On the “strange term” in the homogeneized nonlinear reaction-diffusion equation for critically small catalytic particles

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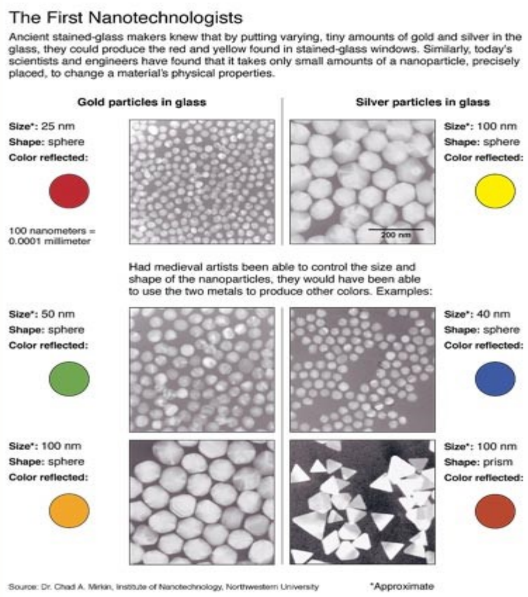
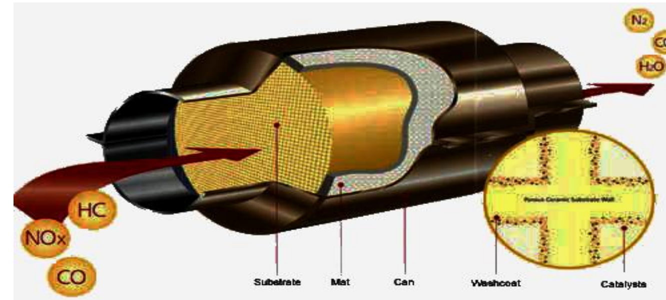
1. Introduction.



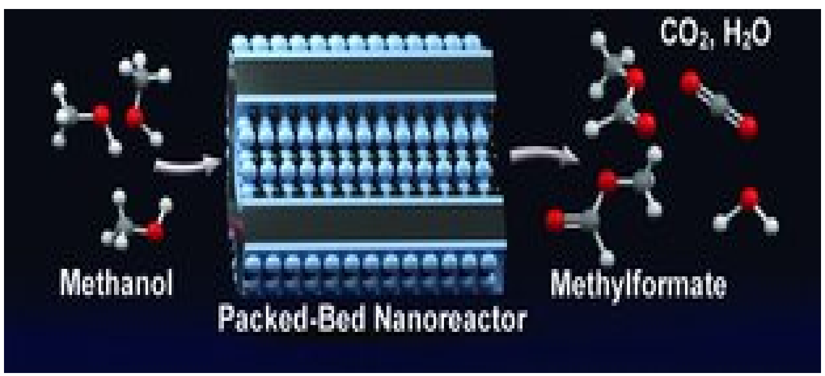
In this talk we aim to give a rigorous presentation of the asymptotic behaviour in reaction-diffusion equations when the size of the particles of a porous medium is very small. As in other domains of the nanotechnology new and unexpected behaviors arise.

Joint works with D. Gómez - Castro (UCM), **A.V. Podolískii** (Moscow State University) and **T.A. Shaposhnikova** (Moscow State University).

Fixed-bed chemical reactors



Catalytic converter for cars



Nanoparticles of silica in a silica nanotube, which includes Pd nanoparticles used to prepare a packed bed nano-reactor

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Lee, K. J., Min, S. H. and Jang, J. (2010). Small, 6: 2378–2382

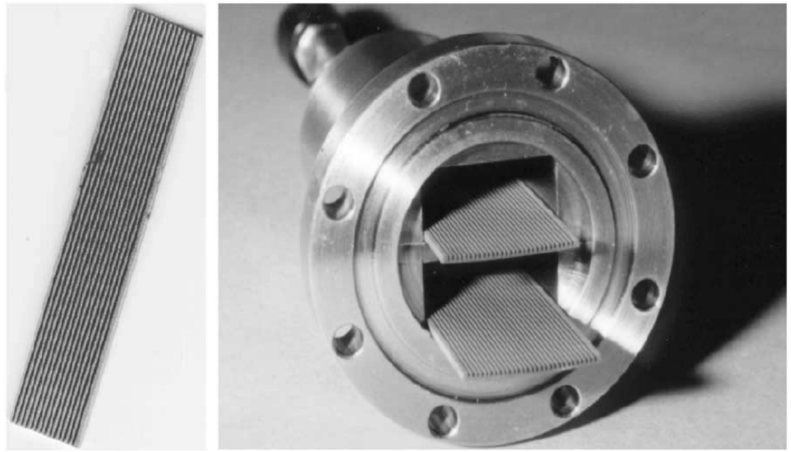


Fig. 10. Microstructured wafer, coated with alumina and gold particles, $Au/Al_2O_3/Al$ (left) and photograph of the reactor system (right).

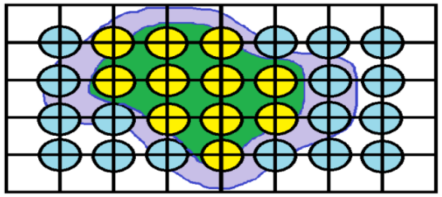
Sabine Schimpf et al. "Supported gold nanoparticles: in-depth catalyst characterization and application in hydrogenation and oxidation reactions". In: *Catalysis Today* 72.1 (2002), pp. 63–78

Gold content, surface area, pore volume, Au particle sizes and degree of dispersion estimated by TEM analysis

Catalyst	Au content (wt.%)	Surface area (m^2/g)	Pore volume (ml/g)	$\bar{d}_{Au}$ (nm)[a]	msd (nm)[b]	D_{Au}[c]
Au/TiO_2-DP	1.7	42	0.35	5.3	0.3	0.36
Au/TiO_2-I	2.9	42	0.38	2.0	0.4	0.47
Au/TiO_2-SG	4.8	117	0.17	1.1	0.2	0.78
Au/ZrO_2-F	1.0	151	0.37	1.4	0.3	0.63
Au/ZrO_2-DP	1.2	127	0.10	3.8[d]	2.9	0.36
Au/SiO_2-IW	1.6	171	0.85	3.9	2.3	0.16
Au/SiO_2-CVD	2.4	NE[e]	NE	1.4	0.3	0.70
$Au/Al_2O_3/Al$	0.02[f]	NE	NE	NE	NE	NE

[a] Mean diameter of log-normal distribution.
[b] Mean square displacement.
[c] Estimated from the ratio of the number of surface atoms to the total number of atoms as calculated for the mean particle size by assuming closed shell particles of spherical shape.
[d] Bimodal particle size distribution (2.1 and 7.4 nm) [20].
[e] Not estimated.
[f] Estimated by peeling off the coating (392.56 mg) of two wafers.

It is well-known that in many models of Chemical Engineering the theory of homogenization yields a global model given by an effective diffusion and a global interior reaction (the same reaction than holds on the boundary of the pellets) when the size of the particles is assumed to converge to zero. It is assumed, usually, that the size of the particles is of the same order as their repetition.



$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & x \in \Omega_\varepsilon\,, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma(u_\varepsilon) & x \in S_\varepsilon\,, \\ u_\varepsilon = 0 & x \in \partial\Omega, \end{cases}$$

- $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n \colon (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}_\varepsilon} \neq \emptyset\}$
- $N_\varepsilon = |\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = const > 0$.

J. M. Vega and A. Liñán. Isothermal nth order reaction in catalytic pellets: Effect of external mass transfer resistance. *Chemical Engineering Science*, 34(11):1319–1322, 1979.

J. I. Díaz. Two problems in homogenization of porous media. *Extracta mathematicae*, 14(2):141–156, 1999.

C. Conca, J. I. Díaz, A. Liñán, and C. Timofte. Homogenization in Chemical Reactive Flows. *Electronic Journal of Differential Equations*, 40:1–22, jun 2004.

Here the diffusion is modeled by the quasilinear operator $\Delta_p u_\varepsilon \equiv div(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$ with $p > 1$. Notice that $p = 2$ corresponds to the linear diffusion operator, and that $p \neq 2$ appears in turbulent regime flows or non-Newtonian flows

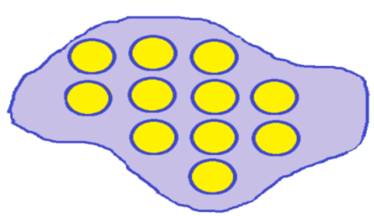
J. I. Díaz. *Nonlinear Partial Differential Equations and Free Boundaries.* Pitman, London, 1985.

In fact we shall consider the structural assumption

$$1 < p < n \text{ and } n \geq 3.$$

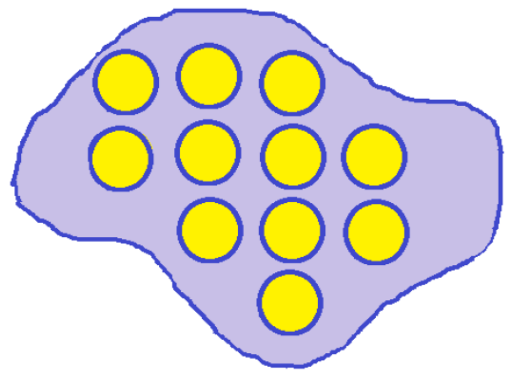
The cases $p \geq n$ are completely different

J. I. Díaz, D. Gómez-Castro, A. V. Podol'skiy, and T. A. Shaposhnikova. Homogenization of the p-Laplace diffusion problem in n-dimensional domains with nonlinear reaction in the boundary of periodically distributed particles. Case $n < p < \infty$. *To appear*, 2017.



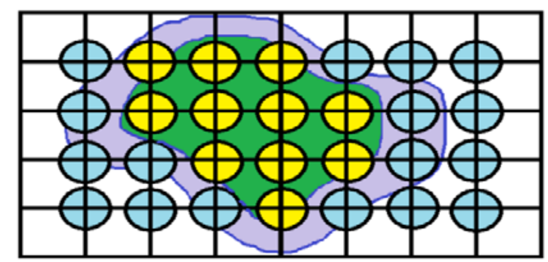
- $\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}$,
- $G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j$
- $\partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega$, $S_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} \partial G_\varepsilon^j$.

The domain $\Omega_\varepsilon \subset \mathbb{R}^n$ is assumed to have a ε-periodical structure (or by isolated particles or by a perforated connected solid component). The particles are balls of radius $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha > 1$.



- $\widetilde{\Omega_\varepsilon} = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$

- $\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}$,
- $G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j$
- $\partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega$, $S_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} \partial G_\varepsilon^j$.



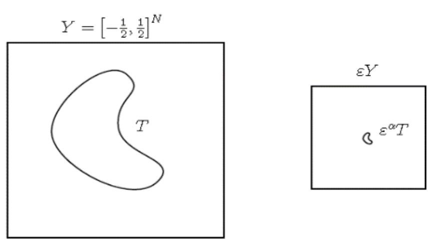
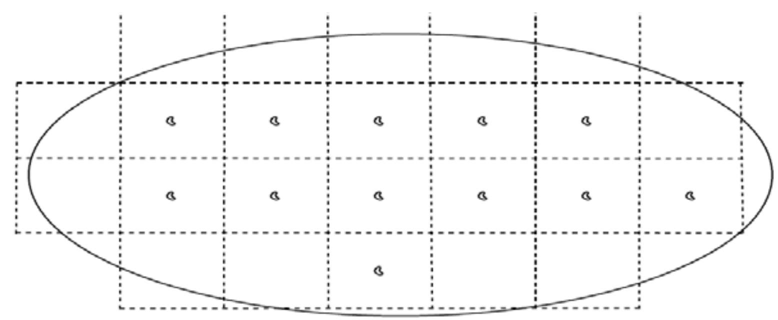


Fig. 1 The reference cell Y and the scalings by ε and ε^α, for $\alpha > 1$. Notice that, for $\alpha > 1$, $\varepsilon^\alpha T$ (for a general particle shaped as T) becomes smaller relative to εY, which scales as the repetition. In our case T will be a ball $B_1(0)$.

- $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega_\varepsilon}} \neq \emptyset\}$
- $N_\varepsilon = |\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = const > 0$.



$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma(u_\varepsilon) & x \in S_\varepsilon, \\ u_\varepsilon = 0 & x \in \partial\Omega, \end{cases} \tag{0.1}$$

Hence, the problem has two different parameters: α and γ. We shall prove that when the size of the particles have a critical value (with respect the normalization factor of the kinetics: see the above boundary condition on the boundary of the particles S_ε)

$$\alpha = \frac{n}{n-p}, \quad \gamma = \alpha(n-1) - n = \alpha(p-1), \tag{0.3}$$

then the homogenized problem involves a different distributed chemical kinetics nonlinearity

$$\begin{cases} -\Delta_p u + \mathcal{A}|H(u)|^{p-2}H(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.4}$$

where

$$\mathcal{A} = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n \tag{0.5}$$

and H is the nondecreasing contraction, from $\mathbb{R}$ to $\mathbb{R}$, given by

$$H(r) = (I + \sigma^{-1} \circ \Theta_{n,p})^{-1}(r), \tag{0.6}$$

with

$$\Theta_{n,p}(s) = \mathcal{B}_0 |s|^{p-2} s \tag{0.7}$$

for $s \in \mathbb{R}$, and

$$\mathcal{B}_0 = \left(\frac{n-p}{C_0(p-1)} \right)^{p-1} \tag{0.8}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Notice that for a Newtonian fluid in $\mathbb{R}^3$ ($n = 3, p = 2$) the critical size corresponds to $\alpha = 3$. Obviously the critical value of α is an increasing function of p. Therefore for non Newtonian dilatant fluids or a Newtonian flow in turbulent regime ($p > 2$) our assumption means $\alpha > 3$, the particles are tiny with respect to their repetition, whereas for pseudoplastic fluids ($p < 2$) the critical particles satisfy $\alpha < 3$, and hence are not so tiny with respect to their repetition.

$$\alpha = \frac{n}{n-p}$$

This change of behaviour is one of the characteristic of the nanotechnological effects (see, e.g. [20]) and does not appear if $1 \leq \alpha < \frac{n}{n-p}$ (see [5, 21]).

C. Conca, J. I. Díaz, A. Liñán, and C. Timofte. Homogenization in Chemical Reactive Flows. *Electronic Journal of Differential Equations*, 40:1–22, jun 2004.

T. A. Shaposhnikova and A. V. Podolskiy. Homogenization limit for the boundary value problem with the with the p-Laplace operator and a nonlinear third boundary condition on the boundary of the holes in a perforated domain. *Functional Differential Equations*, 19(3-4):1–20, 2012.

To mention only some few mathematical works in this direction (some additional references will be presented later) we must start by referencing the pioneering paper, in the case $u = 0$ on S_ε, by Cioranescu and Murat [4] in which a framework more general than our formulation (0.1) was considered but without giving details about the nature of the "new strange term" as precise as the one we present in (0.6). A previous closer conclusion to our main result was proved in [12] and then improved (by using some different techniques of proof) in [23, 13, 17].

[4] D. Cioranescu and F. Murat. A Strange Term Coming from Nowhere. In A. Cherkaev and R. Kohn, editors, *Topics in Mathematical Modelling of Composite Materials*, pages 45–94. Springer Science+Business Media, LLC, New York. 1997.

[12] M. V. Goncharenko. Asymptotic behavior of the third boundary-value problem in domains with fine-grained boundaries. In A. Damlamian, editor, *Proceedings of the Conference "Homogenization and Applications to Material Sciences" (Nice, 1995)*, volume GAKUTO of *GAKUTO*, pages 203–213. Gakkötosho, Tokyo, 1997.

[23] M. N. Zubova and T. A. Shaposhnikova. Homogenization of boundary value problems in perforated domains with the third boundary condition and the resulting change in the character of the nonlinearity in the problem. *Differential Equations*, 47(1):78–90, 2011.

Our main goal is to extend our previous results [9], formulated for there for some non necessarily Lipschitz functions σ and $p \in [2, n)$, to the case a general maximal monotone graph σ and $p \in (1, n)$. Moreover, we shall present the full details of the proofs (in contrast with the short note [9]).

[9] J. I. Díaz, D. Gómez-Castro, A. V. Podol'skii, and T. A. Shaposhnikova. Homogenization of the p-Laplace operator with nonlinear boundary condition on critical size particles: identifying the strange terms for some non smooth and multivalued operators. *Doklady Mathematics*, 94(1):387–392, 2016.

A relevant application of our results is the following. Let us consider the usual formulation in Chemical Engineering (see [22, 7]) with a constant external supply

$$\begin{cases} -\Delta w_\varepsilon = 0 & x \in \Omega_\varepsilon, \\ \partial_\nu w_\varepsilon + \varepsilon^{-\gamma} g(w_\varepsilon) = 0 & x \in S_\varepsilon, \\ w_\varepsilon = 1 & x \in \partial\Omega. \end{cases} \tag{0.12}$$

where g is a nondecreasing real function such that $g(0) = 0$.

$$u = 1 - w \qquad \sigma(u) = g(1) - g(1-u)$$

the problem becomes

$$\begin{cases} -\Delta u_\varepsilon = 0 & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} \sigma(u_\varepsilon) = \varepsilon^{-\gamma} g(1) & x \in S_\varepsilon, \\ u_\varepsilon = 0 & x \in \partial\Omega. \end{cases} \tag{0.13}$$

We will see later (Theorem 5.2) that the new equation for H, when $\alpha = \frac{n}{n-2}$, is

$$\frac{n-2}{C_0} H(s) = \sigma(s - H(s)) - g(1) \tag{0.14}$$

that is

$$H(u) = -\left(g^{-1}\left(\frac{n-2}{C_0} \cdot \right) + Id \right)^{-1} (1-u), \tag{0.15}$$

so that $w_\varepsilon \rightharpoonup w_{\text{crit}}$ the solution of

$$\begin{cases} -\Delta w_{\text{crit}} + \mathcal{A} h(w_{\text{crit}}) = 0 & \Omega, \\ w_{\text{crit}} = 1 & \partial\Omega \end{cases} \tag{0.16}$$

and h is given by

$$h(w) = \left(g^{-1} \left(\frac{n-2}{n} \cdot \right) + Id \right)^{-1} (w). \tag{0.17}$$

In the noncritical cases, $1 < \alpha < \frac{n}{n-2}$, we will show that $w_\varepsilon \rightharpoonup w_{\text{non-crit}}$ the solution of

$$\begin{cases} -\Delta w_{\text{non-crit}} + \hat{\mathcal{A}} g(w_{\text{non-crit}}) = 0 & \Omega, \\ w_{\text{non-crit}} = 1 & \partial\Omega, \end{cases} \tag{0.18}$$

with $\hat{\mathcal{A}} = C_0^{n-1} |\partial G_0|$. Finally, we shall show, in Theorem 5.3

$$w_{\text{crit}} \geq w_{\text{non-crit}} \tag{0.19}$$

so we have a pointwise "better" reaction in the critical case.

2. Statement of the main results

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\ S_\varepsilon = \partial G_\varepsilon,\ \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon. \quad G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j,$$

We define the space $W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ as the completion, with respect to the norm of $W^{1,p}(\Omega_\varepsilon)$, of the set of infinitely differentable functions in $\overline{\Omega}_\varepsilon$ equal to zero in a neighborhood of $\partial\Omega$

$$W_0^{1,p}(\Omega_\varepsilon, \partial\Omega) = \left\{u \in W^{1,p}(\Omega_\varepsilon) : u = 0 \text{ on } \partial\Omega\right\}. \tag{1.2}$$

Concerning the solvability of problem (0.1) we start by introducing the notion of weak solution. We recall that by well-known results (see, e.g., [2]) since we assume that $\sigma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ verifies that

$$\sigma \text{ is a maximal monotone graph of } \mathbb{R}^2,\ 0 \in \sigma(0), \tag{1.3}$$

there exists a function $\Psi : \mathbb{R} \to (-\infty, +\infty]$ such that Ψ is convex lower semicontinuous function with $\Psi(0) = 0$, such that $\sigma = \partial\Psi$ is its subdifferential. We also know that if we define

$$D(\sigma) = \{r \in \mathbb{R} \text{ such that } \sigma(r) \neq \emptyset\},$$

where $\emptyset$ denotes the empty set, and

$$D(\Psi) = \{r \in \mathbb{R} \text{ such that } \Psi(r) < +\infty\},$$

then $D(\sigma) \subset D(\Psi) \subset \overline{D(\Psi)} = \overline{D(\sigma)}$.

In the rest of the paper we shall always assume that

$$f \in L^{p'}(\Omega), \tag{1.4}$$

where, as usual, $p' = p/(p-1)$.

Since u_ε is the minimizer of the following energy functional in $W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ (see [15, 1])

$$E(u) = \int_{\Omega_\varepsilon} |\nabla u|^p + \varepsilon^{-\gamma} \int_{S_\varepsilon} \Psi(u) - \int_{\Omega_\varepsilon} fu, \tag{1.5}$$

we consider the following definition of weak solution

Definition 1.1. We will say that $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of problem (0.1) if, $u_\varepsilon(x) \in D(\Psi)$ for a.e. $x \in S_\varepsilon$, and, for all $v \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$, we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon . \nabla(v - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon)) dA \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon) dx. \tag{1.6}$$

The existence and uniqueness of a weak solution to problem (1.6) is an easy consequence of well-known results:

Proposition 1.1. *There exists a unique $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ weak solution of (1.6). Besides, there exists $K > 0$ independent on ε such that*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \|\Psi(u_\varepsilon)\|_{L^1(S_\varepsilon)} \leq K. \tag{1.7}$$

The homogenized problem will involve the function $H : \mathbb{R} \to \mathbb{R}$ given by (0.6). Let us present some of the properties satisfied by H.

Lemma 1.1. *If σ satisfies (1.3) then function H defined by (0.6) is a nondecreasing nonexpansion on $\mathbb{R}$ (i.e. a nondecreasing Lipschitz continuous function, of Lipschitz constant $L \leq 1$). Moreover, this function H is the unique function $H : \mathbb{R} \to \mathbb{R}$ satisfying the relation*

$$\mathcal{B}_0 |H(r)|^{p-2} H(r) \in \sigma(r - H(r)), \text{ for any } r \in \mathbb{R}. \tag{1.8}$$

Homogenized problem

$$\begin{cases} -\Delta_p u + \mathcal{A}|H(u)|^{p-2}H(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.4}$$

Concerning the homogenized problem (0.4) we point out that since H is a nondecreasing nonexpansion on $\mathbb{R}$ then, given the parameters $\mathcal{A}$ and $\mathcal{B}_0$ given by (0.5) and (0.8) and $f \in L^{p'}(\Omega)$, there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ of problem (0.4). Moreover $|H(u)|^{p-2}H(u) \in L^{p'}(\Omega)$. For the proof it is enough to define $V = W_0^{1,p}(\Omega)$ and the operator $A : V \to V'$ by

$$\langle Av, w \rangle = \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla w dx + \int_\Omega \mathcal{A}|H(v)|^{p-2}H(v) w dx \tag{1.9}$$

for any $w \in V$. Notice that, since H is Lipschitz, then $H(v) \in L^p(\Omega)$ for any $v \in L^p(\Omega)$. Then A is a hemicontinuous strictly monotone coercive operator and the existence and uniqueness of a weak solution u is standard (see, e.g., [15]).

We will make fundamental use of the following reformulation of weak solution. Since the *limit operator* $A : V \to V'$, with $V = W_0^{1,p}(\Omega)$, given by (1.9) is hemicontinuous and monotone we can use the Brezis-Sibony characterization (see Lemme 1.1 of [3], or Theoreme 2.2, Chapter 2 of [15]): $u \in W_0^{1,p}(\Omega)$ is a weak solution of (0.4) if and only if

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u)dx + \int_\Omega \mathcal{B}_0|H(v)|^{p-2}H(v)(v-u)dx \geq \int_\Omega f(v-u)dx \quad (1.10)$$

for any $v \in W_0^{1,p}(\Omega)$.

[3] H. Brézis and M. Sibony. Méthodes d'approximation et d'itération pour les opérateurs monotones. *Archive for Rational Mechanics and Analysis*, 28(1):59–82, 1968.

[15] J. L. Lions. *Quelques méthodes de résolution des Problèmes aux Limites non Linéaires*. Dunod, Paris, 1969.

The main result of this paper is the following convergence result:

Theorem 1.1. *Let $n \geq 3, 1 < p < n$, $\alpha = \frac{n}{n-p}$, $\gamma = \alpha(p-1)$. Let σ be any maximal monotone graph of $\mathbb{R}^2$ with $0 \in \sigma(0)$ and let $f \in L^{p'}(\Omega)$. Let $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ be the (unique) weak solution of problem (0.1). Then there exists an extension $\tilde{u}_\varepsilon$ of u_ε such that $\tilde{u}_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$ where $u \in W_0^{1,p}(\Omega)$ is the (unique) weak solution of the problem (0.4) associated to the function H, defined by (0.6).*

The other key result we will prove in this paper is Theorem 1.2 below, the statement of which requires some preliminary lemmas. The extension $\tilde{u}_\varepsilon$ of solutions u_ε can be obtained by applying the methods of [18]:

[18] A. V. Podol'skii. Solution continuation and homogenization of a boundary value problem for the p-Laplacian in a perforated domain with a nonlinear third boundary condition on the boundary of holes. *Doklady Mathematics*, 91(1):30–34, 2015.

Lemma 1.2. *Let Ω_ε be the domain defined above and let $1 < p < n$, $n \geq 3$. Then, an extension operator $P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega)$ such that*

$$\|P_\varepsilon u\|_{W^{1,p}(\Omega)} \leq C_1 \|u\|_{W^{1,p}(\Omega_\varepsilon)} \tag{1.11}$$

$$\|\nabla(P_\varepsilon u)\|_{L_p(\Omega)} \leq C_2 \|\nabla u\|_{L_p(\Omega_\varepsilon)}. \tag{1.12}$$

Moreover, by applying this extension theorem and the methods introduced in [18] we can prove following useful estimates:

Lemma 1.3. *i) Let $u \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ and $p > 1, n \geq 3$. Then there exists positive constant C such that the following inequality is valid*

$$\|u\|_{L^p(\Omega_\varepsilon)} \leq C \|\nabla u\|_{L^p(\Omega_\varepsilon)}. \tag{1.13}$$

ii) Let $u \in W^{1,p}(Y_\varepsilon)$ such that $\int_{Y_\varepsilon} u = 0$. Then

$$\|u\|_{L^p(Y_\varepsilon)} \leq K_1 \varepsilon \|\nabla u\|_{L^p(Y_\varepsilon)}, \tag{1.14}$$

where constant K_1 is independent of ε.

Thanks to the *a priori* estimate (1.7) and the properties of the above extension operator $P_\varepsilon : W_0^{1,p}(\Omega_\varepsilon, \partial\Omega) \to W_0^{1,p}(\Omega)$ we know that and there exists $u \in W_0^{1,p}(\Omega)$ such that

$$P_\varepsilon u_\varepsilon \rightharpoonup u, \qquad \text{in } W_0^{1,p}(\Omega). \tag{1.15}$$

The difficult task is to show that $u \in W_0^{1,p}(\Omega)$ is the weak solution of problem (0.4) such as it is ensured in Theorem 1.1.

Motivated by this and (1.10), we shall also use the fact that if $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is the weak solution of problem (0.1) then

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v.\nabla(v - u_\varepsilon)dx + \varepsilon^{-v}\int_{S_\varepsilon}(\Psi(v) - \Psi(u_\varepsilon))dA \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon)dx \tag{1.16}$$

for any test function $v \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$.

The problematic term, in order to pass to the limit, is the boundary integrals over S_ε. Here we shall follow a technique of proof introduced by the last author (T.A. Shaposhnikova) in collaboration with different co-authors (see, e.g. Oleinik-Shaposhnikova [16], Shaposhnikova-Zubova [23] and Shaposhnikova-Podol'skii [21]) which can be applied in different frameworks.

Lemma 1.4. *Let $z_\varepsilon \in W_0^{1,p}(\Omega)$, for some $p > 1$, and assume that $z_\varepsilon \rightharpoonup z_0$ in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$. Then,*

$$\left| 2^{2(n-1)}\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} z_\varepsilon dS - \omega_n \int_\Omega z_0 dx \right| \to 0, \qquad as\ \varepsilon \to 0, \tag{1.17}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

This lemma (which we remark is independent of α and γ, see the proof in [23]), is the key point of the homogenization technique in the critical critical case. It is based in the general idea that if P^j_ε is the center of the ball $G^j_\varepsilon = \{x \in Y^j_\varepsilon : |x - P^j_\varepsilon| < a_\varepsilon\}$ and if T^j_ε denotes the ball of radius $\varepsilon/4$ centered at the point P^j_ε then we can get several explicit estimates on the solution $w^j_\varepsilon(x)$ for $j = 1, \ldots, N(\varepsilon)$ of the auxiliary cellular boundary value problem

$$\begin{cases} \Delta_p w^j_\varepsilon = 0 & x \in T^j_\varepsilon, \setminus \overline{G^j_\varepsilon}, \\ w^j_\varepsilon = 1 & x \in \partial G^j_\varepsilon, \\ w^j_\varepsilon = 0 & x \in \partial T^j_\varepsilon. \end{cases} \tag{1.18}$$

One of the many remarkable properties of this cellular problem is that its (unique) weak solution, w^j_ε, is radially symmetric (recall that G_0 is a ball) and satisfies that $\partial_{\nu_p} w^j_\varepsilon$ is constant on ∂T^j_ε and on ∂G^j_ε. Its weak formulation reads

$$\int_{G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2}\nabla w_\varepsilon^j \nabla z ds = \int_{\partial T_\varepsilon^j} z\,\partial_{\nu_p} w_\varepsilon^j ds + \int_{\partial G_\varepsilon^j} z\,\partial_{\nu_p} w_\varepsilon^j ds, \tag{1.19}$$

for any $z \in W^{1,p}(T_\varepsilon^j \setminus \overline{G_\varepsilon^j})$. Furthermore, we can make explicitly several computations. Hence, we have an explicit way to compare the reaction term on S_ε with an auxilary term on balls with radius $C\varepsilon$, and Lemma 1.4 becomes very useful.

Another key idea of our proof is to relate a general test function $v \in W_0^{1,p}(\Omega)$, used to check the limit characterization (1.10), with some suitable *correction*, v_ε, which is a better fitted test function in the microscopic weak formulation (1.16).

In fact, by density, it will be enough to do that with a smooth test function $v \in \mathcal{C}_c^\infty(\Omega)$. We will construct such adaptation among test functions in the form $v_\varepsilon = v - hW_\varepsilon$ where, for the moment $h \in W^{1,\infty}(\Omega)$ without any other property, and, which is crucial, $W_\varepsilon \in W_0^{1,\infty}(\Omega)$ is defined as

$$W_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j},\ j = 1, \ldots, N(\varepsilon) = |\Upsilon_\varepsilon|; \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j=1}^{N(\varepsilon)} T_\varepsilon^j, \end{cases} \tag{1.20}$$

with w_ε^j the solution of the auxiliary cellular boundary value problem (1.18). The following technical result will explain why function H arising in the limit problem (0.4) was taken in this concrete form (more precisely, such that (1.8) holds), different from the boundary kinetics σ.

Theorem 1.2. *Let $1 < p < n$ and $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ be a sequence of uniformly bounded norm, $v \in C_c^\infty(\Omega)$, $h \in W^{1,\infty}(\Omega)$ and let*

$$v_\varepsilon = v - hW_\varepsilon. \tag{1.21}$$

Then

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon) dx = I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + O(\varepsilon) \tag{1.22}$$

$$I_{1,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla(v - u_\varepsilon) dx \tag{1.23}$$

$$I_{2,\varepsilon} = -\varepsilon^{-\gamma} \mathcal{B}_0 \int_{S_\varepsilon} |h|^{p-2} h(v - h - u_\varepsilon) ds \tag{1.24}$$

$$I_{3,\varepsilon} = -A_\varepsilon \varepsilon \sum_{j \in \Upsilon_\varepsilon} \int_{T_\varepsilon^j} |h|^{p-2} h(v - u_\varepsilon) ds, \tag{1.25}$$

where A_ε is a bounded sequence (see (4.18)). Besides, if $\tilde{u}_\varepsilon$ is an extension of u_ε and $\tilde{u}_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ then, for any $v \in W_0^{1,p}(\Omega)$

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla(v - hW_\varepsilon - u_\varepsilon) dx = \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla(v - u) dx. \tag{1.26}$$

The aforementioned corrector term in the form hW_ε, were $h \in W^{1,\infty}(\Omega_\varepsilon, \partial\Omega)$ will be taken to satisfy the condition $h(x) = H(v(x))$ for a.e. $x \in \Omega$ and H given by equation (1.8). These conditions rise naturally so that the term $I_{2,\varepsilon}$ above cancels out with the reaction term.

Remark 1.2. In general, it is expected that the convergence $\tilde{u}_\varepsilon \to u$ can be improved to strong convergence by adding a corrector term. In fact, if σ is smooth, it is known that $u_\varepsilon - H(u_\varepsilon)W_\varepsilon \to u$ strongly in $W_0^{1,p}(\Omega)$ (see, e.g., [23]). It is possible to adapt the arguments to the case of some maximal monotone graphs as, for instance, the one given by the Signorini boundary condition (see [10]).

[23] M. N. Zubova and T. A. Shaposhnikova. Homogenization of boundary value problems in perforated domains with the third boundary condition and the resulting change in the character of the nonlinearity in the problem. *Differential Equations*, 47(1):78–90, 2011.

[10] J. I. Díaz, D. Gómez-Castro, A. V. Podol'skii, and T. A. Shaposhnikova. Homogenization of variational inequalities of Signorini type for the p-Laplacian in perforated domains when $p \in (1, 2)$. *Doklady Mathematics*, To appear, 2017.

Proof of Lemma 1.1. Let $\Theta_{n,p}(s) = \mathcal{B}_0|s|^{p-2}s$ for $s \in \mathbb{R}$. Since σ^{-1} is also a maximal monotone graph of $\mathbb{R}^2$ then, for any $p > 1$ and $\mathcal{B}_0 > 0$ the graph $\sigma^{-1} \circ \Theta_{n,p}$ is also a maximal monotone graph of $\mathbb{R}^2$. Indeed, let $D(\sigma^{-1}) = [a, b]$ for some $-\infty \leq a < b \leq +\infty$ and let $(\sigma^{-1})^0$ the principal section (i.e. the nondecreasing function) of the graph σ^{-1}. This means that

$$(\sigma^{-1})^0(r) = \inf \sigma^{-1}(r), \qquad r \in [a, b]. \tag{2.7}$$

Then, since $\Theta_{n,p}$ is strictly increasing, $\sigma^{-1} \circ \Theta_{n,p}$ is a monotone graph,

$$D(\sigma^{-1} \circ \Theta_{n,p}) = [\Theta_{n,p}^{-1}(a), \Theta_{n,p}^{-1}(b)],$$

and $(\sigma^{-1} \circ \Theta_{n,p})^0 = (\sigma^{-1})^0 \circ \Theta_{n,p}$.
In particular, if σ^{-1} is multivalued in some point $c \in (a, b)$ then $\sigma^{-1} \circ \Theta_{n,p}(c)$ is the full interval

$$\sigma^{-1} \circ \Theta_{n,p}(c) = [(\sigma^{-1})^0(\Theta_{n,p}(c^-)), (\sigma^{-1})^0(\Theta_{n,p}(c^+))]$$

and this implies that $\sigma^{-1} \circ \Theta_{n,p}$ is a maximal monotone graph of $\mathbb{R}^2$ (see [2] Exemple 2.8.1).
Now, since $\sigma^{-1} \circ \Theta_{n,p}$ is also a maximal monotone graph of $\mathbb{R}^2$ we know that $(I + \sigma^{-1} \circ \Theta_{n,p})$ is an injective application such that $R(I + \sigma^{-1} \circ \Theta_{n,p}) = \mathbb{R}$ (see Brezis [2]). Thus if H is defined by (0.6) then H is a non expansion on $\mathbb{R}$ (see Brezis [2] Proposition 2.2). Hence

$$(I + \sigma^{-1} \circ \Theta_{n,p})(H(r)) = r$$

for any $r \in \mathbb{R}$ and, in consequence,

$$H(r) + \sigma^{-1} \circ \Theta_{n,p}(H(r)) = r.$$

In other words,

$$\sigma^{-1} \circ \Theta_{n,p}(H(r)) = r - H(r).$$

4 Proof of Theorem 1.1

Since G_0 is ball, it is easy to see that

$$w_\varepsilon^j(x) = \frac{|x - P_\varepsilon^j|^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}{(C_0\varepsilon^\alpha)^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}, \qquad x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \tag{3.1}$$

is the unique solution of (1.18). Therefore,

Lemma 3.1. *If W_ε is defined by (1.20) the following estimate holds*

$$\int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^q dx \leq K\varepsilon^{n(p-q)/(n-p)}, \tag{3.2}$$

for any $1 \leq q \leq p$. In particular,

$$W_\varepsilon \rightharpoonup 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0. \tag{3.3}$$

Proof of Theorem 1.1. Let $v \in C_c^\infty(\Omega)$ and $h = H(v)$ with $H : \mathbb{R} \to \mathbb{R}$ given by (0.6). Notice that then $h \in W^{1,\infty}(\Omega)$. Let $v_\varepsilon = v - hW_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ with $W_\varepsilon \in W_0^{1,\infty}(\Omega)$ defined by (1.20). Due to (1.16), we know that u_ε satisfies the inequality

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)dx + \tag{3.4}$$

$$\varepsilon^{-\gamma}\int_{S_\varepsilon}(\Psi(v_\varepsilon) - \Psi(u_\varepsilon)) \tag{3.5}$$

$$\geq \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon). \tag{3.6}$$

From here, by Theorem 1.2, we can deduce, since $W_\varepsilon \to 0$ in $L^p(\Omega)$ (due to the compact inclusion), that

$$\lim_{\varepsilon\to 0}\left[I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\Psi(v_\varepsilon) - \Psi(u_\varepsilon))\right] \geq \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon) = \int_\Omega f(v-u). \tag{3.7}$$

Since $H : \mathbb{R} \to \mathbb{R}$ satisfies (1.8), by applying that, if $\xi \in \partial\Psi(s_0) = \sigma(s_0)$ then $\Psi(s) - \Psi(s_0) \geq \xi(s - s_0)$, we can write

$$\begin{aligned} &I_{2,\varepsilon} + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\Psi(v_\varepsilon) - \Psi(u_\varepsilon)) \\ &= \varepsilon^{-\gamma}\int_{S_\varepsilon}\left[\Psi(v - H(v)) - \Psi(u_\varepsilon) - \mathcal{B}_0|H(v)|^{p-2}H(v)(v - H(v) - u_\varepsilon)\right]dA \\ &\leq 0, \end{aligned} \tag{3.8}$$

since $\mathcal{B}_0|H(v(x))|^{p-2}H(v(x)) \in \sigma(v(x) - H(v(x)))$ for any $x \in \overline{\Omega}$. We can pass also to the limit in (1.23) and (1.25) to get that

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla(v-u)dx + \int_\Omega \mathcal{B}_0|H(v)|^{p-2}H(v)(v-u)dx \geq \int_\Omega f(v-u)dx, \quad (3.9)$$

and since $v \in C_c^\infty(\Omega)$ is arbitrary, by density, this also holds for every $v \in W_0^{1,p}(\Omega)$. Hence, we get that u is the unique weak solution of (0.4). □

5 Proof of Theorem 1.2

The proof of Theorem 1.2 for $p = 2$ can be found in [24] and for $2 < p < n$ in [21].

[24] M. N. Zubova and T. A. Shaposhnikova. Averaging of boundary-value problems for the Laplace operator in perforated domains with a nonlinear boundary condition of the third type on the boundary of cavities. *Journal of Mathematical Sciences*, 190(1):181–193, 2013.

[21] T. A. Shaposhnikova and A. V. Podolskiy. Homogenization limit for the boundary value problem with the with the p-Laplace operator and a nonlinear third boundary condition on the boundary of the holes in a perforated domain. *Functional Differential Equations*, 19(3-4):1–20, 2012.

Lemma 4.1. *([10]) Let $1 < p < 2$. Then there exists positive constant $C = C(p)$ such that inequality*

$$||\mathbf{a}-\mathbf{b}|^{p-2}(\mathbf{a}-\mathbf{b}) - (|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b})| \leq C(|\mathbf{a}||\mathbf{b}|)^{\frac{p-1}{2}}, \quad (4.1)$$

is valid for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

By using this result we prove following proposition.

Lemma 4.2. *Let* $1 < p < 2$, $n \geq 3$, $v \in W_0^{1,\infty}(\Omega)$ *and* $\varphi \in W_0^{1,p}(\Omega)$. *Let* $\eta_\varepsilon \in W^{1,p}(\Omega)$ *be such that* $\|\nabla \eta_\varepsilon\|_{L^q(\Omega)} \to 0$, *for some* $q \in [1,p)$, *as* $\varepsilon \to 0$. *Then*

$$\int_{\Omega_\varepsilon} |\nabla(v-\eta_\varepsilon)|^{p-2}\nabla(v-\eta_\varepsilon)\cdot\nabla\varphi dx$$

$$= \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi dx - \int_{\Omega_\varepsilon} |\nabla \eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon\cdot\nabla\varphi dx + O(\varepsilon). \qquad (4.2)$$

Proof. By Lemma 4.1, by applying Hölder's inequality, we have

$$\left| \int_{\Omega_\varepsilon} |\nabla(v-\eta_\varepsilon)|^{p-2}\nabla(v-\eta_\varepsilon).\nabla\varphi dx - \left(\nabla v|^{p-2}\nabla v - |\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon\right).\nabla\varphi dx \right| \qquad (4.3)$$

$$\leq C\int_{\Omega_\varepsilon} |\nabla v|^{\frac{p-1}{2}}|\nabla\eta_\varepsilon|^{\frac{p-1}{2}}|\nabla\varphi| dx \qquad (4.4)$$

$$\leq K\|\nabla v\|_\infty^{\frac{p-1}{2}}\|\nabla\eta_\varepsilon\|_{L^{\frac{p+1}{2}}(\Omega_\varepsilon)}^{\frac{p-1}{2}}\|\nabla\varphi\|_{L^{\frac{p+1}{2}}(\Omega_\varepsilon)}, \qquad (4.5)$$

since $1 < (p+1)/2 < p$. This proves the result. □

Proof of Theorem 1.2. As said before, it is enough to consider the case $p \in (1, 2)$. Applying Lemma 4.2 we obtain

$$\int\limits_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon) dx = J_{1,\varepsilon} + J_{2,\varepsilon} + O(\varepsilon) \tag{4.6}$$

$$J_{1,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v . \nabla(v - hW_\varepsilon - u_\varepsilon) dx \tag{4.7}$$

$$J_{2,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla(hW_\varepsilon)|^{p-2} \nabla(hW_\varepsilon) . \nabla(v - hW_\varepsilon - u_\varepsilon) dx. \tag{4.8}$$

Moreover

$$J_{1,\varepsilon} = I_{1,\varepsilon} + \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v . \nabla(hW_\varepsilon) = I_{1,\varepsilon} + O(\varepsilon). \tag{4.9}$$

(4.10)

On the other hand,

$$J_{2,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2} \nabla W_\varepsilon . \nabla(v - hW_\varepsilon - u_\varepsilon) + O(\varepsilon) \tag{4.11}$$

$$= \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2} \partial_\nu w^j_\varepsilon |h|^{p-2} h (v - u_\varepsilon) ds \tag{4.12}$$

$$+ \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2} \partial_\nu w^j_\varepsilon |h|^{p-2} h (v - h - u_\varepsilon) ds, \tag{4.13}$$

where $\partial_\nu g$ is the usual normal derivative of g. Using (3.1), we get:

$$\partial_\nu w_\varepsilon^j\Big|_{\partial T_\varepsilon^j} = \frac{d}{dr} w_\varepsilon^j\Big|_{r=\varepsilon/4} = -\frac{(n-p)2^{\frac{2n-2}{p-1}} C_0^{\frac{n-p}{p-1}} \varepsilon^{\frac{1}{p-1}}}{(p-1)(1-(C_0\varepsilon^\alpha)^{\frac{n-p}{p-1}}\varepsilon^{-\frac{n-p}{p-1}} 2^{\frac{2n-2p}{p-1}})}, \tag{4.14}$$

$$\partial_\nu w_\varepsilon^j\Big|_{\partial G_\varepsilon^j} = -\frac{d}{dr} w_\varepsilon^j\Big|_{r=a_\varepsilon} = \frac{(n-p)\varepsilon^{\frac{-n}{n-p}}}{(p-1)C_0(1-(C_0\varepsilon^\alpha)^{\frac{n-p}{p-1}}\varepsilon^{-\frac{n-p}{p-1}} 2^{\frac{2n-2p}{p-1}})}. \tag{4.15}$$

Therefore

$$J_{2,\varepsilon} = A_\varepsilon \varepsilon \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T_\varepsilon^j} |h|^{p-2} h(v-u_\varepsilon)ds- \tag{4.16}$$

$$-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\left(\left(\frac{n-p}{p-1}\right)^{p-1} C_0^{1-p}|h|^{p-2}h\right)(v-h-u_\varepsilon)ds-$$

$$-Q_\varepsilon + O(\varepsilon), \tag{4.17}$$

where

$$A_\varepsilon = \left(\frac{n-p}{p-1}\right)^{p-1} \frac{2^{2n-2}C_0^{n-p}}{(1-(C_0\varepsilon^\alpha)^{\frac{n-p}{p-1}}\varepsilon^{-\frac{n-p}{p-1}} 2^{\frac{2n-2p}{p-1}})^{p-1}}, \tag{4.18}$$

$$Q_\varepsilon = \frac{1-(1-a_\varepsilon^{\frac{n-p}{p-1}}\varepsilon^{\frac{p-n}{p-1}}2^{\frac{2n-2p}{p-1}})^{p-1}}{(1-a_\varepsilon^{\frac{n-p}{p-1}}\varepsilon^{\frac{p-n}{p-1}}2^{\frac{2n-2p}{p-1}})^{p-1}C_0^{p-1}}\left(\frac{n-p}{p-1}\right)^{p-1}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}|h|^{p-2}h(v-h-u_\varepsilon)ds. \tag{4.19}$$

It is an easy (but tedious) task to check that

$$\lim_{\varepsilon \to 0} Q_\varepsilon = 0, \tag{4.20}$$

which concludes the proof. □

6. Non critical case and pointwise comparison of homogenized solutions with the critical case

Theorem 5.1. *Let* $n \geq 3, p \in [2, n)$, $1 < \alpha < \frac{n}{n-p}$, $f \in L^\infty(\Omega)$, $r \in \mathcal{C}(\bar{\Omega})$, $\sigma \in \mathcal{C}(\mathbb{R})$ *and* u_ε *be the solution of*

$$\begin{cases} -\Delta_p u_\varepsilon = 0 & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = \varepsilon^{-\gamma} r & x \in S_\varepsilon, \end{cases} \tag{5.1}$$

Then $\tilde{u}_\varepsilon \rightharpoonup u_{non\text{-}crit}$ *in* $W_0^{1,p}(\Omega)$, *the solution of*

$$\begin{cases} -\Delta_p u + \hat{\mathcal{A}}\sigma(u) = f + \hat{\mathcal{A}} r & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{5.2}$$

with $\hat{\mathcal{A}} = C_0^{n-1}|\partial G_0|$.

Proof. Assume first that

$$0 < k_1 \leq \sigma' \leq k_2 \tag{5.3}$$

then the result holds by Theorem 3 in [21].
Applying the estimates in [17] we check that $(P_\varepsilon u_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$, hence there exists a limit $\hat{u}$ such that, up to a subsequence, $P_\varepsilon u_\varepsilon \to \hat{u}$ strongly in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$.
Let M be such that $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq M$ (see [6]). Let σ_δ be a sequence such that $0 < k_{1,\delta} \leq \sigma'_\delta \leq k_{2,\delta}$ and $\sigma_\delta \to \sigma$ in $\mathcal{C}([-M, M])$ as $\delta \to 0$. Let $u_{\varepsilon,\delta}$ be the solution of (5.1) with σ_δ. We can check, again with estimates in [17], that

$$\|u_\varepsilon - u_{\varepsilon,\delta}\|_{L^p(\Omega_\varepsilon)} \leq C\|\sigma - \sigma_\delta\|_{\mathcal{C}([-M,M])}. \tag{5.4}$$

Passing to the limit as $\varepsilon \to 0$, indicating that $P_\varepsilon u_{\varepsilon,\delta} \rightharpoonup u_\delta$ in $W_0^{1,p}(\Omega)$, where u_δ is the solution of (5.2) with σ_δ, we have that

$$\|\hat{u} - u_\delta\|_{L^p(\Omega)} \leq C\|\sigma - \sigma_\delta\|_{\mathcal{C}([-M,M])}. \tag{5.5}$$

It is easy to check that $u_\delta \to u$ in $L^p(\Omega)$ where u is the solution of the problem with σ. Therefore $P_\varepsilon u_\varepsilon \to u$ in $L^p(\Omega)$ as $\varepsilon \to 0$ and $u = \hat{u}$. $\square$

Theorem 5.2. *Let $n \geq 3, p \in [2,n)$, $\alpha = \frac{n}{n-p}$, $f \in L^\infty(\Omega)$, $r \in \mathcal{C}(\bar{\Omega})$, $\sigma \in \mathcal{C}(\mathbb{R})$ and u_ε be the solution of (5.1). Then $\tilde{u}_\varepsilon \rightharpoonup u_{crit}$ in $W_0^{1,p}(\Omega)$ the solution of*

$$\begin{cases} -\Delta_p u + \mathcal{A}H(x,u) = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{5.6}$$

and H is the solution of

$$\mathcal{B}_0 |H(x,s)|^{p-2} H(x,s) = \sigma(s - H(x,s)) - r(x) \tag{5.7}$$

a.e. in Ω.

Sketch of proof. We can apply the same reasoning as before and the fact that $H_\delta \to H$, in the sense of maximal monotone graphs, as $\sigma_\delta \to \sigma$ in $\mathcal{C}([-M,M])$. □

Theorem 5.3. *Assume the conditions of the two previous theorems, $0 \leq f \leq 1$ and $r(x) \equiv g(1) = 1$ constant. Then, we have that*

$$u_{crit} \leq u_{non\text{-}crit}. \tag{5.8}$$

Proof. The condition on f and r guarantee that $0 \leq u \leq 1$ in both cases. It is easy to check that H is increasing, and $H(s) \leq 0$ for $s \in [0,1]$. It is easy to establish the following inequality on the zero order terms

$$\mathcal{B}_0 |H(s)|^{p-2} H(s) \geq \hat{\mathcal{A}}(\sigma(s) - g(1)). \tag{5.9}$$

Therefore, applying the comparison principle (see, e.g., [6]) we have the result. □

Thanks for your attention