Qualitative study of the total variation flow and a related formulation for Bingham fluids

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1 Introduction

One archetype of quasilinear partial differential operators: the p-Laplacian $\Delta_p u = \mathrm{div}(|Du|^{p-2}\, Du)$, $1 < p < \infty$. The equation becomes **singular** if $1 < p < 2$.

Main goal of the lecture: to present several qualitative properties for some stationary and parabolic problems involving the quasilinear p-Laplacian operator for the limit case $p = 1$: mainly, *the total variation flow* equation

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{Du}{|Du|}\right) \qquad (1)$$

[see Kobayashi and Giga (*Journ. Statistical Physics*, **95,** 1999), Andreu, Ballester, Caselles and Mazón (*J. Funct. Anal.* **180**, 2001, 347-403,...] and a related stationary equation

$$-\Delta u - g\mathrm{div}\left(\frac{Du}{|Du|}\right) = f \qquad (2)$$

with g a positive constant, proposed by E. C. Bingham, in 1922 (non-Newtonian fluids) [also in *Image Processing* [Chan et al, *SIAM Journal on Scientific Computing,* **20,** 1999]

I will report: a) some qualitative properties of solutions of (1) (with F. Andreu, J.M. Mazón and V. Caselles, *J. Funct. Anal.* **188**, 2002, 516-547) and b) case of (2) (with R. Cirmi (Univ. di Catania, Italy)).

2 The total variation flow

Let Ω be a bounded set in $\mathbb{R}^N$ ($\partial\Omega$ Lipschitz continuous). We can assume that $0 \in \Omega$.

We are interested in some qualitative properties of the solutions of

$$\begin{aligned} &\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{Du}{|Du|}\right) && \text{in } Q = (0,\infty)\times\Omega \\ &u(0,x) = u_0(x) && \text{in } \Omega \end{aligned}$$

with Dirichlet boundary conditions (problem P_D)

$$u(t,x) = 0 \ \text{ in } \Sigma = (0,\infty)\times\partial\Omega$$

or Neumann boundary conditions (problem P_N)

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \Sigma.$$

In order to introduce the notion of weak solution we recall that a function $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and there are Radon measures $\mu_1, \ldots, \mu_N$ with finite total mass in Ω and

$$\int_\Omega u D_i \varphi dx = -\int_\Omega \varphi d\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$. Notation:

$$|Du|(\Omega) = \sup\{\int_\Omega u \, div \, \boldsymbol{\varphi} \, dx: \\ \boldsymbol{\varphi} \in C_0^\infty(\Omega, \mathbb{R}^N), \|\boldsymbol{\varphi}\|_\infty \leq 1\}.$$

• Useful results: G. Anzellotti, *Ann. di Matematica Pura ed Appl.* IV (135) (1983), 293-318 (see also Kohn-Temam (1983)). • Let

$$X(\Omega) = \{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) : div \ \mathbf{z} \in L^1(\Omega)\}.$$

If $\mathbf{z} \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ the functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ is defined by

$$< (\mathbf{z}, Dw), \varphi >= -\int_\Omega w \, \varphi \, div\mathbf{z} dx - \int_\Omega w \, \mathbf{z} \cdot \nabla \varphi \, dx.$$

Then $(\mathbf{z}, Dw)$ is a Radon measure in Ω,

$$\int_\Omega (\mathbf{z}, Dw) = \int_\Omega \mathbf{z} \cdot \nabla w \, dx$$

for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and

$$\left| \int_B (\mathbf{z}, Dw) \right| \leq \int_B |(\mathbf{z}, Dw)| \leq \|\mathbf{z}\|_\infty \int_B |Dw| \qquad (3)$$

for any Borel set $B \subseteq \Omega$. In addition, $(\mathbf{z}, Dw)$ is absolutely continuous with respect to $|Dw|$, with Radon-Nikodym derivative $\theta(\mathbf{z}, Dw, x)$ which is a $|Dw|$ measurable function from Ω to $\mathbb{R}$ and

$$\int_B (\mathbf{z}, Dw) = \int_B \theta(\mathbf{z}, Dw, x)|Dw| \qquad (4)$$

for any Borel set $B \subseteq \Omega$. Moreover

$$\|\theta(\mathbf{z}, Dw, .)\|_{L^\infty(\Omega,|Dw|)} \leq \|\mathbf{z}\|_{L^\infty(\Omega,\mathbb{R}^N)}. \qquad (5)$$

• The weak trace on $\partial\Omega$ of the normal component of $\mathbf{z} \in X(\Omega)$ can be defined (Anzelloti, loc.cit.): there exists a linear operator $\gamma : X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(\mathbf{z})\|_\infty \leq \|\mathbf{z}\|_\infty$$

$$\gamma(\mathbf{z})(x) = \mathbf{z}(x) \cdot \mathbf{n}(x) \quad \text{for all } x \in \partial\Omega \text{ if } \mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z}, \mathbf{n}](x)$. Moreover, the following *Green's formula*, for $\mathbf{z} \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, is established:

$$\int_\Omega w \, div\mathbf{z} dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \mathbf{n}] w \, dH^{N-1}. \quad (6)$$

- The "energy space" we shall use is
  $L^1_w(0,T,BV(\Omega)) \equiv$
  $\{v : (0,T) \to BV(\Omega) : v \in L^1((0,T)\times\Omega)$
  $t \to < Dv(t), \phi >$ is measurable $\forall \phi \in C^1_0(\Omega,\mathbb{R}^N)$
  and $\int_0^T |Dv(t)|(\Omega)\,dt < \infty\}$.

Definition 1. *Let* $u_0 \in L^2(\Omega)$. *A function* $u : (0,T)\times\Omega \to R$ *is a weak solution of* (P_D) *(respectively,* (P_N)*) if* $u \in C([0,T], L^2(\Omega)) \cap H^1(\delta, T; L^2(\Omega))$, $\forall \delta \in (0,T)$, $u(0) = u_0$, $u \in L^1_w(0,T;BV(\Omega))$, *and there exists* $z \in L^\infty((0,T)\times\Omega : R^N)$ *with* $\|z\|_\infty \le 1$ *such that* $u_t = divz$ *in* $D'((0,T)\times\Omega)$ *and*

$$\int_\Omega (u(t)-w)u_t(t) = \int_\Omega (\mathbf{z}(t), Dw) - |Du(t)|(\Omega) - \int_{\partial\Omega} [\mathbf{z}(t),\mathbf{n}]w - \int_{\partial\Omega} |u(t)|$$

(respectively,

$$\int_\Omega (u(t)-w)u_t(t) = \int_\Omega (\mathbf{z}(t), Dw) - |Du(t)|(\Omega)$$

in case of the Neumann problem) for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ *and a.e. on* $(0,T)$.

Theorem 1. (Andreu, Ballester, Caselles and Mazón (2001)) *Let* $u_0 \in L^2(\Omega)$. *Then for every* $T > 0$ *there exists a unique* $u(t)$ *weak solution of (*P_D*) in* $(0, T) \times \Omega$. *Moreover, it is characterized in the sense that there exists* $\mathbf{z}(t) \in X(\Omega)$, *such that* $\|\mathbf{z}(t)\|_\infty \leq 1$, $u'(t) = div\mathbf{z}(t)$ *in* $D'(\Omega)$ *a.e.* $t \in (0, +\infty[$ *and*

$$\int_\Omega (\mathbf{z}(t), Du(t)) = |Du(t)|(\Omega), \tag{7}$$

and

$$[\mathbf{z}(t), \mathbf{n}] \in sign(-u(t)) \quad H^{N-1} - a.e. \text{ on } \partial\Omega. \tag{8}$$

Finally, we have the following comparison principle: if $u(t), \hat{u}(t)$ *are solutions corresponding to initial data* $u_0, \hat{u}_0$, *respectively, then*

$$\|(u(t) - \hat{u}(t))^+\|_2 \leq \|(u_0 - \hat{u}_0)^+\|_2 \tag{9}$$

for all $t \in [0, T]$. ■

Theorem 2. (Andreu, Ballester, Caselles and Mazón(2000)) *Let* $u_0 \in L^2(\Omega)$. *Then for every* $T > 0$ *there exists a unique weak solution of (*P_N*) in* $(0, T) \times \Omega$. *Moreover, if* $u(t), \hat{u}(t)$ *are weak solutions corresponding to initial data* $u_0, \hat{u}_0$, *respectively, then*

$$\|(u(t) - \hat{u}(t))^+\|_2 \leq \|(u_0 - \hat{u}_0)^+\|_2 \tag{10}$$

for all $t \in [0, T]$. ■

• Concerning the asymptotic behaviour for $t \to \infty$, it was shown by R. Hardt and X. Zhou (Comm. Partial Diff. Eqs., **19** (1994)) that if $u(t)$ satisfies (P_D) then $u(t) \to 0$ in $L^1(\Omega)$.
A stronger result can be obtained via the *comparison principle*

Theorem 3 . *Let* $u_0 \in L^\infty(\Omega)$ *and let* $u(t, x)$ *be the unique solution of problem* (P_D). *Then,*

$$\|u(t)\|_\infty \leq \left(\|u_0\|_\infty - \frac{N}{d(\Omega)} t \right)^+, \tag{11}$$

$d(\Omega) := \sup_{x \in \Omega} |x|$. *In particular,*

$$T^*(u_0) \leq \frac{d(\Omega)\|u_0\|_\infty}{N}, \tag{12}$$

where $T^*(u_0) := \inf\{t > 0\colon u(t) = 0\}$(*the finite extinction time*).

The proof will be obtained by comparison with uniform super and subsolutions of the form $U(t, x) = \alpha(t)$. Something new appears for the study of our operator since, in the p-Laplacian case, the conditions on $\alpha(t)$ to generate a supersolution are

$$\alpha(t) \geq 0 \text{ and } u_0(x) \leq \alpha(0), \text{ a.e.} x \in \Omega,$$
$$\alpha'(t) \geq 0 \tag{13}$$

and, in fact, those conditions are also sufficient for the total variation flow. Nevertheless, in the limit case $p = 1$, condition (13) can be substituted by a different one:

Proposition 1. *Let* $u_0 \in L^\infty(\Omega)$ *and let* $u_1(t,x)$ *be the solution of* (P_D). *Let* $u_2(t,x) = \alpha(t)$. *Then,*
(i) *if* $\alpha(t) \geq 0$, $u_0(x) \leq \alpha(0)$, *a.e.*$x \in \Omega$ *and*

$$\alpha'(t) \geq -\frac{N}{d(\Omega)}$$

we have $u_1(t) \leq u_2(t)$ *a.e. on* Ω,
(ii) *if* $\alpha(t) \leq 0$ *and* $u_0(x) \geq \alpha(0)$, *a.e.*$x \in \Omega$ *and*

$$\alpha'(t) \leq \frac{N}{d(\Omega)}.$$

we have $u_1(t) \geq u_2(t)$ *a.e. on* Ω.

Proof of the Proposition: Let us prove i) under the additional condition $|\alpha'(t)| \leq \frac{N}{d(\Omega)}$. By Theorem 1 there exists $\mathbf{z}_1(t) \in X(\Omega)$ such that $\|\mathbf{z}_1(t)\|_\infty \leq 1$, $u_1'(t) = div\mathbf{z}_1(t)$

in $\mathcal{D}'(\Omega)$ a.e. $t \in]0, +\infty[$ and satisfying

$$\int_\Omega (\mathbf{z}_1(t), Du_1(t)) = |Du_1(t)|(\Omega) \tag{14}$$

$$[\mathbf{z}_1(t), \mathbf{n}] \in sign(-u_1(t)) \quad H^{N-1} - a.e. \text{ on } \partial\Omega.$$

Take $\mathbf{z}_2(t)(x) := \dfrac{\alpha'(t)x}{N}$ (so, $\|\mathbf{z}_2(t)\|_\infty \leq 1$). Since $div\mathbf{z}_2(t) = \alpha'(t) = u_2'(t)$, applying Green's formula (6), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u_1(t) - u_2(t))^+]^2 =$$

$$= \int_\Omega \big(div\mathbf{z}_1(t) - div\mathbf{z}_2(t)\big)(u_1(t) - u_2(t))^+ =$$

$$- \int_\Omega \big(\mathbf{z}_1(t) - \mathbf{z}_2(t), D(u_1(t) - u_2(t))^+\big)$$
$$+ \int_{\partial\Omega} [\mathbf{z}_1(t) - \mathbf{z}_2(t), \mathbf{n}](u_1(t) - u_2(t))^+ \, dH^{N-1}.$$

If $R_t(r) := (r - \alpha(t))^+$, then

$$\int_\Omega \big(\mathbf{z}_1(t) - \mathbf{z}_2(t), D(u_1(t) - u_2(t))^+\big)$$
$$= \int_\Omega \big(\mathbf{z}_1(t) - \mathbf{z}_2(t), DR_t(u_1(t))\big)$$
$$= \int_\Omega \big(\mathbf{z}_1(t), DR_t(u_1(t))\big) - \int_\Omega \big(\mathbf{z}_2(t), DR_t(u_1(t))\big).$$

Now, by Proposition 2.7 of Anzelloti, we have

$$\int_\Omega \big(\mathbf{z}_1(t), DR_t(u_1(t))\big)$$
$$= \int_\Omega \theta(\mathbf{z}_1(t), DR_t(u_1(t)), x)|DR_t(u_1(t))|$$
$$= \int_\Omega \theta(\mathbf{z}_1(t), Du_1(t), x)|DR_t(u_1(t))|.$$

From (14), we have $\theta(\mathbf{z}_1(t), Du_1(t), x) = 1$ a.e. with respect to the measure $|Du_1(t)|$. Now, since the measure $|DR_t(u_1(t))|$ is absolutely continuous respect to the measure $|Du_1(t)|$, we also have $\theta(\mathbf{z}_1(t), Du_1(t), x) = 1$ a.e. with respect to the measure $|DR_t(u_1(t))|$. Consequently

$$\int_\Omega \big(\mathbf{z}_1(t), DR_t(u_1(t))\big) = \int_\Omega |DR_t(u_1(t))|.$$

Moreover, since $\|\mathbf{z}_2(t)\|_\infty \leq 1$, we have

$$\int_\Omega \left(\mathbf{z}_1(t) - \mathbf{z}_2(t), D(u_1(t) - u_2(t))^+\right) \geq 0.$$

On the other hand,

$$|[\mathbf{z}_2(t), \mathbf{n}]| \leq 1, [\mathbf{z}_1(t), \mathbf{n}] \in sign(-u_1(t))$$

and $u_2(t) \geq 0$ implies that

$$\int_{\partial\Omega} [\mathbf{z}_1(t) - \mathbf{z}_2(t), \mathbf{n}](u_1(t) - u_2(t))^+ \, dH^{N-1} \leq 0.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u_1(t) - u_2(t))^+]^2 \leq 0$$

and the proof concludes. Now, if $\alpha'(t) \geq -\frac{N}{d(\Omega)}$ we can write

$$\alpha'(t) - \operatorname{div}\left(\frac{D\alpha(t)}{|D\alpha(t)|}\right) = f(t)$$

for some $f(t) \geq 0$ and conclusion follows from the inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u_1(t) - u_2(t))^+]^2 \leq \int_\Omega [0 - f(t))^+]^2 \leq 0.$$

The proof of ii) is similar. ■

Proof of Theorem 3: Take

$$\alpha(t) := \left(\|u_0\|_\infty - \frac{N}{d(\Omega)}t\right)^+$$

and apply Proposition 1. ■

The previous estimate can be refined if the support of u_0 is contained in a ball $B(0,r) \subset\subset \Omega$. For that, we compute explicitly the evolution of the characteristic function of a ball.

Proposition 2. *Assume that* $B(0,r) \subset\subset \Omega$ *and let* $u_0 = k\chi_{B(0,r)}$. *Then the solution of problem* (P_D) *is given by*

$$u(t,x) = \mathrm{sign}(k)\left(|k| - \frac{N}{r}t\right)^+ \chi_{B(0,r)}(x).$$

For the proof, let $T = \frac{N}{|k|r}$, take

$$\mathbf{z}(t) := \begin{cases} -\dfrac{x}{r} & \text{if } x \in B(0,r),\ 0 \le t \le T, \\ -r^{N-1}\dfrac{x}{|x|^N} & \text{if } x \in \Omega \setminus B(0,r),\ 0 \le t \le T, \\ 0 & \text{if } x \in \Omega \text{ and } t > T, \end{cases}$$

and check that $u'(t) = div\mathbf{z}(t)$ in $\mathcal{D}'(\Omega)$ a.e. $t \in [0, +\infty[$, $\int_\Omega (\mathbf{z}(t), Du(t)) = |Du(t)|(\Omega)$ and $[\mathbf{z}(t), \mathbf{n}] \in sign(-u(t)) \quad H^{N-1} - a.e. \text{ on } \partial\Omega.$ ■

Remarks.

1. Notice that by Proposition 2, there is not propagation of the support. This must be compared to the p-Laplacian case:

$$P_D^p \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|Du|^{p-2} Du) & \text{in } Q = (0, \infty) \times \Omega, \\ \\ u(t, x) = 0 & \text{in } \Sigma = (0, \infty) \times \partial\Omega, \\ \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

with $1 < p < \infty$ and, for instance, $u_0 = k\chi_{B(0,r)}$, $B(0, r) \subset\subset \Omega$: if $p > 2$ then there is *finite speed of propagation* (*supp u(t,.) is a compact* $\subset\subset \Omega$, at least for t small), but if $1 < p \leq 2$ $ku(t, x) > 0, \forall x, \forall t > 0.$

2. The above result shows that there is *no spatial smoothing effect*, for $t > 0$, similar to the case of the linear heat equation and many other quasilinear parabolic equations. In our case, the solution is discontinuous and has the minimal required spatial regularity: $u(t, .) \in BV(\Omega) \backslash W^{1,1}(\Omega)$.

The method of super and subsolutions "fails" if u_0 is unbounded and also for the Neumann problem. Nevertheless, a different method can be applied: *the (global) energy method* (see the monograph, S.N. Antontsev, J.I. Díaz and S.Shmarev, *Energy Methods for Free Boundary Problems. Applications to Nonlinear PDEs and Fluid Mechanics*, Birkhäuser, Boston, Progress in Nonlinear Differential Equations and Their Applications, 2001)

Theorem 4. a) *Let* $u_0 \in L^N(\Omega) \cap L^2(\Omega)$*, and let* $u(t,x)$ *be the solution of problem (*P_D*). Then* $u(t) \in L^N(\Omega)$ *for t>0 and* $T^*(u_0) < \infty$.
b) *Suppose* $N = 2$ *and* $u_0 \in L^2(\Omega)$. *Let* $u(t,x)$ *be the unique weak solution of problem (*P_N*). Then there exists a finite time* T_0 *such that*

$$u(t) = \overline{u}_0 := \frac{1}{\lambda(\Omega)} \int_\Omega u_0(x)\, dx \quad \forall\, t \geq T_0.$$

Proof of a). Let $q \geq 1$, and $\varphi(r) := |r|^{q-1}r$. Then, taking $w = u(t) - \varphi(u(t))$ as test function, after some technical arguments, it yields

$$\frac{1}{q+1}\frac{d}{dt}\int_{\Omega} |u(t)|^{q+1} + |D\varphi(u(t))|(\Omega) \qquad (15)$$

$$+ \int_{\partial\Omega} |u(t)|^{q} \, dH^{N-1} \leq 0.$$

If we denote

$$v(t)(x) := \begin{cases} \varphi(u(t))(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

then, by Sobolev's inequality for BV functions (see Theorem 5.6.1 of Evans and Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, 1992)

$$\| |u(t)|^q \|_{L^{N/N-1}(\Omega)} = \|v(t)\|_{L^{N/N-1}(\mathbb{R}^N)} \leq C \|Dv(t)\|_{BV(\mathbb{R}^N)}.$$

Therefore, from (15), we obtain that

$$\frac{1}{q+1}\frac{d}{dt}\int_{\Omega} |u(t)|^{q+1} + \frac{1}{C}\| |u(t)|^q \|_{L^{N/N-1}(\Omega)} \leq 0.$$

Then, taking $q = N-1$, we get

$$\frac{d}{dt}\int_{\Omega} |u(t)|^{N} + M\left(\int_{\Omega} |u(t)|^{N}\right)^{\frac{N-1}{N}} \leq 0.$$

From where the conclusion follows.
Proof of b). Taking $w = \overline{u_0}$ as test function it yields

$$\int_\Omega (u(t) - \overline{u_0})\, u_t(t) = -\left|Du(t)\right|(\Omega).$$

Now, by Poincaré inequality for BV functions (see Evans and Gariepy, *loc.. cit.*) and having in mind that we have *conservation of mass*, we obtain

$$\|u(t) - \overline{u_0}\|_2 \leq C \left|Du(t)\right|(\Omega).$$

Thus, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - \overline{u_0})^2 + \frac{1}{C}\|u(t) - \overline{u_0}\|_2 \leq 0.$$

Therefore, the function $y(t) := \int_\Omega (u(t) - \overline{u_0})^2$ satisfies the inequality $y'(t) + My(t)^{1/2} \leq 0$. ■

Remark.
3. The energy method can be applied to more general quasilinear equations of the form

$$\frac{\partial u}{\partial t} = \mathrm{div}\mathbf{A}(x, t, u, Du)$$

with $\mathbf{A}(x, t, u, \mathbf{p}) \cdot \mathbf{p} \geq |\mathbf{p}|$. ■

A finer study near *the finite extinction time is possible.*

Theorem 5. i) *Let* $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$ *and let* $u(t,x)$ *be the solution of* (P_D). *Let*

$$w_D(t,x) := \begin{cases} \dfrac{u(t,x)}{T^*(u_0) - t} & \text{if } \; 0 \le t < T^*(u_0) \\ 0 & \text{if } \; t \ge T^*(u_0). \end{cases}$$

Then, there exists an increasing sequence $t_n \to T^*(u_0)$, *and a solution* $v_D^* \neq 0$ *of the stationary problem*

$$(S_D) \begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) = v & \textit{in } \Omega \\ \\ v = 0 & \textit{on } \partial\Omega \end{cases}$$

such that

$$\lim_{n\to\infty} w_D(t_n) = v_D^* \text{ in } L^p(\Omega)$$

for all $1 \le p < \infty$.

ii) *Suppose* $N = 2$. *Let* $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$ *and let* $u(t,x)$ *be the weak solution of problem* (P_N). *Let*

$$w_N(t,x) := \begin{cases} \dfrac{u(t,x) - \overline{u_0}}{T^*(u_0) - t} & \text{if } 0 \le t < T^*(u_0) \\ 0 & \text{if } \; t \ge T^*(u_0). \end{cases}$$

Then, there exists an increasing sequence $t_n \to T^*(u_0)$, *and a solution* $v_N^* \neq 0$ *of the stationary problem*

$$(S_N) \begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) = v & \text{in } \Omega \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$\lim_{n\to\infty} w_N(t_n) = v_N^* \quad \text{in} \quad L^p(\Omega),$$

for all $1 \leq p < \infty$.

Idea of the proof of i). Let $g(t) := (T^*(u_0) - t)^+$. Then, for $0 \leq t < T^*(u_0)$,

$$w(t) = \frac{u(t)}{g(t)} \ \text{ and } w'(t) = \frac{u'(t)}{g(t)} + \frac{w(t)}{g(t)}.$$

We make a change of scale in time $t = \varphi(\tau)$, such that $\varphi(+\infty) = T^*(u_0)$. To do that we take

$$\varphi(\tau) := T^*(u_0)\left(1 - e^{-\tau}\right).$$

Hence, if we define

$$v(\tau) := w(\varphi(\tau)) = \frac{u(\varphi(\tau))}{T^*(u_0)} e^{\tau},$$

$v(\tau)$ is a strong solution of the problem

$$v'(\tau) + \partial\Phi\left(v(\tau)\right) \ni v(\tau),$$

where $\Phi : L^2(\Omega) \to (-\infty, +\infty]$ is defined by

$$\Phi(u) = \begin{cases} |Du|(\Omega) + \int_{\partial\Omega} |u| & \text{if } u \in BV(\Omega) \cap L^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

then we have $\mathcal{A} \cap (L^2(\Omega) \times (L^2(\Omega)) = \partial\Phi$. Let us see that there exists an increasing sequence $\tau_n \to +\infty$ and a function $v^* \in BV(\Omega)$, such that $\lim_{n\to\infty} v(\tau_n) = v^*$ in $L^p(\Omega)$ [which implies the existence of $t_n \to T^*(u_0)$ such that $\lim_{n\to\infty} w(t_n) = v^*$ in $L^p(\Omega)$].

We have

$$\frac{1}{2}\frac{d}{d\tau}\int_\Omega v(\tau)^2 + |Dv(\tau)|(\Omega) + \int_{\partial\Omega} |v(\tau)| = \int_\Omega v(\tau)^2.$$

On the other hand,

$$\|v(\tau)\|_\infty = \frac{e^\tau}{T^*(u_0)} \|u(\varphi(\tau))\|_\infty.$$

Hence, we get

$$\|v(\tau)\|_\infty \le C \quad \text{for all} \quad \tau \ge \tau_0 > 0 \qquad (16)$$

since we can prove (by applying the smoothing effect of Ph. Benilan and M.G. Crandall, [in *Contributions to Analysis and Geometry*, D.N. Clark et al. eds., John Hopkins University Press, 1981, 23-39]) that

$$\|u(t)\|_\infty \le \frac{2\|u_0\|_\infty}{\tau}(T^*(u_0) - t) \quad \text{for} \quad \tau \le t \le T^*(u_0).$$

By Lemma 3.3 of Brezis (*Operateurs Maximaux Monotones*,...,1973
we have

$$\frac{d}{d\tau}\left(\Phi(v(\tau))\right) = -\int_\Omega v'(\tau)^2 + \int_\Omega v(\tau)v'(\tau),$$

from where it follows that

$$\begin{aligned} &|Dv(\tau)|(\Omega) + \int_{\partial\Omega} |v(\tau)| - \frac{1}{2}\int_\Omega v(\tau)^2 \\ &\leq |Dv(0)|(\Omega) - \frac{1}{2}\int_\Omega v(0)^2 + \int_{\partial\Omega} |v(0)| \ \forall\ \tau \geq 0. \end{aligned}$$

Thus, the orbit $\{v(\tau),\ \tau \geq 0\}$ is bounded in $BV(\Omega)$. Hence, by the *compact embedding theorem for BV-functions* (see, e.g., Ambrosio-Fusco-Pallara, Oxford Mathematical Monographs, 2000) $\{v(\tau),\ \tau \geq 0\}$ is relatively compact in $L^p(\Omega)$ for $1 \leq p < \frac{N}{N-1}$, and consequently, there exists $\tau_n \to \infty$ and $v^* \in L^p(\Omega) \cap BV(\Omega)$, such that $v(\tau_n) \to v^*$ in $L^p(\Omega)$. Moreover, by (16) we can assume that $v(\tau_n) \to v^*$ in $L^q(\Omega)$ for all $1 \leq q < \infty$. On the other hand, by using the energy inequality of Theorem 3 we have that

$$\|v(\tau)\|_N \geq C \qquad \forall\ \tau \geq 0.$$

Then, we get $v^* \neq 0$. Finally, v^* is a solution of the stationary problem (S_D) since $T(t)v^* = v^*$, where $(T(t))_{t\geq 0}$

is the semigroup in $L^2(\Omega)$ generated by $\mathcal{A}-I$. The proof of part ii) is, essentially, similar. ■

Remarks.

4. Previous versions of this type of behaviors: J.G. Berryman and C. J. Holland, *Arch. Rational. Mech. Anal.* **74,** (1980), 279-288 (for $u_t - \Delta u^m = 0$, $0 < m < 1$), J.I. Díaz and A. Liñán (Movimiento de descarga de gases en conductos largos: modelización y estudio de una ecuación doblemente no lineal. In the book *Reunión Matemática en Honor de A.Dou* (J.I.Díaz y J.M.Vegas eds.) Universidad Complutense de Madrid, 1989, 95-119 (for $u_t - \Delta_p u^m = 0$, $0 < (p-1)m < 1$).

5. Notice that by Theorem 5, there exists solutions of the "singular eigenvalue type" problem (S_D) which are not strictly positive (in contrast with the Krein-Rutman theorem).

Concerning the study of (S_D) under symmetry assumptions we have:

Proposition 3. *Let* $\Omega = B(0,R)$, $R > 0$, *and* $u_0 \geq 0$ *be a radial function in* $B(0,R)$. *If* v^* *is the asymptotic profile of the solution of* (P_D) *then* $v^*(x) = g(|x|)$ *for a decreasing function* $g : [0,R] \to [0, \|u_0\|_\infty]$ *satisfying*

$g(r) = \frac{1}{r}$ or $g'(r) = 0$, a.e. in $r \in (0, R)$. $\blacksquare$

We finish this section by giving some explicit solutions of (S_D) in the radial case.

Proposition 4. *The following functions are solutions of (S_D) in $B(0, R)$:*

$$u_1(x) = \frac{N-1}{|x|},$$

$$u_2(x) = \frac{Per(B(p,r))}{|B(p,r)|}\chi_{B(p,r)}(x), \ \forall \ B(p,r) \subseteq B(0,R),$$

$$u_3(x) = \begin{cases} \dfrac{N}{r} & \text{if} \quad x \in B(0,r) \subseteq B(0,R) \\ \\ \dfrac{N-1}{|x|} & \text{if} \quad x \in B(0,R) \setminus B(0,r). \end{cases}$$

Moreover, if $R_1 < R_2 \leq R$, $B_1 = B(0, R_1)$, $B_2 = B(0, R_2)$. Then the "tower function"

$$u_4(x) = \frac{Per(B_1)}{|B_1|}\chi_{B_1}(x) + \frac{Per(B_2) - Per(B_1)}{|B_2| - |B_1|}\chi_{B_2 \setminus B_1}(x)$$

is also solution of (S_D) in $B(0, R)$. $\blacksquare$

The proof uses several techniques from the geometrical measure theory.

3 On the Bingham stationary model

We shall study some qualitative properties on the spatial structure of solutions of problem

$$(BS)\begin{cases} -\Delta u - g\mathrm{div}\left(\frac{Du}{|Du|}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Given $f \in L^2(\Omega)$, the existence and uniqueness of a solution $u \in H_0^1(\Omega)$ was shown by Duvaut- Lions (1969). The regularity $H^2(\Omega)$ was obtained later by Brezis (1971). Let us define the *plastic region* by

$$\Omega_0 = \{x \in \Omega : |Du| = 0\}.$$

Theorem 6. *Assume $f \in L^\infty(\Omega)$ and let $c := \|f\|_\infty$. Let $\omega_N = |B(0,1)|$,*

i) if $|\Omega| \leq \omega_N(\frac{Ng}{c})^N$ then $u(x) = 0$, a.e. $x \in \Omega$,

ii) if $f(x) \equiv c$ and $|\Omega| > \omega_N(\frac{Ng}{c})^N$ then $|\Omega_0| \geq \omega_N(\frac{Ng}{c})^N$.

The main ingredients of the proof are the consideration of the special case $\Omega = B(0,R)$ and a comparison in terms of the *decreasing symmetric rearrangement*

Proposition 5. *Let* $\Omega = B(0,R)$ *and* $f(x) \equiv c$

i) if $R \leq \dfrac{Ng}{c}$ *then* $u(x) = 0$, *a.e.* $x \in \Omega$,

ii) if $R > \dfrac{Ng}{c}$ *then* $\Omega_0 = B(0, \dfrac{Ng}{c})$.

Idea of the proof of Proposition 5. By the equivalent formulation in terms of a *Lagrange multiplier*, there exists $\mathbf{p} \in \Lambda := \{\mathbf{q} \in L^\infty(\Omega)^N : \|\mathbf{q}\|_\infty \leq 1\}$ such that

$$\begin{cases} -\Delta u - g\,\mathrm{div}\mathbf{p} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \mathbf{p}\cdot Du = |Du| & a.e. \text{ in } \Omega. \end{cases}$$

Then, by approaching (when $p \searrow 1$) by the solutions of

$$(BS_p) \begin{cases} -\Delta u - g\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we prove that if $R \leq \dfrac{Ng}{c}$ then $\|\mathbf{p}\|_\infty < 1$, and so $u(x) = 0$, *a.e.* $x \in \Omega$. If $R > \dfrac{Ng}{c}$ it is possible to construct (explicitly) the solution. So, for instance, for $N = 2$,

$$u(r) = \begin{cases} (R-r)(\frac{c}{4}(R+r) - g) & \text{if } \frac{2g}{c} \leq r \leq R, \\ \frac{c}{4}(R - \frac{2g}{c})^2 & \text{if } 0 \leq r \leq \frac{2g}{c}, \end{cases}$$

(see also Glowinski, R. Lions, J.L. and Tremolières, R., *Numerical Analysis of Variational Inequalities,* North-Holland, Amsterdam, 1981).

Proposition 6. *Let* $f \in L^2(\Omega)$, $f \geq 0$. *Let* $f^* \in L^2(\Omega^*)$ *its decreasing symmetric rearrangement. Let* U *be the solution of* BS *associated to* Ω^* *and* f^*. *Then*

$$u^*(x) \leq U(x), \text{ a.e. } x \in \Omega^*$$

and

$$|Du^*(x)| \leq |DU(x)|\,, \text{ a.e. } x \in \Omega^*.$$

(The proof is an easy variation of J.I.D: "Desigualdades de tipo isoperimétrico para problemas de Plateau y capilaridad", *Revista de la Academia Canaria de Ciencias*, Vol. III, No.1, 127-166, 1991)

Remarks.

6. The proof of Theorem 6 is now immediate from Propositions 5 and 6.

7. The radial solutions can be used as super and subsolutions in order to get pointwise estimates on the location of the plastic region.

8. In the radial case we conclude that $|\Omega_0| = \omega_N(\frac{Ng}{c})^N$

independently of R (once that $R > \dfrac{Ng}{c}$). This is entirely different to the case of the free boundary for

$$\begin{cases} -\Delta_p u + u = 1 & \text{in } \Omega = B(0,R), \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

assumed $p > 2$. In that case the "solid region" is $\Omega_1 = \{x \in \Omega : u = 1\}$ and $|\Omega_1| \nearrow$ if $R \nearrow$ (see: J.I.D. *Nonlinear PDEs and free boudaries,* Pitman, London, 1985).

9. Recent numerical experiences in J.W. He and R. Glowinski: "Steady Bingham fluid flow in cylindrical pipes: a time dependent approach to the iterative solution", *Numerical Algebra with Applications, 2000,* **7,** 381-428.

10. Estimates on $|\Omega_0|$ for different special geometries of Ω in P. Mossolov and V. Miasnikov: "Variational methods in the theory of the fluidity of a viscous-plastic medium", *Journal of Mechanics and Applied Mathematics, 1965,* **73,** 468-492.

11.To finish, let us consider the evolution problem

$$(BE)\begin{cases} u_t - \nu\Delta u - g\mathrm{div}\left(\frac{Du}{|Du|}\right) = f(t,x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0,x) = u_0(x) & \text{on } \Omega, \end{cases}$$

for $\nu \geq 0$ and $g > 0$ and $f(t,x) \neq 0$.

• Conditions on f for the existence of a *finite extinction time* ? ($f \equiv 0$ in Section 2).

• Necessary condition:

$$f\,(t,x) \in B(0) \quad \text{a.e. } x \in \Omega,\ t \text{ large}$$

where

$$B : D(B) \subset L^2(\Omega) \to \mathcal{P}(\Omega),$$
$$Bu = -\nu\Delta u - g\mathrm{div}\left(\frac{Du}{|Du|}\right).$$

The abstract results for multivalued operators can not be applied (H. Brezis, *Proc. Int. Congress Math.* Vancouver, 1974, J.I.D. *Rev. Real Acad. Ciencias*, 74, 1980, 865-880)

As in Proposition 1,

$$B(0) \supset \{c \in \mathbb{R}\colon\ |c| \leq g\frac{N}{d(\Omega)}\},$$

$d(\Omega) := \sup_{x\in\Omega} |x|$.

Proposition 7. *Let* $u_0 \in L^\infty(\Omega)$*,* $f \in L^\infty(Q)$ *and let* $u(t,x)$ *be the unique solution of problem* (BE)*. Assume*

$$ess\sup\left\{\|f\,(t,\cdot)\|_{L^\infty(\Omega)} : t \in]T_f, T[\right\} < g\frac{N}{d(\Omega)}.$$

Then, for any $t \in]T_f, T[$ *we have*

$$\|u(t)\|_\infty \leq \left(\|u\,(T_f,\cdot)\,\|_{L^\infty(\Omega)} - (\frac{gN}{d(\Omega)} - c)t\right)^+$$

with $c := ess\sup\left\{\|f(t,\cdot)\|_{L^\infty(\Omega)} : t\in]T_f, T[\right\}$. *In particular,*

$$T^*(u_0,f) \leq \frac{\|u(T_f,\cdot)\|_{L^\infty(\Omega)}}{(\frac{gN}{d(\Omega)}-c)}. \tag{17}$$

Compare (as in Proposition 1) with uniform super and subsolutions satisfying

$$\overline{\alpha}'(t) \geq -(\frac{gN}{d(\Omega)}-c)$$

and

$$\underline{\alpha}'(t) \leq (\frac{gN}{d(\Omega)}-c),$$

respectively. ■

Notice that

$$\|u(T_f,\cdot)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + \int_0^{T_f} \|f(s,\cdot)\|_{L^\infty(\Omega)}\,ds$$

and estimate (17) becomes explicit.