g(t, u(t-\tau))\right] \\ \qquad = f(t, u(t)) - B(t)\dfrac{d}{dt}\left[g(t, u(t-\tau))\right], t > 0, \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0. \end{cases}$$

However, the presence of the term $f(t, u(t))$ makes the (formal) integration of both sides of the equation hard to deal with: instead of an explicit expression of u, it becomes an *integral equation* with u as the unknown, and it would be necessary to choose the right function space in which the equation not only made sense but had a unique fixed point as well. In any case, the neutral formulation can be used to give a precise meaning to the equation, but we will not follow this approach here.

Instead, we will change our strategy and make use of a very useful, but little-known mathematical device: *Alekseev's nonlinear variation of constants formula.* We now briefly recall this result in a very simple setting, which will suffice for our purposes. (Laksmikantham)

ALEKSEEV, V. M. (1961). An estimate for the perturbations of the solutions of ordinary differential equations (Russian). *Westnik Moskov Unn. Ser* 1 28–36.

V. Laksmikantham, S. Leela, Differential an Integral Inequalities, vol I., Academic Press, New York, 1969.

Proposition 6 (Alekseev's formula) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 Let $y = \phi(t, t_0, \xi)$ represent the unique solution of the ODE*

$$\begin{cases} y' = f(t, y(t)), \\ y(t_0) = \xi, \end{cases}$$

and let $\Phi(t, t_0, \xi) = \partial_\xi \phi(t, t_0, \xi)$, where ∂_ξ denotes partial differentiation. Then ϕ is C^2, Φ is C^1 and if $G : \mathbb{R} \to \mathbb{R}$ is L^1_{loc}, the solution $u(t)$ of the so-called "perturbed problem"

$$\begin{cases} u' = f(t, u(t)) + G(t), \\ u(t_0) = \xi, \end{cases}$$

has the integral representation

$$u(t) = y(t) + \int_{t_0}^{t} \Phi(t, s, y(s)) G(s) ds,$$

where $y(t) = \phi(t, t_0, \xi)$ is the "unperturbed" or "reference" solution.

Remark 8 *Alekseev's formula is usually stated under stronger regularity conditions on* G. *However, it is very simple to check by direct differentiation that the function* $u(t)$ *defined by*

$$u(t) = y(t) + \int_{t_0}^{t} \Phi(t, s, y(s))G(s)ds,$$

is an absolutely continuous solution of the (Carathéodory) equation

$$\begin{cases} u' = f(t, u(t)) + G(t), \\ u(t_0) = \xi, \end{cases}$$

Alekseev's formula is usually applied to the more ambitious setting of having G *depending on* t and u, *which is typical of control theory.* $u(t) = y(t) + \int_{t_0}^{t} \Phi(t, s, y(s))G(s)ds$, *then becomes an integral equation and a more delicate analysis is required.*

Fortunately, we can consider the retarded term as an external "forcing"

$$G(t) = B'(t)g(t, \xi(t - \tau)),$$

and by setting $t_0 = 0$, $\xi = u(0) = \xi(0)$, $y(t) = \phi(t, 0, \xi)$, write (formally):

$$u(t) = y(t) + \int_{0}^{t} \Phi(t, s, y(s))B'(s)g(s, \xi(s - \tau))ds,$$

and integrate by parts:

$$\begin{aligned} u(t) &= y(t) + [\Phi(t,s,y(s))B(s)g(s,\xi(s-\tau))]_{s=0}^{s=t} \\ &\quad - \int_0^t B(s)\frac{d}{ds}\left[\Phi(t,s,y(s))g(s,\xi(s-\tau))\right] ds \\ &= y(t) + \Phi(t,t,y(t))B(t)g(t,\xi(t-\tau)) \\ &\quad - \int_0^t B(s)\frac{d}{ds}\left[\Phi(t,s,y(s))g(s,\xi(s-\tau))\right] ds. \end{aligned}$$

On the other hand, as we saw before, for $\xi \in W^{1,q}(-\tau, 0)$ and $g \in C^1$ the composite function $s \mapsto g(s, \xi(s-\tau))$ is also $W^{1,q}(-\tau, 0)$ and so is its product by the C^1 function $\Phi(t, s, y(s))$. Therefore, its derivative belongs to $L^q(-\tau, 0)$ and the indefinite integral, as in all the previous cases, is an absolutely continuous function. This means that the integration by parts is legitimate and we may state the following result, which is an extension of the previous ones. We may summarize the previous comments in the following way:

The initial value problem

$$\begin{cases} u'(t) = f(t, u(t)) + B'(t)g(t, u(t-\tau)), \quad 0 < t < \tau, \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0, \end{cases}$$

with $f \in C^2(\mathbb{R}^2)$, $g \in C^1(\mathbb{R}^2)$ and initial function ξ in $W^{1,q}(-\tau, 0)$ can be given a precise integral sense in $[0, \tau]$ by means of the neutral equivalent equation

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - B(t)g(t, u(t-\tau))\right] \\ \qquad\qquad = f(t, u(t)) - B(t)\dfrac{d}{dt}\left[g(t, u(t-\tau))\right], t > 0, \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0. \end{cases}$$

and its unique solution u admits the integral representation

$$u(t) = y(t) + B(t)g(t, \xi(t-\tau)) - \int_0^t B(s)\frac{d}{ds}\left[\Phi(t, s, y(s))g(s, \xi(s-\tau))\right] ds,$$

(where $y(t) = \phi(t, 0, \xi(0))$) as well as the "asymptotic expansion"

$$u(t) = B(t)g(t, \xi(t-\tau)) + AC,$$

which gives the qualitative picture of the behavior of the solution near singularities of B.

1.10. Aplicación a una EDP lineal con retardo

In order to avoid technicalities, let us consider the delayed linear heat equation with Neumann boundary conditions

$$(P_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = B'(t)u(t-\tau, x), & \text{for } (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t,x) = 0, & \text{for } (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(\theta, x) = \xi(\theta, x), & \text{for } (\theta, x) \in (-\tau, 0) \times \Omega, \end{cases}$$

where Ω is a connected domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary, and concentrate on a simplified version of the single-singularity case

$$B(t) = \frac{a}{|t - t^*|^\alpha} + m(t),$$

with $m \in C^1([0,\tau])$, $\alpha \in (0,1)$ and $t^* \in (0,\tau)$. It is well known (Ha,Wu) that on $[0, t^*)$ the initial value problem is well defined for continuous initial functions ξ and has a unique solution. The possibility of extending the solution beyond t^* will be discussed later. Also, other function spaces and boundary conditions are easily treated by these methods.

K. S. Ha: Nonlinear Functional evolutions in Banach spaces, Kluwer, AA Dordrecht, 2003.

J. Wu, Theory and applications of functional partial differential equations, Springer Verlag, New York, 1996.

Separable solutions

Assume that the initial function is separable: $u(x,t) = \xi(t)\phi_0(x)$ for $t \in [-\tau, 0]$ and $x \in \Omega$. It is then natural to look for solutions of the same type $u = w(t)\phi(x)$, thus obtaining

$$w'(t)\phi(x) = w(t)\Delta\phi(x) + B'(t)w(t-\tau)\phi(x).$$

In order to have a separable solution we divide by $w(t)\phi(x)$ and observe that the assumed identity

$$\frac{w'(t)}{w(t)} = \frac{\Delta\phi(x)}{\phi(x)} + B'(t)\frac{w(t-\tau)}{w(t)},$$

can only hold if there exists a real constant λ such that

$$\Delta\phi = \lambda\phi,$$

(that is, ϕ is an *eigenfunction* of Δ with the given boundary conditions, with associated eigenvalue λ) and w satisfies the delay-differential equation

$$\begin{cases} w'(t) = \lambda w(t) + B'(t)w(t-\tau), & \text{for } t \geq 0, \\ w(\theta) = w_0(\theta), & \text{for } t \in [-\tau, 0], \end{cases}$$

which is of the type alreadystudied

This obviously requires that $\phi_0(x) = \phi(x)$ be already an eigenfunction. Assuming this is the case, we have an explicit representation of these separable solutions from the formula

$$\begin{cases} u(t) = e^{\lambda t}\xi(0) + B(t)\xi(t-\tau) \\ \qquad + \int_0^t e^{\lambda(t-s)} B(s)\left[-\lambda\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases}$$

$$\begin{aligned} u(t,x) &= w(t)\phi(x) \\ &= B(t)\xi(t-\tau)\phi(x) \\ &+ \phi(x)\int_0^t e^{\lambda(t-s)} B(s)\left[-\lambda\xi(s-\tau) + \xi'(s-\tau)\right] ds, \quad t \in [0,\tau]. \end{aligned}$$

If $\xi(t^* - t) > 0$ then $w(t) = B(t)\xi(t-\tau) \to \infty$ as $t \to t^*$, and the same will happen for

the separable solution on the region $\{\phi > 0\}$, while $u(t,x) = w(t)\phi(x) \to -\infty$ as $t \to t^*$ when $\phi(x) < 0$. Clearly, the opposite behavior takes place when $\xi(t^* - t) < 0$. In any case, we have *single instant time blow-up* outside the nodal region $\{x \in \Omega : \phi(x) = 0\}$, meaning that the *explosion time* is the same for all the points involved.and uniformly in $\mathbf{x}$ The most important case from the practical viewpoint is that of $\phi(x) \equiv 1$, the first eigenfunction of Δ with Neumann boundary conditions.

Some linear and nonlinear delay-PDEs with blowing-up solutions via comparison arguments

We consider now the sign condition $B'(t) \geq 0$ on $[0, t^*)$, whose importance comes from the fact that some *comparison arguments* can be applied in this case, thus enlarging considerably the set of equations for which we get blowing-up solutions. Although our arguments also apply to the case of $(NLDDE)$ here we merely state a simple version of more general results for (NLP_N), which will be enough for our purposes

Proposition 9 *For $i = 1, 2$, consider the delayed reaction-diffusion equations*

$$(NLP_N) \quad \begin{cases} \frac{\partial u^i}{\partial t} - \Delta u^i = f^i(t, u^i(t,x), u^i(t-\tau, x)), & (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial u^i}{\partial n}(t,x) = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u^i(\theta, x) = \xi^i(\theta, x), & (\theta, x) \in (-\tau, 0) \times \Omega, \end{cases}$$

where f^i are locally Lipschitz in all its arguments and nondecreasing in its third variable, i.e.

$$p^1 \leq p^2 \Longrightarrow f^1(t, u^1, p^1) \leq f^2(t, u^2, p^2),$$
$$\text{for a.e. } t \geq 0, \text{ for any } u^i, p^i \in \mathbb{R}.$$

Let ξ^1 and ξ^2 be two initial functions, in $C([-\tau, 0] : L^p(\Omega))$ for some $p \in [1, +\infty]$, ordered as follows:

$$0 \leq \xi^1(\theta, x) \leq \xi^2(\theta, x), \quad \text{for any } \theta \in [-\tau, 0] \text{ and a.e. } x \in \Omega.$$

Then there exists the corresponding weak solutions $u^1(t,x)$, $u^2(t,x)$, in $C([-\tau, T^i_{\max}) : L^p(\Omega))$, for some $T^i_{\max} \in (0, +\infty]$, and they satisfy

$$0 \leq u^1(t,x) \leq u^2(t,x), \quad \text{for all } t \in \left[-\tau, T^i_{\max}\right), \text{ a.e. } x \in \Omega.$$

Proof. The existence of solutions is consequence of well-known results Most of the comparison results in the indicated literature are presented for the simper case in which $f^1 \equiv f^2$ (see also other general references The case of

$$p^1 \leq p^2 \Longrightarrow f^1(t, u^1, p^1) \leq f^2(t, u^2, p^2),$$
$$\text{for a.e. } t \geq 0, \text{ for any } u^i, p^i \in \mathbb{R}.$$

with $f^1 \neq f^2$ is well known in the literature without delay (Diaz) and can be easily adapted to the case of delayed equations. □

Other boundary conditions are possible. The condition to be taken into account is that $-\Delta$ with the given boundary condition generates a *positive semigroup* on the usual function spaces.

Applying this result to our case, we have the following

Theorem 10 *Assume that the initial function satisfies*

$$\xi(\theta, x) \geq \xi_0(\theta)\phi(x), \quad \textit{for } \theta \in [-\tau, 0],\ x \in \Omega,$$

where ϕ *is an eigenfunction of* Δ *with the Neumann boundary condition,* $\xi_0 \in W^{1,q}(-\tau, 0)$ *and* $\xi_0(t^* - \tau) > 0$. *Assume*

$$B(t) = \frac{a}{|t - t^*|^\alpha} + m(t),$$

and let $f(t, u, p)$ *be locally Lipschitz in all its arguments, nondecreasing in its third variable, and such that*

$$p^1 \leq p^2 \Longrightarrow B'(t)p^1 \leq f(t, u, p^2),$$
$$\textit{for a.e. } t \geq 0, \textit{ for any } u, p^1, p^2 \in \mathbb{R}.$$

Then, if $u(x, t)$ *is the solution of*

$$(NLP_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(t, u(t, x), u(t - \tau, x)), & (t, x) \in (0, +\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t, x) = 0, & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(\theta, x) = \xi(\theta, x), & (\theta, x) \in (-\tau, 0) \times \Omega, \end{cases}$$

we have

$$u(x, t) \geq \left[\frac{a}{|t - t^*|^\alpha}\xi_0(t - \tau) + n(t)\right]\phi(x),$$

where n *is an absolutely continuous function on* $[-\tau, 0]$. *In particular,* u *blows up at some finite time* $T_{\max} \leq t^*$ *in the sense that*

$$\lim_{t \to t^*} u(t, x) = \infty, \quad a.e.\, x \in \{x \in \Omega : \phi(x) > 0\}.$$

Proof. Let $w(t)$ denote the solution of the initial value problem

$$\begin{cases} w'(t) = \lambda w(t) + B'(t)w(t-\tau), & \text{for } t \geq 0, \\ w(\theta) = \xi_0(\theta), & \text{for } \theta \in [-\tau, 0], \end{cases}$$

where λ is the eigenvalue associated to the eigenfunction ϕ. As before, the hypotheses on ξ_0 imply that $w(t)$ admits the asymptotic expansion

$$w(t) = B(t)\xi(t-\tau) + n(t),$$

where $n(t)$ is absolutely continuous. On the other hand, the comparison result stated above implies that $u(t,x) \geq w(t)\phi(x)$, and the result is proved. ■

Remark 11 *The theorem holds for the Dirichlet boundary condition without any change. For (possibly nonlinear) Robin boundary condition $\partial u/\partial n + k(t,x,u) = 0$, for some nondecreasing function $k(.,.,u)$ of u (a requirement imposed for the applicability of comparison arguments (Díaz).*

] J. I. Díaz, Nonlinear Partial Differential Equations and Free Boundaries. Vol.I. Elliptic equations.
Research Notes in Mathematics n^o 106, Pitman, London, 1985.

Remark 12 *For both Dirichlet and Neumann boundary conditions, if the initial function satisfies $\xi(t,x) \geq \mu > 0$ in Ω, we can always choose ϕ to be the* first eigenfunction, *which does not change sign by the Krein-Milman theorem. We have then* instantaneous blow-up on the whole domain Ω.

Remark 13 *In the region $\{x \in \Omega : \phi(x) < 0\}$ the comparison argument does not give us any useful information, unless some symmetric condition $u(\theta,x) \leq \tilde{\xi}_0(\theta)\phi(x)$, $\tilde{\xi}_0(t^* - \tau) < 0$ holds.*

1.12 Continuación más allá de la explosión

The question of existence of solutions on the whole interval $[0, \tau]$ is more delicate since it involves performing some kind of integration by parts in order to define a suitable notion of generalized solution. The special structure of the right hand side of our equation, however, simplifies the situation, since the method of steps is directly applicable. Using the notation of abstract evolution equations in Banach spaces X, our basic equation is written as follows

$$\begin{cases} u'(t) = Au(t) + B'(t)u(t-\tau), & \text{in } X, \text{ for } t \geq 0, \\ u(\theta) = \xi(\theta), & \text{for } \theta \in [-\tau, 0], \end{cases}$$

where $u(t)$ is the function $u(t)(x) = u(t, x)$, the same for ξ and A is the abstract operator

on X associated to $-\Delta$ with Neumann boundary conditions. On the basic interval $[0, \tau]$ we may express the solution by means of the variation of constants formula:

$$u(t) = e^{At}\xi(0) + \int_0^t e^{A(t-s)}B'(s)\xi(s-\tau)ds, \quad 0 \leq t \leq \tau.$$

Assume, again,

$$B(t) = \frac{a}{|t-t^*|^\alpha} + m(t),$$

. The fact that B' is not a function (and so $u' \notin L^1(0,\tau : X)$) requires integration by parts. By proceeding formally we arrive to a direct extension of equation

$$\begin{cases} u(t) = e^{\lambda t}\xi(0) + B(t)\xi(t-\tau) \\ \qquad + \int_0^t e^{\lambda(t-s)} B(s) \left[-\lambda\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases}$$

by substituting λ by A :

$$\begin{cases} u(t) = e^{At}\xi(0) + B(t)\xi(t-\tau) \\ \qquad + \int_0^t e^{A(t-s)} B(s) \left[-A\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases}$$

which we may use as definition of "generalized solution in $W^{-1,p'}(0,\tau : X)$" for some

$p = p(\alpha) > 1$ small enough. As an illustration, let us state a simple sufficient condition for a "generalized solution in $W^{-1,p'}(0,\tau : L^2(\Omega))$" to exist:

Theorem 14 *Let* $\xi \in C^2([-\tau, 0] \times \bar{\Omega})$ *satisfying* $\partial\xi/\partial n = 0$ *on* $\partial\Omega$ *for all* $\theta \in [-\tau, 0]$. *Assume*

$$B(t) = \frac{a}{|t - t^*|^\alpha} + m(t),$$

Then the integral in

$$\begin{cases} u(t) = e^{At}\xi(0) + B(t)\xi(t-\tau) \\ \qquad + \int_0^t e^{A(t-s)} B(s) \left[-A\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases}$$

is well defined and the equation

$$\begin{cases} u'(t) = Au(t) + B'(t)u(t-\tau), \quad \text{in } X, \text{ for } t \geq 0, \\ u(\theta) = \xi(\theta), \quad \text{for } \theta \in [-\tau, 0], \end{cases}$$

has a "generalized solution in $W^{-1,p'}(0, \tau : L^2(\Omega))$*" for some* $p = p(\alpha) > 1$ *small enough, and, so defined, at least, on* $[0, \tau]$.

Proof. The hypotheses imply that $\xi(t, .)$ belongs to the domain of A and the function $s \mapsto \Delta\xi(s-\tau, \cdot) + \partial_t\xi(s-\tau, \cdot)$ is continuous from $[0, \tau]$ into $C(\bar{\Omega})$. Therefore, its product by the L^2 function B is in L^2, and the integral is well defined.. ■

2. Control de la explosión en EDO no lineales sin retardo (Orlando 2014, 48-62)

- Casal, A.C., Díaz, J.I. and Vegas, J.M.: Controlled explosions of blowing-up trajectories in semilinear problems and a nonlinear variation of constant formula, XXIII Congreso de Ecuaciones Diferenciales y Aplicaciones, XIII Congreso de Matemática Aplicada, Castellón, 9-13 septiembre 2013. e-Proccedings.
- Casal, A.C., Díaz, J.I. and Vegas, J.M.: Complete recuperation after the blow up time for semilinear problems, AIMS Procceding 2015, 223-229. (2015).

2.1. Introducción

We consider blowing-up solutions $y^0(t)$, $t \in [0, T_{y_0})$, of some ODEs

$$P(f, y_0) = \begin{cases} \frac{dy}{dt}(t) = f(y(t)) & \text{in } \mathbb{R}^d, \\ y(0) = y_0, \end{cases}$$

- $d \geq 1$
- $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz function superlinear near the infinity

$$f(y)y \geq C\,|y|^{p+1} \text{ if } |y| > k, \text{ for some } p > 1 \text{ and } C, k > 0.$$

- (i.e. there is a complete blow-up in the norm of $y(t)$ after T_{y_0})
- In Control Theory, (as in Díaz, Fursikov, (1994)) it can be seen how to avoid the blow-up phenomenon:

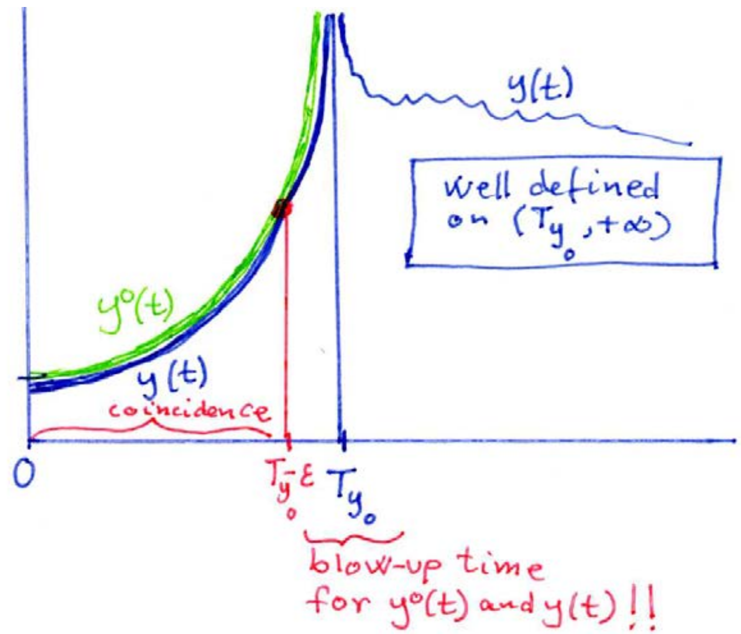
$$P(f, y_0, u) = \begin{cases} \frac{dy}{dt}(t) = f(y(t)) + u(t) & \text{in } \mathbb{R}^d, \\ y(0) = y_0, \end{cases}$$

No delay in the statement !!

- with a suitable control $u \in L^1_{loc}(0, +\infty : \mathbb{R}^d)$
- for any small enough $\epsilon > 0$ there exists a continuous deformation $y(t)$ of $y^0(t)$, as solution of the control perturbed problem
- defined on the whole interval $[0, +\infty)$
- such that $y(t) = y^0(t)$ for any $t \in [0, T_{y_0} - \epsilon]$

J. I. Díaz and A. V. Fursikov, A simple proof of the approximate controllability from the interior for nonlinear evolution problems. *Applied Mathematics Letters* **7**(5) (1994), 85–87.

Definition. *We say that the trajectory* $y^0(t)$ *of problem* $P(f, y_0)$, *with blow-up time* T_{u^0}, *has a controllable explosion if for any small enough* $\epsilon > 0$ *we can find a continuous deformation,* $y(t)$, *of the trajectory* $y^0(t)$, *built as solution of the control perturbed problem* $P(f, y_0, u)$, *for a suitable control* $u \in W_{loc}^{-1,q'}(0, +\infty : \mathbb{R}^d)$ [the dual space of $W_{0,loc}^{1,q}(0, +\infty : \mathbb{R}^d)$], *for some* $q > 1$, *such that* $y(t) = y^0(t)$ *for any* $t \in [0, T_{y_0} - \epsilon]$, $y(t)$ *also blows-up at* $t = T_{y_0}$ (the controlled explosion) *but* $y(t)$ *can be extended beyond* T_{y_0} *as a function* $y \in L^1_{loc}(0, +\infty : \mathbb{R}^d)$.



The controled solution is a less explosive function than the original one !!

Theorem 1. *Assume f locally Lipschitz continuous and superlinear. Then, for any $y_0 \in \mathbb{R}^d$ the blowing up trajectory $y^0(t)$ of the associated problem $P(f : y_0)$ has a controlled explosion by means of the control problem $P(f, y_0, u)$.* For the proof, we use two important tools

- A suitable delayed feedback problems (in the spirit of Casal, Díaz, Vegas, (2009))

- A powerful *nonlinear variation of constants formula*

Proposition. [Alekseev's formula] *Let* $f : \mathbb{R} \to \mathbb{R}$ *be* C^2*. Let* $y = \phi(t, t_0, \xi)$ *represent the unique solution of the ODE*

$$\begin{cases} y' = f(y(t)), \\ y(t_0) = \xi, \end{cases} \tag{1}$$

and let $\Phi(t, t_0, \xi) = \partial_\xi \phi(t, t_0, \xi)$*, where* ∂_ξ *denotes partial differentiation. Then* ϕ *is* C^2, Φ *is* C^1*, and for any* $G : \mathbb{R} \to \mathbb{R}$ *in* L^1_{loc}*, the solution* $z(t)$ *of the so-called "perturbed problem"*

$$\begin{cases} z' = f(z(t)) + G(t), \\ z(t_0) = \xi, \end{cases} \tag{2}$$

has the integral representation

$$z(t) = y(t) + \int_{t_0}^{t} \Phi(t, s, z(s))G(s)ds, \tag{3}$$

where $y(t) = \phi(t, t_0, \xi)$ *is the "unperturbed" or "reference" solution.*

Remark 1. Notice that $\Phi(t, t_0, \xi)$ satisfies $\Phi(t, t, \xi) = 1$. Notice also that Alekseev's formula is usually stated under stronger regularity conditions on G, and for $d \geq 1$.

- This type of formula was first established for nonlinear terms of class C^2 (Alekseev, (1961), Laksmikantham, Leela, (1969))
- We show that the formula holds also for Lipschitz functions f (first assumed globally Lipschitz)

For instance, given such a f and

- a family of maximal monotone operators $\beta(t,y)$, on $H=\mathbb{R}^d$,
- $\beta(t,\cdot)\in L^1_{loc}(0,+\infty:\mathbb{R}^d)$,

and the perturbed problem

$$P^*(f,\beta,\xi)=\begin{cases}\frac{dy}{dt}(t)\in f(y(t))+\beta(t,y(t)), \text{ in } \mathbb{R}^d,\\ y(t_0)=\xi.\end{cases} \tag{4}$$

if f is globally Lipschitz function, the solutions of $P^*(f,\beta,\xi)$ are well defined, as absolutely continuous functions on $[0,T]$, for any given $T>0$ (see Brezis, Operateurs maximaux monotones...).

H. Brezis, *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematical Studies, Amsterdam, 1973.

- Let $y^0(t)=\phi(t,t_0,\xi)$, the unique solution of the ODE

$$P^*(f,0,\xi)=\begin{cases}y'(t)=f(y(t)) \text{ in } \mathbb{R}^d,\\ y(t_0)=\xi.\end{cases} \tag{5}$$

- Call $\Phi(t,t_0,\xi)=\partial_\xi\phi(t,t_0,\xi)$. We prove:

Theorem 2. *The flow map ϕ is Lipschitz continuous, Φ is absolutely continuous and the solution $y(t)$ of the "perturbed problem" $P^*(f, \beta, \xi)$ has the integral representation*

$$y(t) = y^0(t) + \int_{t_0}^{t} \Phi(t, s, y(s))\beta(s, y(s))ds, \text{ for any } t \in [0, T], \tag{6}$$

where $y^0(t) = \phi(t, t_0, \xi)$ is the solution of the "unperturbed" problem $P^*(f, 0, \xi)$.

2.2 Explosiones controladas para EDO con términos superlineales

1. $f \in C^2$ **and superlinear (e.g.** $f(y) = |y|^{p-1} y$ **with** $p > 1$**).**

Assume, for simplicity, $d = 1$.

Theorem 3. *Assume* $f \in C^2$ *and superlinear. Then, for any* $y_0 \in \mathbb{R}^d$ *the blowing up trajectory* $y^0(t)$ *of the associated problem* $P(f : y_0)$ *has a controlled explosion.*

Proof. Step 1 (the strategy). Define $\tau = T_{y_0} - \epsilon$, make $\widetilde{t} = t - \tau$ and consider the delayed problem

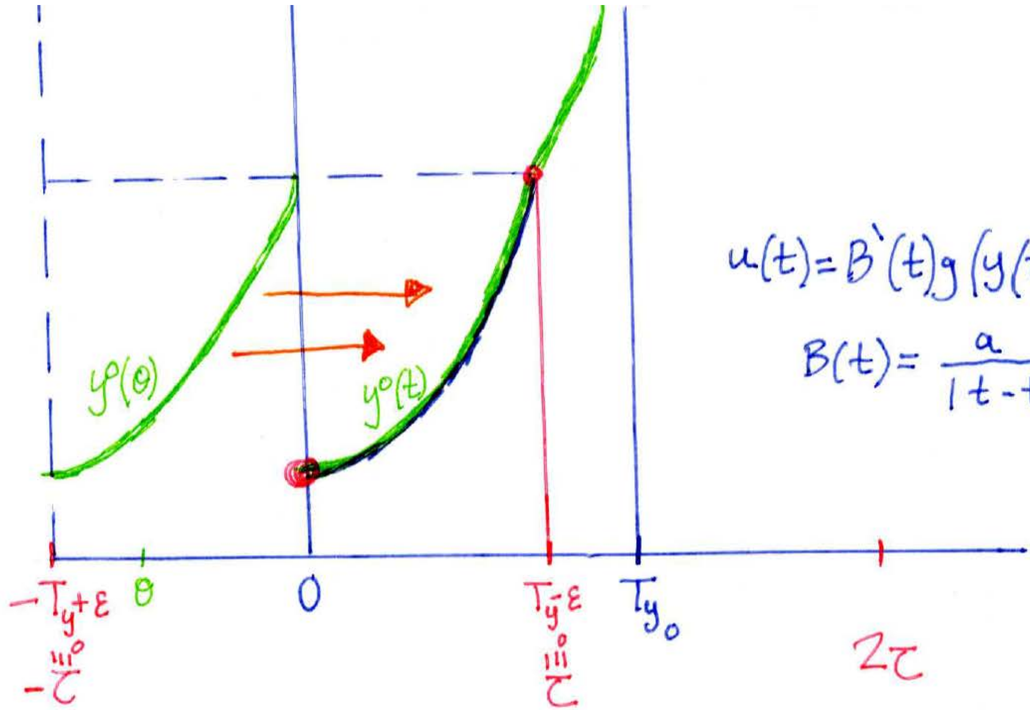
$$\widetilde{P}(f, y^0, B) = \begin{cases} y'(t) = f(y(t)) + B'(t)g(y(t-\tau)), & 0 < t < \tau \\ y(\theta) = y^0(\theta), & -\tau \leq \theta \leq 0 \end{cases} \tag{7}$$

where, for simplicity we denote again $\widetilde{t}$ by t, so that, for any $-\tau \leq \theta \leq 0$ we are identifying $y^0(\theta)$ with $y^0(\theta + T_{y_0} - \epsilon)$, for some suitable functions $B(t)$ and where $g(r)$ is any C^2 function (for instance $g(r) = r$).

- Our goal is to show that we can chose the control term $u(t) := B'(t)g(y(t-\tau))$ such that
- $u \in W^{-1,q'}(0, \tau : \mathbb{R}^d)$.
- the solution of $\widetilde{P}(f, y^0, B)$ is defined and integrable on the whole interval $[0, \tau)$

Since $y(t-\tau) = y^0(t-\tau)$ for any $t \in [0, T_{y_0} - \epsilon]$, this will prove the result by iteration on the intervals $\tau < t < 2\tau, ..., n\tau < t < (n+1)\tau$, $n \in \mathbb{N}$.

Escala inicial del tiempo



$$u(t) = B'(t)\, g\big(y(t-\tau)\big)$$

$$B(t) = \frac{a}{|t-t^*|^{\alpha}} + m(t), \quad \alpha \in \left(0, \tfrac{1}{q}\right),\ q>1, \quad a>0$$

$$t^* = \varepsilon \in \big(0, \tau \equiv (T_{y_0} - \varepsilon)\big).$$

(en la nueva escala del tiempo renombrada de nuevo como en la antigua escala: t)

Problema con retardo en una nueva escala del tiempo

Step 2 (choice of function B and reformulation as neutral equation). Given $q > 1$, $a > 0$ and $\alpha \in (0, \frac{1}{q})$ and a continuous function m (to be taken, for instance, in order to have $B(0) = 0$) we define

$$B(t) = \frac{a}{|t - t^*|^\alpha} + m(t), \quad t \in [0, \tau], \tag{8}$$

with $t^* = \epsilon$ (i.e. $t = T_{y^0}$ in the original time scale). We assume that $t^* \in (0, \tau)$, i.e. $2\epsilon < T_{y^0}$. We can reformulate $\widetilde{P}(f, y^0, B)$ as the neutral problem

$$\begin{cases} \frac{d}{dt}\left[y(t) - B(t)g(y(t-\tau))\right] \\ \qquad\qquad = f(y(t)) - B(t)\frac{d}{dt}\left[g(y(t-\tau))\right], t > 0, \\ y(\theta) = y^0(\theta), \quad -\tau \le \theta \le 0 \end{cases} \tag{9}$$

We will change our strategy and apply the *Alekseev's nonlinear variation of constants formula* (Alekseev, (1961)).

Continuation of Step 2 We can consider the retarded term as an external "forcing"

$$G(t) = B'(t)g(\xi(t-\tau)), \tag{10}$$

and by setting $t_0 = 0$, $\xi = z(0) = y^0(0)$, $y(t) = \phi(t, 0, \xi)$, we can write (formally):

$$z(t) = y(t) + \int_0^t \Phi(t, s, z(s))B'(s)g(y^0(s-\tau))ds, \tag{11}$$

and integrate by parts:

$$\begin{aligned} z(t) &= y(t) + \left[\Phi(t, s, z(s))B(s)g(y^0(s-\tau))\right]_{s=0}^{s=t} \\ &\quad - \textstyle\int_0^t B(s)\frac{d}{ds}\left[\Phi(t, s, z(s))g(y^0(s-\tau))\right] ds \\ &= y(t) + \Phi(t, t, z(t))B(t)g(y^0(t-\tau)) \\ &\quad - \textstyle\int_0^t B(s)\frac{d}{ds}\left[\Phi(t, s, z(s))g(y^0(s-\tau))\right] ds. \end{aligned} \tag{12}$$

By the remark above, $\Phi(t,t,z(t)) = 1$. On the other hand, as we saw before, for $y^0 \in W^{1,q}(-\tau,0)$ and $g \in C^1$ the composite function $s \mapsto g(y^0(s-\tau))$ is also $W^{1,q}(-\tau,0)$ and so is its product by the C^1 function $\Phi(t,s,z(s))$. Therefore, its derivative belongs to $L^q(-\tau,0)$ and the indefinite integral, as in all the previous cases, is an absolutely continuous function. This means that the integration by parts is legitimate and we may state the following result, which is an extension of the previous ones. Summarizing:

- The initial value problem

$$\widetilde{P}(f,y^0,B) = \begin{cases} y'(t) = f(y(t)) + B'(t)g(y(t-\tau)), & 0 < t < \tau \\ y(\theta) = y^0(\theta), & -\tau \le \theta \le 0 \end{cases} \tag{13}$$

with $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$ and initial function y^0 in $W^{1,q}(-\tau,0)$ has a precise integral sense in $[0,\tau]$ by means of the neutral equivalent equation.

- Its unique solution z admits the integral representation

$$z(t) = y(t) + B(t)g(y^0(t-\tau)) - \int_0^t B(s)\frac{d}{ds}\left[\Phi(t,s,z(s))g(y^0(s-\tau))\right] ds, \tag{14}$$

(where $y(t) = \phi(t,0,y^0(0))$).

Then, for every $\xi \in W^{1,r}(0,\tau)$ (where $1/q+1/r=1$) the neutral Cauchy problem has a unique solution given by the identity (14). Therefore $z \in L^q(0,\tau)$ and $z(t) - B(t)g(y^0(t-\tau))$ is an absolutely continuous function and we may write symbolically

$$z(t) = B(t)g(y^0(t-\tau)) + AC \tag{15}$$

where "AC" means "an absolutely continuous function". As a consequence, the singularities of the solution on $[0,\tau]$ are also singularities of B. Thus, in particular, let $t^* = \epsilon$ (notice that $t^* = T_{y^0}$ in the original scale of time), $0 < \alpha < 1$, let m be continuous on $[0,\tau]$ and let

$$B(t) = \frac{a}{|t-t^*|^\alpha} + m(t), \tag{16}$$

Since the initial function y^0 satisfies $y^0(t^*-\tau) = y^0(\epsilon) \neq 0$, then t^* is also a singularity of z (the controlled explosion) and

$$z(t) \simeq \frac{a}{|t-t^*|^\alpha} g(y^0(\epsilon)), \quad \text{as } t \to t^*, \tag{17}$$

It blows up
but now it is
integrable !!

is an asymptotic expansion of z near $t^* = T_{y_0}$, which gives the qualitative picture of the behavior of the solution near singularities of B. Obviously, from the choice of α we get that the control $u(t) := B'(t)g(y(t-\tau))$ is in $W^{-1,q'}(0,\tau:\mathbb{R}^d))$, for any function $g \in C^1$.

Example. The proof of Theorem 1 is constructive and so, if we consider a special $P(f, y_0)$ case, as, for instance, the one corresponding to $f(y) = y^3$ and $y_0 = 1$ then we can identify easely the associate *control problem* $P(f, y_0, u)$. Ideed, in this case,

$$\phi(t, t_0, \xi) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\frac{1}{2\xi^2} - (t - t_0)}},$$

and $T_{y^0} = 1/2$. Thus we can take, e.g., $\epsilon = 1/8$ (so that $2\epsilon < T_{y^0}$), $\tau = T_{y_0} - \epsilon = 3/8$, $\alpha = 1/5$, $a = 1$, $g(s) = s$, $B'(t) = -(1/5)sign(t - 1/2)/\left|t - 1/2\right|^{6/5}$ and thus the searched control $u(t)$ is given by $u(t) = B'(t)y(t - 6/8)$ (for $t > 0$) with y solution of the problem

$$\begin{cases} y'(\tilde{t}) = y(\tilde{t})^3 - \frac{sign(\tilde{t}-1/8)}{5\left|\tilde{t}-\frac{1}{8}\right|^{6/5}}(y(\tilde{t} - 3/8)), & 0 < \tilde{t} < \tau \\ y(\theta) = y^0(\theta), & -3/8 \leq \theta \leq 0 \end{cases} \tag{18}$$

where $y^0(\theta) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\frac{1}{2} - (\theta + \frac{3}{8})}}$ if $\theta \in [-\frac{3}{8}, 0]$.

Moderate initial history

Monstrous delayed control !!!

- The proof of Theorem 1 is exactly the same as that of Theorem 3, once Theorem 2 is proved.

- Since we need only a control of the growing of the solution near the blow-up time T_{y_0}, it is enough to prove only for globally Lipschitz functions f.

- It can be extended easily for $d > 1$.

Proof of Theorem 2. Let $f_n \in C^1(\mathbb{R}^d : \mathbb{R}^d)$ be a sequence approximating f in $W^{1,s}(\mathbb{R}^d : \mathbb{R}^d)$, for any $s \in [1, +\infty)$, and such that

$$\|\partial_x f_n(\cdot)\|_{L^\infty(\mathbb{R}^d : \mathcal{M}_{d\times d})} \leq \|\partial_x f(\cdot)\|_{L^\infty(\mathbb{R}^d : \mathcal{M}_{d\times d})} := M \text{ for any } n \in \mathbb{N} \tag{18}$$

(see, Adams, Sobolev Spaces, Academic Press, 1975).

- Let $y_n^0 = \phi_n(t, t_0, \xi)$ be the unique solution of the unperturbed ODE

$$P^*(f_n, 0, \xi) = \begin{cases} y'(t) = f_n(y(t)) \quad \text{in } \mathbb{R}^d, \\ y(t_0) = \xi, \end{cases} \tag{19}$$

and let $\Phi_n(t, t_0, \xi) = \partial_\xi \phi_n(t, t_0, \xi),$. Let us consider the sequence of perturbed problems

$$P^*(f_n, \beta, \xi) = \begin{cases} \frac{dy_n}{dt}(t) \in f_n(y_n(t)) + \beta(t, y_n(t)), \text{ in } \mathbb{R}^d, \\ y(t_0) = \xi. \end{cases} \tag{20}$$

Then, by the classical version of the Alekseev formula (also valid for $d \geq 1$) we know that

$$y_n(t) = y_n^0(t) + \int_{t_0}^t \Phi_n(t, s, y_n(s))\beta(s, y_n(s))ds, \text{ for any } t \in [0, T], \tag{21}$$

- Since $f_n \to f$ and f is locally Lipschitz we know that $y_n^0(\cdot) \to y^0(\cdot)$ and $y_n(\cdot) \to y(\cdot)$ strongly in $AC([0,T] : \mathbb{R}^d)$ for any fixed $T > 0$ (Theorem 4.2 of Brezis book).
- Since any maximal monotone operator is strongly-weakly closed, at least, $\beta(\cdot, y_n(\cdot)) \rightharpoonup \beta(\cdot, y(\cdot))$ in $L^2(0, T : \mathbb{R}^d)$.
- From the classical Peano theorem, there exists a $\Phi(t, s, y)$ such that

$$\Phi_n(t, \cdot, y_n(\cdot)) \to \Phi(t, \cdot, y(\cdot)), \text{ for a.e. } t \in (0, T),$$

strongly in $L^2(0, T : \mathcal{M}_{d \times d})$.

- $\Phi_n(t, t_0, \xi)$ is the solution of the problem

$$\begin{cases} \Phi'(t) = H_n(t, t_0, \xi)\Phi(t) & \text{in } \mathcal{M}_{d\times d}, \\ \Phi(t_0) = I, \end{cases}$$

where

$$H_n(t, t_0, \xi) = \partial_x f_n(\phi_n(t, t_0, \xi)).$$

- Since M is given by (18)

$$\|H_n(t, t_0, \xi)\|_{L^\infty(t_0,T:\mathcal{M}_{d\times d})} \leq M \quad \text{for any } t_0 \in (0, T) \text{ and for any } \xi \in \mathbb{R}^d.$$

- Thus, by Gronwall inequality, there exists a positive constant $\widetilde{M} = \widetilde{M}(t_0, \xi)$ such that

$$\|\Phi_n(\cdot, t_0, \xi)\|_{W^{1,\infty}(0,T)} \leq \widetilde{M}$$

which implies that there exists a Lipschitz function $\Phi(t, s, \xi)$ such that $\Phi_n(t, \cdot, y_n(\cdot)) \rightharpoonup \Phi(t, \cdot, y(\cdot))$ in $W^{1q}(0, T : \mathcal{M}_{d\times d})$ for any $q \in (1, \infty)$. This leads to the strong convergence in $L^2(0, T : \mathcal{M}_{d\times d})$. Then we can pass to the limit in formula (21) and get that

$$y(t) = y^0(t) + \int_{t_0}^{t} \Phi(t, s, y(s))\beta(s, y(s))ds, \text{ for any } t \in [0, T].$$

Remark 2. Notice that since our main interest is to study the asymptotic, near T_{y_0}, we do not need to identify the limit matricial function $\Phi(t, s, y)$. This is a complicated task over the set of points $y \in \mathbb{R}^d$ where f is not Frechet differentiable in y (see a nonlinear characterization in Mirica, On differentiability with respect to initial data in the theory of differential equations, *Rev. Roumaine des Math. Pures Appl., (2003)*).

Remark 3. Several applications to the case of the some nonlinear blowing-up parabolic problems of the type

$$(P_N) \quad \begin{cases} \frac{\partial y}{\partial t} - \Delta y = |y|^{p-1}\, y + u(t,x) & \text{for } (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial y}{\partial n}(t,x) = 0, & \text{for } (t,x) \in (0,+\infty) \times \partial\Omega, \\ y(0,x) = y_0(x), & \text{for } x \in \Omega, \end{cases} \tag{22}$$

once we assume $p > 1$, for suitable conditions on $y_0 \in L^2(\Omega)$ and for an appropriate choice of the control function (taken as a suitable delayed feedback control) can be given in a similar way to the results presented in Casal, Díaz, Vegas, *Dynam. Systems Appl. 18 (2009)*. By limitations in the length of this work, those results will be given elsewhere.

3. Aplicación a EDPs estacionarias con condiciones dinámicas
(Post-Athenas II: Rakotoson 60, 2017, 63-76)

Controlled explosions: dynamics after blow-up time for semilinear problems with a dynamic boundary condition

A.C. Casal, G. Díaz, J.I. Díaz, J.M. Vegas

3.1. Introducción

This talk collects some results by the authors presented at the AIMS 2016 (Orlando, USA), a detailed manuscript [CDDV2018] will be submitted to publication in a few days.
It is well known that in many nonlinear dynamical problems the maximal horizon $T_\infty < \infty$ of solutions is defined trough the blow-up time $\|u(\cdot, T_\infty)\|_{L^\infty} = \infty$.

Nevertheless, in the case of some ordinary differential equations it is possible to control the horizon time in such way that the solution left well defined after the blow-up time T_∞ (see [CDV2015] Casal, A.C., Díaz, J.I. and Vegas, J.M.). The main goal of this contribution is to extend such control process to some illustrative semilinear boundary value problem as

$$\begin{cases} -\Delta u(x,t) + u(x,t)^m = 0, & x \in \mathbf{B}_R(0) \times]0,\infty[, \\ \dfrac{\partial u}{\partial t}(x,t) + \dfrac{\partial u}{\partial \mathbf{n}}(x,t) = u(x,t)^p + \alpha(t), & (x,t) \in \partial \mathbf{B}_R(0) \times]0,\infty[, \\ u(x,0) = u_0(x), & x \in \mathbf{B}_R(x_0), \end{cases}$$

where $\mathbf{n}$ is the unit outer normal and the constants positive m and p satisfies

$$p > \frac{m+1}{2} > 1.$$

An important motivation comes of the study of the dynamics of the concentration of lithium in porous electrodes.

Some more details about related previous results:

It is well know than one of the more relevant qualitative behaviors of nonlinear evolution problems is the possibility to get the finite time blow-up of the L^∞-norm of the solution. Without any aims to be exhaustive, we mention as general references are the books:

- Bebernes, J. and Eberly,D.: Mathematical Problems from Combustion Theory, Springer, New York, 1989,
- Hu, B., Blow-up Theories for Semilinear Parabolic Equations, Lecture Notes in Mathematics 2018, Springer-Verlag, Berlin, 2011,
- Quittner, P. and Souplet, P.: Superlinear Parabolic Problems, Birkhauser, Berlin, 2007,
- Samarskii, A. A., Galaktionov, V. A., Kurdyumov, S. P., and Mikhailov, A. P.: Blow-Up in Quasilinear Parabolic Equations ,Walter de Grueter, Berlin, 1995,
- Straughan, B.: Explosive Instabilities in Mechanics, Springer, Berlin, 1998,
- Zel'dovich, Ya. B., Barenblatt, G. I., Librovich, V. B., and Makhviladze, G. M.: The Mathematical Theory of Combustion and Explosions, Donald H. McNeill, trans., Consultants Bureau (Plenum), New York-London, 1985,

as well as the surveys

- Brezis, H., Cazenave, Th., Martel, Y., and Ramiandrisoa, A.: Blow up for $u_t - \Delta u = g(u)$ revisited. Adv. Differ. Equat., 1, 73-90 (1996).
- Galaktionov, V.A., and Vázquez, J.L.: The problem of blow-up in nonlinear parabolic equations, Discrete Contin. Dyn. Syst., 8, 2, 399-433 (2002).

In this paper we are specially interested in conditions on the involved nonlinear terms which allow to ensure that the solution can be continued beyond the finite time blow-up of the L^∞-norm of the solution.

This corresponds to the absence of the most usual case in which the so called *complete blow-up phenomenon* holds (see,.e.g. Baras, P., and Cohen, L.: Complete Blow-Up after T_{max} for the Solution of a Semilinear Heat Equation, J. Funct. Anal. 71, 142-174 (1987)).

This philosophy was initiated in the previous paper by some the authors of this paper (now also in collaboration with G. Díaz) dealing with some ordinary and partial differential equations with time-delay

- Casal, A.C., Díaz, J.I. and Vegas, J.M.: Blow-up in some ordinary and partial differential equations with time-delay, Dynam. Systems Appl., 18 1, 29-46 (2009).

- Casal, A.C., Díaz, J.I. and Vegas, J.M.: Controlled explosions of blowing-up trajectories in semilinear problems and a nonlinear variation of constant formula, XXIII Congreso de Ecuaciones Diferenciales y Aplicaciones, XIII Congreso de Matemática Aplicada, Castellón, 9-13 septiembre 2013. e-Proccedings.

- Casal, A.C., Díaz, J.I. and Vegas, J.M.: Complete recuperation after the blow up time for semilinear problems, AIMS Procceding 2015, 223-229. (2015).

In the present paper we shall consider the case in which the dynamics takes place mainly on the boundary of a set of $\mathbb{R}^{\mathrm{N}}$, which, by simplicity will be assumed to be a ball $\mathbf{B}_{\mathrm{R}}(0)$. More precisely we shall consider some semilinear elliptic equations with a dynamic boundary condition of the following type:

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \times]0,\infty[\\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial \mathbf{n}} = f(u) + \alpha(t) & \text{on } \partial\times]0,\infty[, \\ u(x,0) = u_0(|x|) > 0, & x \in \partial\mathbf{B}_{\mathrm{R}}. \end{cases} \tag{1}$$

under some structural assumptions which will be fulfilled in the special case of $g(u) = u^m$ and $f(u) = u^p$ with suitable $m, p > 1$. Here $\alpha(t)$ is the control function which we search in order to get a solution u such that $u \in \mathrm{L}^1_{loc}(0,\infty : \mathrm{L}^\infty(\mathbf{B}_{\mathrm{R}}))$.

A quite complete list of references dealing with nonlinear problems with dynamic boundary conditions, starting already in 1901, can be found, e.g., in the survey papers

- Bejenaru, I., Díaz, J.I. and Vrabie, I.I.: An abstract approximate controllability result and applications to elliptic and parabolic systems with dynamical boundary conditions, Electr. J. Differ. Eqns., 50, 1-19, (2001),
- Bandle, C., von Below, J. and Reichel, W.: Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up, Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali, Rendiconti Lincei Matematica E Applicazioni 2006, 35-67 (2006).

The study of the special case in which only the nonlinear dynamic boundary conditions is the origin of blow-up phenomena was considered in

- Kirane, M.: Blow up for some equations with semilinear dynamical boundary conditions of parabolic and hyperbolic type, Hokkaido Math. J., 21, 2, 222-229 (1992),

and later by several other authors: see e.g.,

- Kirane, M., Nabana, E., and Pokhozhaev, S. I.: The Absence of Solutions of Elliptic Systems with Dynamic Boundary Conditions, Differential Equations, 38 6 (2002) 808-815
- Kirane, M. and Tatar, N.: Absence of local and global solutions to an elliptic system with time-fractional dynamical boundary conditions, Siberian Mathematical Journal, 48 3, 477-488 (2007).

Notice that this is a different situation to the case in which there is a nonlinear parabolic equation with a source term jointly with a dynamic boundary condition: see, e.g.,

- Amann H. and Fila M.: A Fujita-type theorem for the Laplace equation with a dynamical boundary condition, Acta Math. Univ. Comenian., 66, 2, 321-328 (1997),
- Joachim von Below & Gaëlle Pincet Mailly (2003) Blow Up for Reaction Diffusion Equations Under Dynamical Boundary Conditions 28:1-2, 223-247,
- Bandle, C., von Below, J. and Reichel, W.: Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up, Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali, Rendiconti Lincei Matematica E Applicazioni 2006, 35-67 (2006),
- Rault, J.F., Phenomene d'explosion et existence globale pour quelques problemes paraboliques sous les conditions au bord dynamiques. These, Université du Littoral, Côte d'Opale, 2010. https://tel.archives-ouvertes.
- Vázquez, J.L. and Vitillaro, E.: On the Laplace equation with dynamical boundary conditions of reactive diffusive type, J. Math. Anal. Appl., 354 2, 674-688, (2009),
- Fiscella, A. and Vitillaro, E. (2015). Blow-up for the wave equation with nonlinear source and boundary damping terms. Proceedings of the Royal Society of Edinburgh: Section A Mathematics,145(4):759–778.

In all the cases, the blow-up takes place on the boundary, as it also holds for the case of a nonlinear parabolic equation with a source term jointly with a static possibly nonlinear Robin type boundary condition: see, e.g.,

- Levine, H, and Payne, L.: Nonexistence theorems for the hear equations with nonlinear boundary conditions and for the porous medium equation backward in time, J. Differential Equations, 16 (2), 319-334 (1974),
- López Gómez, J., Márquez, V. and Wolanski, N.I.: Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition, J. Diff. Eq. 92 (2),1991, 384-401 (1991),
- the survey Fila, M. and Filo, J.: Blow-up on the boundary: a survey. In Singularities and differential equations. Banach Center Publications, volume 33. Institute of Mathematics, Polish Academy of Sciences, Warszawa, 67-77 (1996),
- and many other more recent papers.

From the point of view of Control Theory, one of the pioneering works on control for blow-up problems for nonlinear parabolic equations with a source term was the book

- Lions, J.L. Contrôle des systémes distribués singuliers, Gauthier-Villars, Bordas, Paris,1983,

see also

- Díaz, J.I. and Lions, J.-L.: Sur la controlabilite de problemes paraboliques avec phenomenes d'explosion, C. R. Acad. Scie. de Paris. t. 327, Serie I, 173-177 (1998),
- Díaz, J.I. and Lions, J.-L., On the approximate controllability for some explosive parabolic problems. In: Hoffmann, K.-H., et al. (eds.) Optimal Control of Partial Differential Equations (Chemnitz, 1998), Internat. Ser. Numer. Math., vol. 133, Birkhauser, Basel, 115-132 (1999),
- Fern andez-Cara, E. and Zuazua, E.: Null and approximate controllability for weakly blowing up semilinear heat equations. Ann. Inst. Henri Poincar e Anal. Non Lineaire, 17, 583-616 (2000),
- Coron, J.M., E. Trelat, E. : Global steady-state controllability of 1-D semilinear heat equations, SIAM J. Control and Optimization, 43 (2), 549-569 (2004).
- Coron, J.M., Guerrero, S. and Rosier, L: Null controllability of a parabolic system with a cubic coupling term, SIAM J. Control and Optimization, 48, 8, 5629-5653 (2010),
- Vo, T.M.N.: Construction of a control for the cubic semilinear heat Equation, Vietnam J. Math., 44, 587-601 (2016).

In these, and many other works, the goal was to avoid the occurrence of the blow-up phenomenon by means of suitable controls (see also, e.g. some numerical experiences in He, J.W. and Glowinski, R.: Neumann Control of Unstable Parabolic Systems: Numerical Approach, J. Optim. Theory Appl., 9 c6, 1, 1-55 (1998)).

The possibility to choose the blow-point time and points were considered in

- F. Merle, Construction of Solutions with Exactly k Blow-up Points for the Schrödinger Equation with Critical Nonlinearity, Commun. Math. Phys. 129, 223-240 (1990),
- Li,X.: A modeling study of the pore size evolution in lithium-oxygen baterry electrodes, Journal of The Electrochemical Society, 162, A1636-A1645, (2015).

The optimal control for problems with a dynamic boundary condition was considered in

- Ahmed N. U. and Kerbal S.: Necessary conditions of optimality for systems governed by B-evolutions. In Ladde G. S. (ed.) et al., Dynamic Systems and Applications. Vol. 2.Proceedings of the 2nd International Conference (Atlanta, GA, USA, May 24-27, 1995),Dynamic Publishers, Atlanta, GA, 293-300, (1996).

The approximate controllability for the case of dynamic boundary conditions leading to global solutions was considered in

- Bejenaru, I., Díaz, J.I. and Vrabie, I.I.: An abstract approximate controllability result and applications to elliptic and parabolic systems with dynamical boundary conditions, Electr. J. Differ. Eqns., 50, 1-19, (2001),

3.2. Perfiles explosivos y resultados previos

We begin with a suitable application of the well-known Keller–Osserman condition

$$\int^{\infty} \frac{ds}{\sqrt{G(s)}} < \infty \tag{1}$$

where $G(s) = \int_0^s g(\tau)d\tau$ for a continuous and increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$, with $g(0^+) = 0$. Since

$$\Psi(\delta) = \int_\delta^\infty \frac{ds}{\sqrt{2G(s)}}, \quad \delta > 0 \tag{2}$$

is decreasing we can define the function $U(x,t)$ given implicitely by

$$(T_\infty - t) + (R - |x|) = \int_{U(x,t)}^\infty \frac{ds}{\sqrt{2G(s)}}$$

or

$$U(x,t) = \Psi^{-1}\left[(T_\infty - t) + (R - |x|)\right], \quad (x,t) \in [-R, +R] \times [0, T_\infty[, \tag{3}$$

here R and T_∞ are arbitrary positive constants fixed previously. From definition it follows

$$\begin{cases} U(x,t) = \Psi^{-1}\left[T_\infty - t\right], & x \in \{\pm R\} \times [0, T_\infty[, \\ U(x,0) = \Psi^{-1}\left[T_\infty + (R - |x|)\right], & x \in [-R, +R]. \end{cases} \tag{4}$$

Moreover, the capital property $\Psi(\infty) = 0$ implies

$$\lim_{(x,t)\to\{\pm R\}\times\{T_\infty\}} U(x,t) = +\infty. \tag{5}$$

Straightfordwad computation prove

$$-\mathrm{U}_{xx}(x,t)+g\big(\mathrm{U}(x,t)\big)=0,\quad x\in[-\mathrm{R},+\mathrm{R}]\setminus\{0\},\ \ t<\mathrm{T}_\infty. \tag{6}$$

and

$$\mathrm{U}_t(\pm\mathrm{R},t)=\sqrt{2\mathrm{G}\big(\mathrm{U}(\pm\mathrm{R},t)\big)},\quad t<\mathrm{T}_\infty$$

whence the master equation

$$\mathrm{U}_t(\pm\mathrm{R},t)\pm\overbrace{\frac{\partial\mathrm{U}(\pm\mathrm{R},t)}{\partial x}}^{\sqrt{2\mathrm{G}\left(\mathrm{U}(\pm\mathrm{R},t)\right)}}=2\sqrt{2\mathrm{G}\big(\mathrm{U}(\pm\mathrm{R},t)\big)},\quad t<\mathrm{T}_\infty \tag{7}$$

holds.

Function Ψ^{-1} provides the unique explosive profile on the boundary of the large solution

$$\begin{cases} -\Delta\mathrm{U}_\infty+g\big(\mathrm{U}_\infty\big)=0 & \text{in } \mathbf{B}_\mathrm{R}(0)\subset\mathbb{R}^\mathrm{N},\\ \mathrm{U}_\infty=\infty & \text{on } \partial\mathbf{B}_\mathrm{R}(0),\end{cases} \tag{8}$$

provided

$$\limsup_{\gamma\to\infty}\frac{\Psi(\eta\gamma)}{\Psi(\gamma)}<1\quad\text{for } \eta>1. \tag{9}$$

More precisely

$$\lim_{x \to \partial \mathbf{B}_{\mathrm{R}}(0)} \frac{\mathrm{U}_{\infty}(x)}{\Psi^{-1}(\mathrm{R} - |x|)} = 1. \tag{10}$$

(see [ADR2015, Theorem 1.1]). In fact, U_{∞} is a radially symmetric function denoted by

$$\mathrm{U}_{\infty}(x) = \mathrm{U}_{\infty}(\mathrm{R} - |x|), \quad x \in \mathbf{B}_{\mathrm{R}}(0).$$

Example 1

For the power like case $g(s) = s^m$, condition (1) becomes $m > 1$. Then

$$\Psi_m(\delta) = \frac{\sqrt{2}}{m-1} \frac{1}{\delta^{\frac{m-1}{2}}}, \quad \delta \geq 0. \tag{11}$$

Moreover, (9) also holds. Then

$$\mathrm{U}_{\infty}(x) = \left(\frac{2(m+1)}{(m-1)^2} \right)^{\frac{1}{m-1}} (\mathrm{R} - |x|)^{-\frac{2}{m-1}} + o(\mathrm{R} - |x|).$$

□

Alarcón, S., Díaz, G. and Rey, J.M.: Large solutions of elliptic semilinear equations in the borderline case. An exhaustive and intrinsic point of view, *Journal of Mathematical Analysis and Applications*, **431**, 365-405, (2015).

Example 2

Other illustrative choice satisfying (1) and (9) is the function $g(s) = se^{2s}$ for which

$$\mathrm{U}_\infty(x) = \sqrt{2}\,\mathrm{erfc}^{-1}\left(\frac{\mathrm{R} - |x|}{\sqrt{\pi}}\right) + o(\mathrm{R} - |x|)$$

where $\mathrm{erfc}(\gamma) = 1 - \mathrm{erf}(\gamma) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_\gamma^\infty e^{-s^2} ds.$ □

We note that these explosive profiles do not depend on the geometrical properties of $\mathbf{B}_\mathrm{R}(0)$ as curvature or dimension. These influences can appear in lower term of the explosive expansion near $\partial \mathbf{B}_\mathrm{R}(0)$ (see again [ADR2015]).
In the master equation (7) the dynamical blow up term is balanced with the blow up absorpion term. When the dynamical blow up term is absolutly dominant at the infinity the master equation becomes

$$\mathrm{W}_t(\pm\mathrm{R}, t) = f\left(\mathrm{W}(\pm\mathrm{R}, t)\right), \quad t < \mathrm{T}_\infty \tag{12}$$

where the function $\mathrm{W}(\pm\mathrm{R}, t)$ is given implicitely by

$$\mathrm{T}_\infty - t = \int_{\mathrm{W}(\pm\mathrm{R},t)}^\infty \frac{ds}{f(s)},$$

assumed the version

$$\int^\infty \frac{ds}{f(s)} < \infty \tag{13}$$

of the Keller-Osserman condition.

Thus,

$$W(\pm R, t) = \Phi^{-1}(T_\infty - t), \quad t < T_\infty$$

for the decreasing function

$$\Phi(\delta) = \int_\delta^\infty \frac{ds}{f(s)}, \quad \delta > 0 \tag{14}$$

As for $\Psi(\delta)$ we will require the version

$$\limsup_{\gamma \to \infty} \frac{\Phi(\eta\gamma)}{\Phi(\gamma)} < 1 \quad \text{for } \eta > 1. \tag{15}$$

of the property (9).

Example 3

For the power like case $f(s) = s^p$, condition (13) becomes $p > 1$. Then

$$\Phi_m(\delta) = \frac{p-1}{\delta^{p-1}}, \quad \delta \geq 0. \tag{16}$$

Then

$$W_\infty(\pm R, t) = \frac{1}{(p-1)^{\frac{1}{p-1}}} \frac{1}{(T_\infty - t))^{\frac{1}{p-1}}}.$$

□

Let us consider the dynamic problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \mathbf{B}_{\mathrm{R}} \times]0,\infty[, \quad \mathbf{B}_{\mathrm{R}} \doteq \mathbf{B}_{\mathrm{R}}(0) \subset \mathbb{R}^{\mathrm{N}}, \; \mathrm{N} > 1, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial \mathbf{n}} = f(u) & \text{on } \partial \mathbf{B}_{\mathrm{R}} \times]0,\infty[, \\ u(x,0) = u_0 > 0, & x \in \partial \mathbf{B}_{\mathrm{R}}, \end{cases} \tag{17}$$

where g and f are continuous increasing functions and $\mathbf{n}$ is the unit outer normal vector.

Theorem 1 (Existence of a finite blow up time)

Let us assume

$$p > \frac{m+1}{2} > 1$$

for the power-like choices $g(s) = s^m$ and $f(s) = s^p$. Then (17) has a unique solution on $\overline{\mathbf{B}}_{\mathrm{R}} \times [0, \mathrm{T}_\infty[$, $\mathrm{T}_\infty \leq \Phi_m(u_0) < \infty$ (see (16)), such that

$$\begin{cases} 0 \leq u(x,t) < \left(\dfrac{2(m+1)}{(m-1)^2}\right)^{\frac{1}{m-1}} \dfrac{1}{(\mathrm{R}-|x|)^{\frac{2}{m-1}}}, & (x,t) \in \mathbf{B}_{\mathrm{R}} \times [0,\mathrm{T}_\infty[, \\ \lim\limits_{t \nearrow \mathrm{T}_\infty} u(x,t) = \left(\dfrac{2(m+1)}{(m-1)^2}\right)^{\frac{1}{m-1}} \dfrac{1}{(\mathrm{R}-|x|)^{\frac{2}{m-1}}}, & x \in \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

Moreover

$$\lim_{t \nearrow \mathrm{T}_\infty} \frac{u(x,t)}{(\mathrm{T}_\infty - t)^{\frac{1}{p-1}}} = (p-1)^{\frac{1}{p-1}}$$

for $x \in \partial \mathbf{B}_{\mathrm{R}}$. We note that $u(\cdot,t)$ is integrable near T_∞ if $p > 2$. □

The above illustrative result is extended to some general terms $g(u)$ and $f(u)$

Theorem 2 (Existence of a finite blow up time)

Assume (1), (13) *and*

$$\lim_{\tau\to\infty} \frac{f(\tau)}{\sqrt{2\mathrm{G}(\tau)}} = \infty \tag{18}$$

with $\mathrm{G}(s) = \int_0^s g(s)ds$.

Then (17) *has a unique solution on* $\overline{\mathbf{B}}_{\mathrm{R}} \times [0, \mathrm{T}_\infty[$, $\mathrm{T}_\infty \leq \Phi(u_0) < \infty$, *such that*

$$\begin{cases} 0 \leq u(x,t) < \mathrm{U}_\infty(x), & (x,t) \in \mathbf{B}_{\mathrm{R}} \times [0, \mathrm{T}_\infty[, \\ \lim\limits_{t \nearrow \mathrm{T}_\infty} u(x,t) = \mathrm{U}_\infty(x), & x \in \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

(*see* (8)). *Moreover under* (15) *the inequality*

$$\lim_{t\nearrow \mathrm{T}_\infty} \frac{u(x,t)}{\Phi^{-1}(\mathrm{T}_\infty - t)} = 1,$$

holds for each $x \in \partial\mathbf{B}_{\mathrm{R}}$.

Remark 1

Among other contributions, Theorem 2 says that the blow-up takes place only on the boundary $\partial\mathbf{B}_{\mathrm{R}}$ at $t = \mathrm{T}_\infty$.

In Theorems 1 and 2 the dynamical blow up term $f(u)$ is absolutely greater than the aborption term $g(u)$. Next, we consider the case where the domination is balanced.

Theorem 3 (Existence of a finite blow up time)

For power-like choice $g(s) = s^m,\ m > 1$ *one obtains*

$$\lim_{t \nearrow \mathrm{T}_\infty} \frac{u(x,t)}{(\mathrm{T}_\infty - t)^{\frac{2}{m-1}}} = \left(\frac{m-1}{\sqrt{2}} (\ell - 1) \right)^{-\frac{2}{m-1}}$$

for each $x \in \partial \mathbf{B}_{\mathrm{R}}$, *provided that the continuous and increasing function* f *satisfies*

$$\lim_{\tau \to \infty} f(\tau) \tau^{-\frac{m+1}{2}} = \ell \sqrt{\frac{m+1}{2}},\ \ell > 1.$$

We note that $u(\cdot, t)$ *is integrable near* T_∞ *if* $m > 3$.

Again this illustrative result is a particular case of some general terms $g(u)$ and $f(u)$

Theorem 4 (Existence of a finite blow up time)

Let us assume

$$\lim_{\tau \to \infty} \frac{f(\tau)}{\sqrt{2\mathrm{G}(\tau)}} = \ell > 1 \tag{19}$$

and (9). *Then*

$$\lim_{t \nearrow \mathrm{T}_\infty} \frac{u(x,t)}{\Psi^{-1}((\ell - 1)(\mathrm{T}_\infty - t))} = 1, \quad x \in \partial \mathbf{B}_{\mathrm{R}}.$$

Remark 2

- We may apply Theorem 4 to the functions given in Example 2.
- We may extend the explosive boundary behavior

$$\lim_{t \nearrow \mathrm{T}_\infty} \frac{u(x,t)}{\Psi^{-1}\big((\ell-1)(\mathrm{T}_\infty - t)\big)} = 1$$

whenever additional assumptions on general functions Ψ satisfying the borderline case assumption

$$\limsup_{\gamma \to \infty} \frac{\Psi(\eta_0 \gamma)}{\Psi(\gamma)} = 1 \quad \text{for some } \eta_0 > 1$$

(see [ADR2015]). It enables us consider functios as $g(s) = s\big(\log s\big)^\tau,\ s > 1$ with $\tau > 2$.

□

3.4. Explosiones controladas

In our main contribution we prove that we may to govern the blow up by means of a suitable control for which exists a kind of extension of the solution after the blow up.

Theorem 5

If

$$p > \frac{m+1}{2} > 1,$$

there exists a function $\alpha(t) \geq 0$ *such that* $\alpha(\mathrm{T}_\infty) = +\infty$ *for which the solution of the problem*

$$\begin{cases} -\Delta u(x,t) + u(x,t)^m = 0, & x \in \mathbf{B}_{\mathrm{R}} \times]0,\infty[, \\ \dfrac{\partial u}{\partial t}(x,t) + \dfrac{\partial u}{\partial \mathbf{n}}(x,t) = u(x,t)^p + \alpha(t), & (x,t) \in \partial \mathbf{B}_{\mathrm{R}} \times]0,\infty[, \\ u(x,0) = u_0(x), & x \in \mathbf{B}_{\mathrm{R}}, \end{cases} \tag{20}$$

is well defined for all $t \geq 0$.

Ideas of the Proof As it is well known the solution of

$$\frac{d}{dt}\mathrm{V}(t) = \mathrm{V}(t)^p$$

blows up at some finite time T_∞ depending on the initial data. As in [CDV2015] we can avoid the blow up phenomenon by introducing a sharp control function $\alpha(t)$ vanishing in the interval $[0, \mathrm{T}_\infty - \delta]$, with δ is small. So that the solution V_α of

$$\frac{d}{dt}\mathrm{V}_\alpha(t) = \mathrm{V}_\alpha(t)^p + \alpha(t), \quad t \geq 0.$$

recovers the dynamic of V on $[0, \mathrm{T}_\infty - \delta]$.

Essentially the behaviour of the control is not far

$$\alpha(t) = \frac{d}{dt}\mathrm{V}(t) - \mathrm{V}(t)^p$$

when it is possible. The detailed obtainment of $\alpha(t)$ is based on a suitable reasoning involving the variation of constants formula applied on a multivalued problem.

With this reasoning, if u is the solution of (20), since $\frac{\partial u}{\partial \mathbf{n}} \geq 0$ we get that

$$\frac{\partial}{\partial t} u(\mathrm{R}, t) \leq u(\mathrm{R}, t)^p + \alpha(t).$$

So, we have the comparison

$$0 \leq u(\mathrm{R}, t) \leq \mathrm{V}(t)$$

and then $u(\mathrm{R}, t)$ is well defined for all $t \geq 0$. □

Remark 3

- For the domination balanced case the construction of the control function $\alpha(t)$ is also available.
- Some related questions, including some borderline cases, are studied in [CDDV2018].

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