Theoretical advances in the fixed-boundary gravimetric boundary value problem

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SESSION 25: MATHEMATICS OF PLANET EARTH

1. Mathematics of the Planet Earth.

International Union of Geodesy and Geophysics (IUGG)

International Association of Cryospheric Sciences (IACS)

International Association of Geodesy (IAG)

International Association of Geomagnetism and Aeronomy (IAGA)

International Association of Hydrological Sciences (IAHS)

International Association of Meteorology and Atmospheric Sciences (IAMAS)

International Association for the Physical Sciences of the Oceans (IAPSO)

International Association of Seismology and Physics of the Earth's Interior (IASPEI)

International Association of Volcanology and Chemistry of the Earth's Interior (IAVCEI)

Commission on Geophysical Risk and Sustainability (GeoRisk)

Commission on Mathematical Geophysics (CMG)

Committee on the Study of Earth's Deep Interior (SEDI)

Union Commission for Data and Information (UCDI)

[International Council for Science (ICSU)]

International Mathematical Union (IMU)

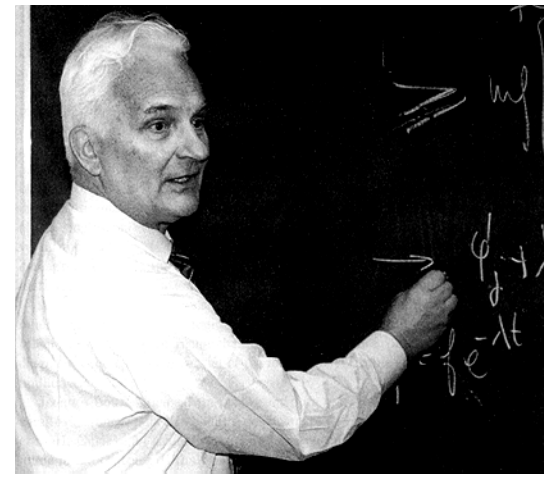
International Astronomical Union (IAM)

J.L. Lions:

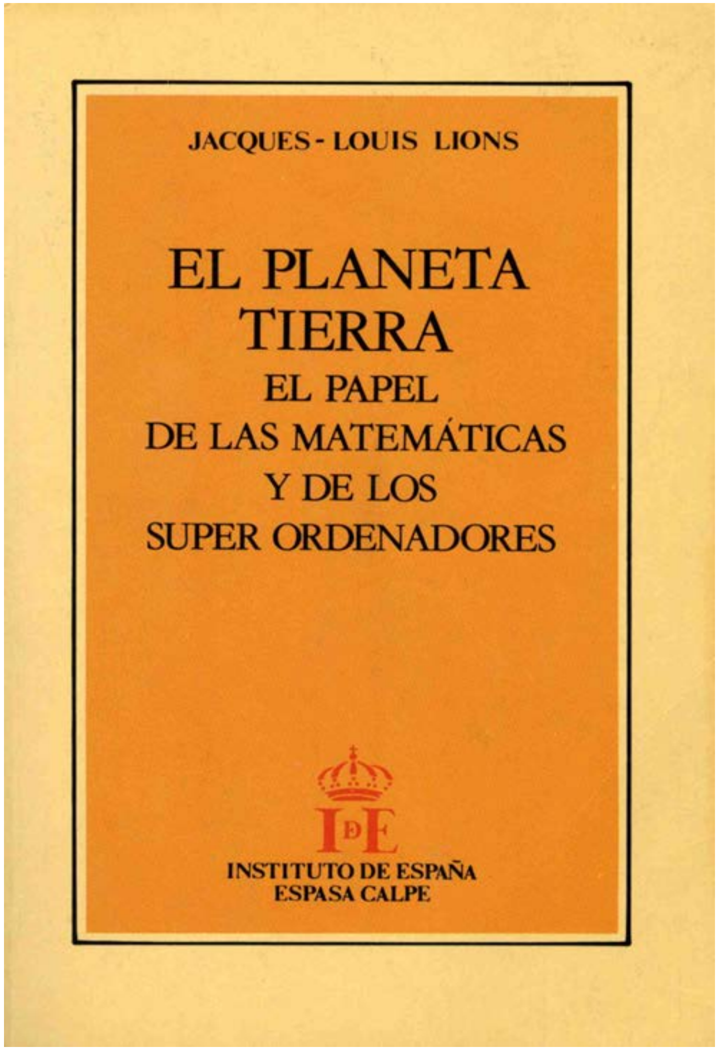
a mathemaician with a global concept of Mathematics and Science

Jacques-Louis Lions
(2 mai 1928 - 16 mai 2001)

1947-50 Élève puis agrégé pré-parateur à l'E.N.S.
1951-54 Attaché de recherches au CNRS.
Ses enseignements :
54-63 : Faculté des sciences de Nancy
63-78 : Faculté des sciences de Paris
66-86 : École Polytechnique
73-98 : Collège de France
Ses présidences :
80-84 : INRIA
84-92 : CNES
91-94 : Union Mathématique Internationale
96-98 : Académie des Sciences



Proposer (as President of the IMU) of the WORLD MATHEMATICAL YEAR 2000



Lions, J.-L., El planeta Tierra. El papel de las Matemáticas y de los superordenadores. Espasa-Calpe, Madrid, 1990 (M.Artola, J.I.D.)

Desde la realización de la máquina de calcular de Blaise Pascal, hace ya dos siglos y medio, se cuenta con una metodología general para el estudio predictivo y cuantitativo de *sistemas complejos*. Tal estudio reposa en tres pilares de carácter universal: modelización y análisis matemático; cálculo numérico y simulación por medio de (super) ordenadores; y acción sobre el sistema con el fin de asegurar el funcionamiento deseado (teoría de Control).

¿Es aplicable esta *trilogía* al **Sistema del Planeta Tierra?** ¿Es posible su comprensión con el fin de actuar sobre él? ¿Qué contribuciones puede aportar esta *trilogía* al estudio de la evolución climatológica?

El autor analiza los aspectos matemáticos de estas cuestiones presentándolos de una manera fácilmente accesible. En un apéndice final, con la colaboración de J. I. Díaz, el autor presenta una lista de problemas aún abiertos y que previsiblemente serán objeto de atención en próximos años.

J. L. Lions es miembro de la Academia de Ciencias de París y profesor en el College de France desde 1973. En la actualidad es también presidente del Centre National d'Études Spatiales y miembro de los consejos científicos de Electricité de France y de la Meteorología Nacional Francesa. Recientemente ha sido elegido presidente de la Unión Matemática Internacional. Es miembro de numerosas academias de otros paises: URSS, Brasil, Bélgica, Portugal, Academia de Boston, etc. Es también doctor *Honoris Causa* por numerosas universidades, entre ellas las Universidades Complutense y Politécnica de Madrid y de Santiago de Compostela.

My collaboration with J.L. Lions: UCM Summer Courses

(El Escorial 1992, Almería 1993)

Díaz, J.I., Lions, J.-L., eds., ***Mathematics, Climate and Environment***, Research Notes in Applied Mathematics 27, Masson, Paris, 1993.

Environment, ***Economics and Their Mathematical Models***, Research Notes in Applied Mathematics 35, Masson, Paris, 1994.

NATO Advanced Study Institute (Santa Cruz de Tenerife (Spain), January, 11-21, 1995).

Díaz, J. I., ed., ***The Mathematics of Models for Climatology and Environement***, NATO ASI Series, Springer Verlag, 1997.

Committee on Atmosphere, Ocean and Environment of the **International Mathematical Union**, June 1995- January 2004
(other members: C. Conca, J. Bona, M. Mimura and A.J. Majda (Chairman)).

J. I. Díaz (Edit.), Ocean Circulation and Pollution Control. A Mathematical and Numerical Inquiry, Lecture Notes, EMS Volume, Springer-Verlag 2003
(2nd Diderot Videoconference of the European Mathematical Society [Amsterdam-Madrid-Venice]

SEMINARIO INTERNACIONAL COMPLUTENSE

EARTH SCIENCES AND MATHEMATICS

Madrid, September, 13, 14 and 15, 2006

Organizing Committee
J. I. Díaz (UCM), J. Fernández (CSIC-UCM), J. A.Tejada (UCM), F. Luzón Martínez (Univ. Almería)

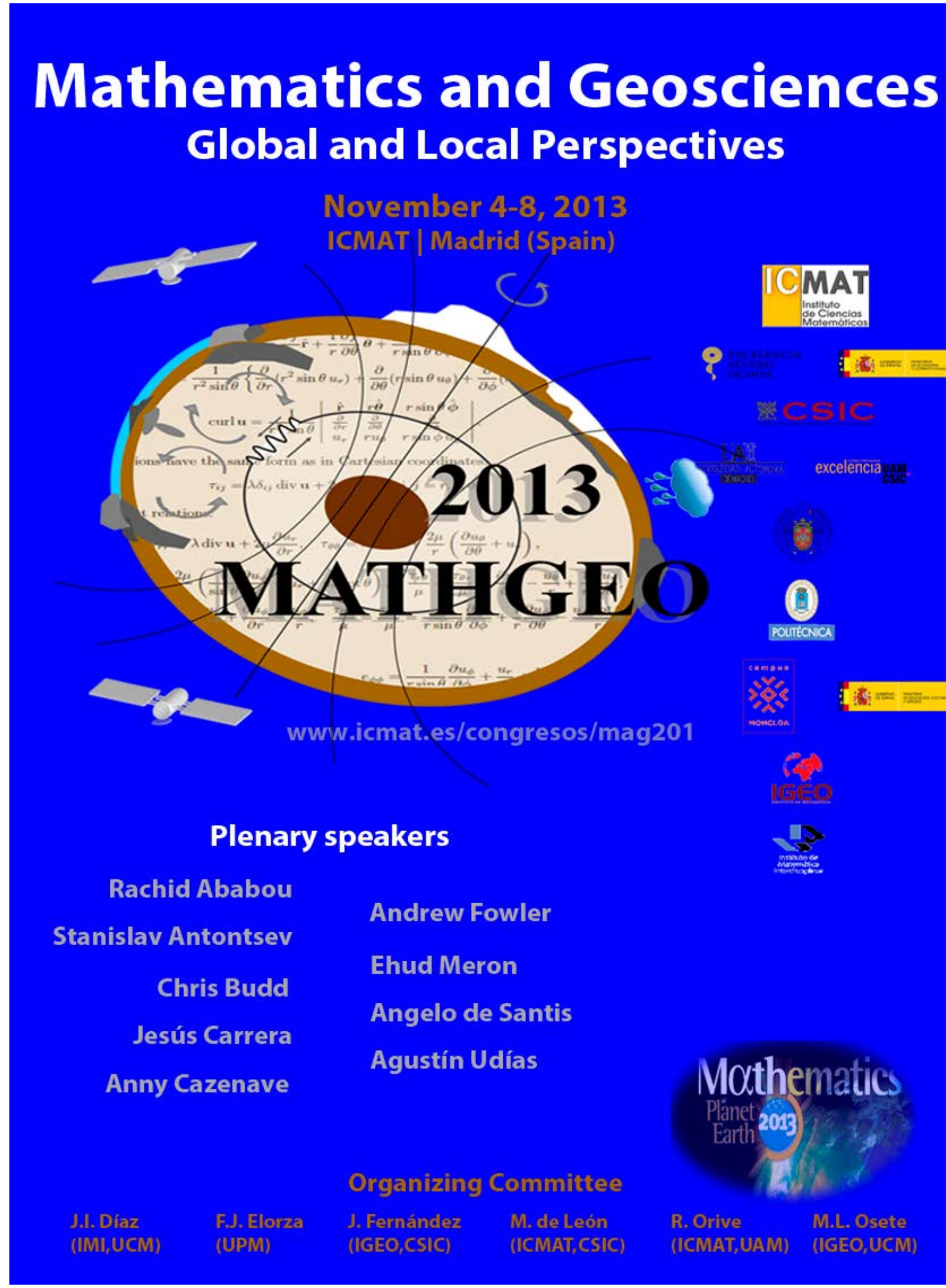
A. G. Camacho, J. I. Díaz, J. Fernández
Earth Sciences and Mathematics, Volume I and II
Birkhauser, 2008.

H. Brezis and J. I. Díaz, Matemáticas y Medio Ambiente, Volumen especial de Rev. R. Acad. Cien.Serie A Matem, 96, nº 3, 2003,

(Joint conference: Academie des Sciences (France), Real Academia de Ciencias (Spain), Paris, June 2002)

J.I. Díaz, A. Elipe, A. Quarteroni and L. Rández (Editors)
Maths and Water
Monografías de la Real Academia de Ciencias de Zaragoza, **31,** Zaragoza, 2009

A 2013 initiative in Spain.



2. On the fixed-boundary gravimetric boundary value problem

G. Díaz, J. I. Díaz, J. Otero. Some remarks on the Backus problem in Geodesy. ***Física de la Tierra***: ***Observaciones Geodésicas y Gravimetria***, Vol.8, No 8, 179-194, 1996

G. Díaz, J. I. Díaz, J. Otero. On an oblique boundary value problem related to the Backus problem in Geodesy, ***Nonlinear Analysis: Real World Applications***, 7 (2006) 147-166

G. Díaz, J. I. Díaz, J. Otero: Construction of the maximal solution of Backus' problem in Geodesy and Geomagnetism. ***Stud. Geophys. Geod***., 55 (2011), 415−440

IAMG 2013: New results on the degenerate case (geophysics).

Problem statement

G. Backus (1968): Application of a non-linear boundary-value problem for the Laplace's equation to gravity and geomagnetic intensity surveys

The problem is to find u so that for a given function $g > 0$ on a closed surface S in $\mathbb{R}^3$ (Earth's surface)

$$\text{BP}: \begin{cases} \Delta u = 0 & \text{in } \Omega = \text{Ext}(S), \\ |\nabla u| = g & \text{on } S, \\ u(x) \to 0 & \text{as } x \to \infty. \end{cases}$$

where Δ is the Laplace operator.

The fixed-boundary gravimetric bvp I

- In **geodesy**, the BP is called the fixed-boundary gravimetric boundary value problem and g is the modulus of gravity (*Koch and Pope, 1972*). In this simple model we are excluding the centrifugal potential.

$$S, g = |\nabla u|_S \longrightarrow u$$

- So we distinguish it from the free boundary problem of Molodensky (*Hörmander, 1976*), where S is unknown and the complete gradient of u is known on the surface.

$$\mathbf{g} = \nabla u|_S \longrightarrow S, u$$

The fixed-boundary gravimetric bvp II

Natural condition in geodesy

The normal derivative of u on S is negative, i.e. ∇u points towards the interior of S

Exact formulation of the Backus problem in geodesy

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \quad u(x) \to 0 \quad \text{as } x \to \infty, \\ \dfrac{\partial u}{\partial n} \leq 0\,, \ |\nabla u| = g \quad \text{on } S \end{cases}$$

where n is the normal of S pointing to the exterior Ω

Applications: geodesy

- Koch and Pope (1972), Bjerhammar and Svensson (1983), Grafarend (1989), Heck (1989), Sacerdote and Sansò (1989), Holota (1997, 2005), Čunderlík et al. (2008) ...
  Main achievement: Local solvability
- The earth's gravity points towards the interior: the restriction $u' \leq 0$ is natural ($\to$ maximal solution)
- Since $u' \leq 0 \Rightarrow u > 0$, the solution must be of the form

$$\boxed{u = \frac{c}{r} + \sum_{n=1}^{\infty} u_n\,, \quad \left(c = \frac{1}{4\pi}\int_S u\, ds > 0\right)} \tag{2}$$

In contrast to that, in geomagnetism the solution must satisfy Eq.(8) with $c = 0$ (cf. *Campbell, 1997*)

$$\int_S u \, ds = 0 \,. \tag{9}$$

Hence u changes its sign on S, so that neither u nor $-u$ belong to K and then $\partial u / \partial r$ changes its sign on S too. As an example, Fig. 1 displays the field of a magnetic dipole. It is observed that:

1. ∇u is tangential to the unit sphere along the equator E.
2. $\nabla u|_E$ is orthogonal to E.
3. $\partial u / \partial r$ changes its sign on S through E from plus to minus in the direction of the vector field $\nabla u|_E$.

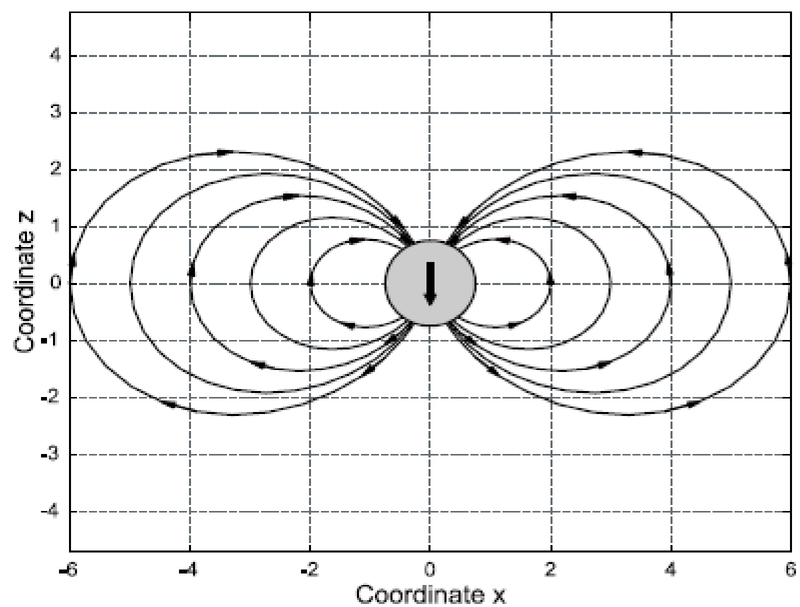
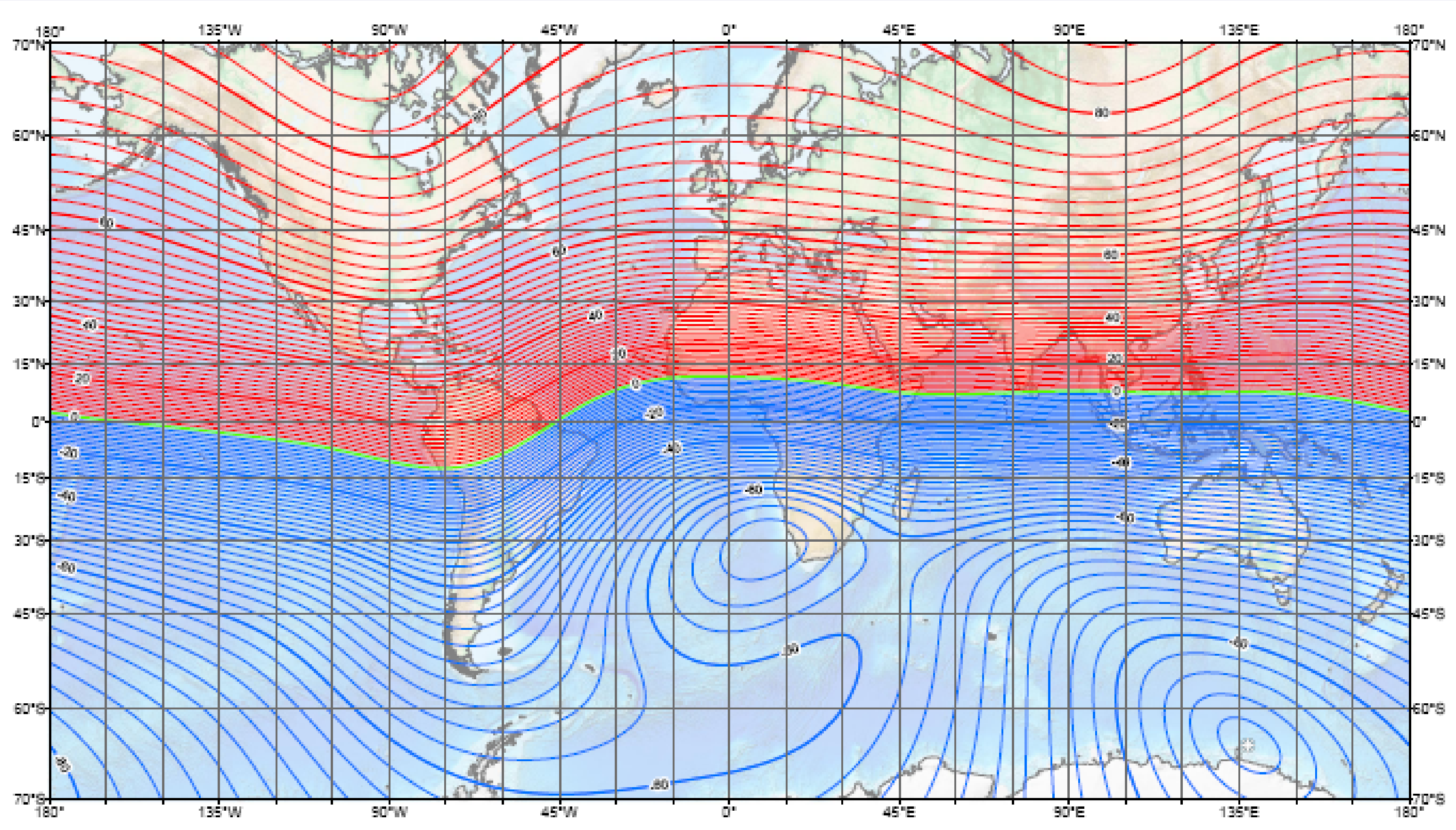


Fig. 1. Magnetic dipole field using the Matlab function lforce2d by A. Abokhodair (*http://www.mathworks.com/matlabcentral/fileexchange*).

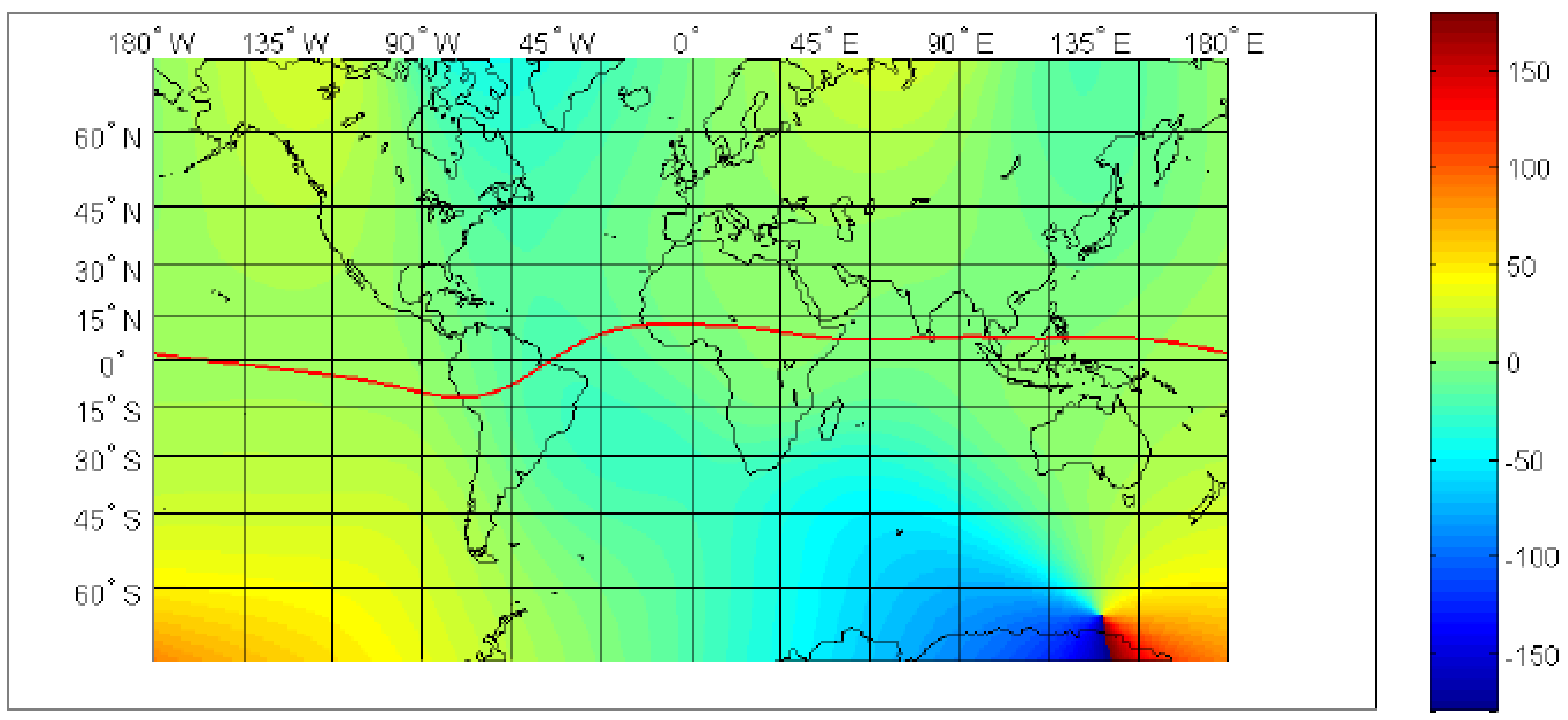
BP in geophysics: dip equator

Inclination of the Earth magnetic field (epoch 2010.0) (downward is positive): u_n' changes its sign on S across the dip equator.



BP in geophysics: emergent field

Declination of the Earth magnetic field (epoch 2010.0) (positive east of true north): $u'_t\,|_{\text{dip}}$ points to the part of S where ∇u is exterior with respect to Ω



BP in geophysics: an attempt to approach

Formulation of the Backus problem in geomagnetism

To find u so that for a given function $g > 0$ on a closed surface S in $\mathbb{R}^3$

$$\Delta u = 0 \quad \text{in} \ \ \Omega, \quad u(x) \to 0 \quad \text{as} \ \ x \to \infty, \text{and}$$

1. $|\nabla u| = g$ on S,
2. $Z = \{z \in S : \ u'_n(z) = 0\}$ is a simple closed curve,
3. $\nabla u(z)$, $z \in S$, points to the part of S where ∇u is exterior with respect to Ω, i.e. where $u'_n < 0$.

BP is a non linear oblique derivative problem

- $g > 0$: we write the boundary condition in the form of an oblique derivative

$$\boxed{\frac{\partial u}{\partial l} = g\,,\ \ l = \underline{\nabla u/|\nabla u| \equiv e_u}}$$

- The direction of the vector field l depends on the solution u itself through its gradient, $l = l(x, \nabla u)$. In the linear case $l = l(x)$.
- In operator form the boundary condition is

$$\boxed{G(x, \nabla u) = 0 \text{ where } G : S \times \mathbb{R}^3 \to \mathbb{R}, G(x, p) = |p| - g(x)}$$

BP is degenerate

- Linearized problem:

$$\Delta v = 0 \quad \text{in } \Omega, \quad v(x) \to 0 \quad \text{as } x \to \infty,$$

$$\boxed{\frac{\partial v}{\partial l} = g - G[u]} \quad \text{on } S$$

where $l = e_u = \alpha n + X$, decomposed into normal and tangential components.

- For each u, this is a *linear oblique derivative problem*, degenerate if the set of tangency for l is not empty

$$Z_u = \{x \in S : \alpha(x) = \langle n(x), e_u(x) \rangle = 0\} \neq \emptyset$$

Classical solvability condition

- Nonlinear elliptic boundary value problems of the general form:

  $$F[u] = 0 \quad \text{in } \Omega, \quad G[u] = G(x, u, \nabla u) = 0 \quad \text{on } \partial\Omega.$$

  The classical solvability condition for these problems is the constancy of the sign ($\gtrless 0$) of

  $$\Phi = \langle n(x), G_p(x, p) \rangle$$

  (*Lieberman and Trudinger, 1986*; *Ishii, 1991*).

- BP: $\Phi = \langle n(x), p/|p| \rangle$ is not of constant sign.
  Problems with Φ of changing sign on $S \times \mathbb{R}^3$ have received little attention, with exceptions in the eighties (*Godin P. (1985), Subelliptic Non Linear Oblique Derivative Problems, Amer. J. Math.*, **107**(3), *591–615*).

Initial remark

- Basic non-uniqueness: $G(x, -p) = G(x, p)$.
- **Notation**: $\mathcal{H}(\Omega) \equiv$ real space of functions which are harmonic in Ω and tend to zero at infinity
- Simple starting remark:

$$\boxed{|\nabla u| = |\nabla v| \Longleftrightarrow \langle \nabla(u - v) \, , \nabla(u + v) \rangle = 0 \, .}$$

- Thus, the question of uniqueness of solutions of BP leads to the consideration of the nullspace ($\mathcal{N}$) of an oblique derivative problem for the Laplace equation:

$$\boxed{\text{ODP} : \psi \in \mathcal{H}(\Omega) \text{ and } \langle \nabla \psi \, , \nabla \phi \rangle = f \quad \text{on} \;\; S,}$$

where $\phi \in \mathcal{H}(\Omega)$ is given.

Trivial nullspace

- If $\phi'_n < 0$ then $\psi = 0$ by Hopf's Lemma. In this case the nullspace of ODP is trivial: $\mathcal{N} = \{0\}$.
- **Notation :** $\mathcal{H}^-(\Omega) = \{u \in \mathcal{H}(\Omega) : \partial u/\partial n < 0 \text{ on S}\}$.
- The following is a classical result (G. Bakus)

Theorem

BP has at most one solution in the class $\mathcal{H}^-(\Omega)$

- G. Díaz et al. (2006) have extended this theorem to the class:

$$\mathcal{H}_0^-(\Omega) = \{u \in \mathcal{H}(\Omega) : \partial u/\partial n \leq 0 \text{ on S}\}$$

Comparison principle

This extension is based on the following comparison theorem:

Theorem

Let $u \in \mathcal{H}(\Omega)$ and $v \in \mathcal{H}_0^-(\Omega)$ be such that $|\nabla u| \leq |\nabla v|$ on S. Then $u \equiv v$ or $u < v$ in $\overline{\Omega}$.

Theorem

The fixed-boundary gravimetric boundary value problem

$$u \in \mathcal{H}(\Omega), \text{ and } \quad \frac{\partial u}{\partial n} \leq 0\,, \; |\nabla u| = g \quad on \;\; S$$

has at most one solution.

- In other words: BP has at most one solution in the class $\mathcal{H}_0^-(\Omega)$

Nontrivial nullspace

- If $\mathcal{N} \neq \{0\}$, let $\psi_0 \in \mathcal{N} \setminus \{0\}$. Then, the functions

$$u = \frac{1}{2}(\phi + \psi_0) \in \mathcal{H}(\Omega) \quad \text{and} \quad v = \frac{1}{2}(\phi - \psi_0) \in \mathcal{H}(\Omega)$$

  are different and satisfy $|\nabla u| = |\nabla v|$ on S.

- We take the function ϕ so that the set of tangency Z_ϕ between $\nabla \phi$ and S is not empty. Moreover,
  - Z is a closed submanifold of codimension 1 (a simple closed curve) and $\nabla \phi \mid_S$ is transversal to S
  - ϕ'_n changes its sign on S through Z from plus to minus in the direction of the vector field $\nabla \phi$ (it is of emergent type).

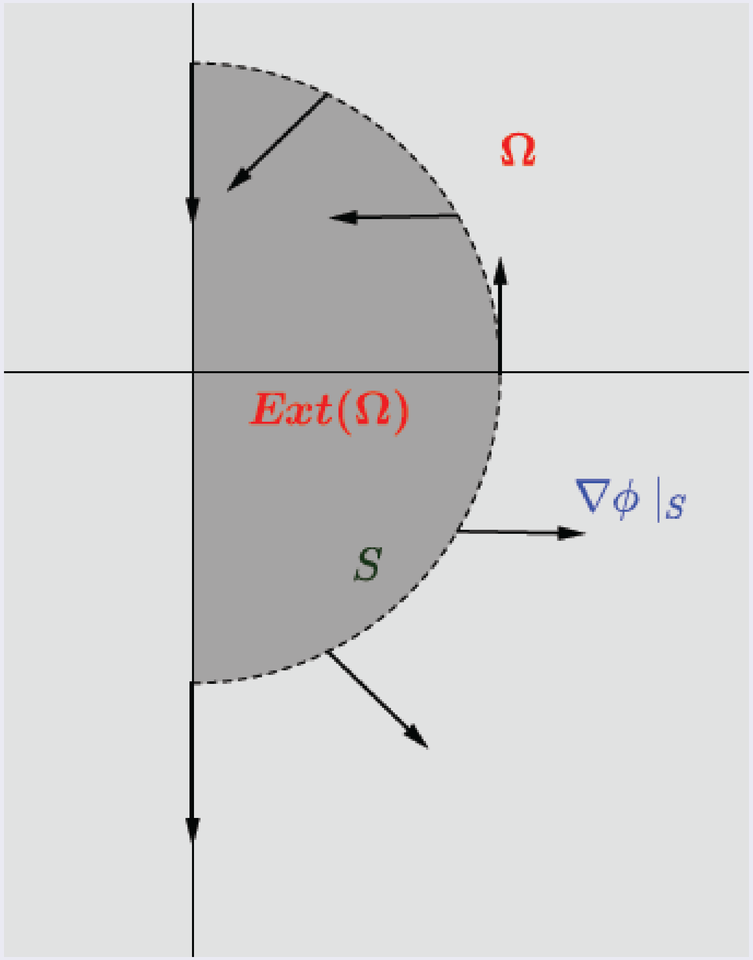
Nontrivial nullspace: an example

Example

$$\phi = \frac{z}{r^3} \text{ (magnetic dipole)}$$

$$\phi'_n = -2\frac{z}{r^4}\begin{cases} > 0 & \text{if } z < 0, \\ = 0 & \text{if } z = 0, \\ < 0 & \text{if } z > 0. \end{cases}$$

$\nabla\phi$ is of emergent type



Nonuniqueness of Backus' problem

- **The theory of degenerate oblique derivative problems** (*Popivanov and Palagachev, 1997; Paneah, 2000*), states that for this type of vector fields, the oblique derivative problem admits an infinite-dimensional nullspace.
- For the function in the example (magnetic dipole) the nullspace of ODP was constructed by *G. Backus* in 1970 (also: *Jorge and Magnanini*, 1993; *Akhmet'ev and Khokhlov*, 2004).
- We now reach the known conclusion that **the Backus' problem is not uniquely solvable in $\mathcal{H}(\Omega)$**. Our approah has been the infinite dimensionality of ODP when the vector field is of emergent type.

In geodesy: non-degenerate

From the practical point of view it is relevant to consider the linearized Backus' problem (LBP)

$$\Delta v = 0 \quad \text{in } \Omega, \quad v(x) \to 0 \quad \text{as } x \to \infty,$$
$$\langle \nabla v \,, e_u \rangle = g - G[u] \quad \text{on } S,$$

where $e_u = \nabla u / |\nabla u|$.

- In **geodesy**, u is the "normal" potential associated with a rotating equipotential ellipsoid. If we assume that S is smooth enough, $u \in \mathcal{H}^-(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ and $g \in C^\alpha(S)$, then LBP has a (unique) solution $v \in C^{1,\alpha}(\overline{\Omega})$ (*Freeden and Gerhards, Geomathematically oriented potential theory*, 2013).
- For a numerical solution see: Čunderlík et al., *Numerical solution of the linearized fixed gravimetric boundary-value, 2008.*

In geophysics: degenerate

- If u is the scalar potential of the dipole model of the Earth's magnetic field, then LBP is degenerate because the set of tangency Z_u is not empty (dip equator). In addition, e_u is of emergent type.
- The nullspace is infinite dimensional
- Other important features from the theory of degenerate oblique boundary problems are:
- The values of v should be prescribed on Z_u in order to get uniqueness (*Egorov and Kondrat'ev*, 1969).
- Whereas in the non-degenerate case any solution gains one derivative from the boundary data ($\alpha \to 1 + \alpha$), here the solution loses regularity near the set of tangency

Open problem

The geomagnetism provides a beautiful degenerate oblique derivative problem that deserves future attention

Decomposition

Assumed existence, we denote by U the unique solution of BP in $\mathcal{H}_0^-(\Omega)$. A consequence of the comparison principle is the following.

Theorem

$\pm U$ are the maximal and minimal solutions of the Backus problem, i.e.

$$-U < u < U \quad in \quad \overline{\Omega},$$

for any other solution of BP.

- To construct U, we have used the following decomposition (*Díaz et al.*, 2011)

$$\mu^{1/2} U = \frac{1}{r} + v\,,$$

where μ is an unknown positive constant and v is a harmonic function such that

$$\int v\, ds = 0\,.$$

Equivalent nonlinear problem

To solve BP in $\mathcal{H}_0^-(\Omega)$ is then equivalent to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ 2\,\dfrac{\partial v}{\partial r} = 1 + |\nabla v|^2 - \mu[v]\, g^2 & \text{on } S, \\ v(x) \to 0 & \text{as } x \to \infty\,, \end{cases}$$

where

$$\mu = \mu[v] = \mu_0 \Big[1 + \frac{1}{4\pi} \int_S |\nabla v|^2 \, ds \Big]\,,$$

and with the additional property that on S

$$\partial v / \partial r \leq 1\,.$$

Successive approximations

$$\mu_0^{-1} = \frac{1}{4\pi}\int_S g^2\, ds$$

$$\mu_0 \to v_1 \to \mu_1 \to u_1$$

$$\mu_n = \mu_0\left[1 + \frac{1}{4\pi}\int_S |\nabla v_n|^2\, ds\right]$$

$$(v_1, \mu_1) \to v_2 \to \mu_2 \to u_2$$

$$(v_2, \mu_2) \to v_3 \to \mu_3 \to u_3$$

$$2v_{n+1} = -\frac{1}{r} + |\nabla v_n|^2 - \mu_n F$$

$$(n \geq 0,\ v_0 = 0)$$

$$\vdots$$

$$(v_{n-1}, \mu_{n-1}) \to v_n \to \mu_n \to u_n$$

$$U_n = \mu_n^{-1/2}\left(r^{-1} + v_n\right)\ (n \geq 0)$$

$$F \in \mathcal{H}(\Omega), \quad \frac{\partial F}{\partial r} = f \quad \text{on } S$$

$$G_n \in \mathcal{H}(\Omega), \quad \frac{\partial G_n}{\partial r} = |\nabla v_n|^2 \quad \text{on } S,$$

Numerical example

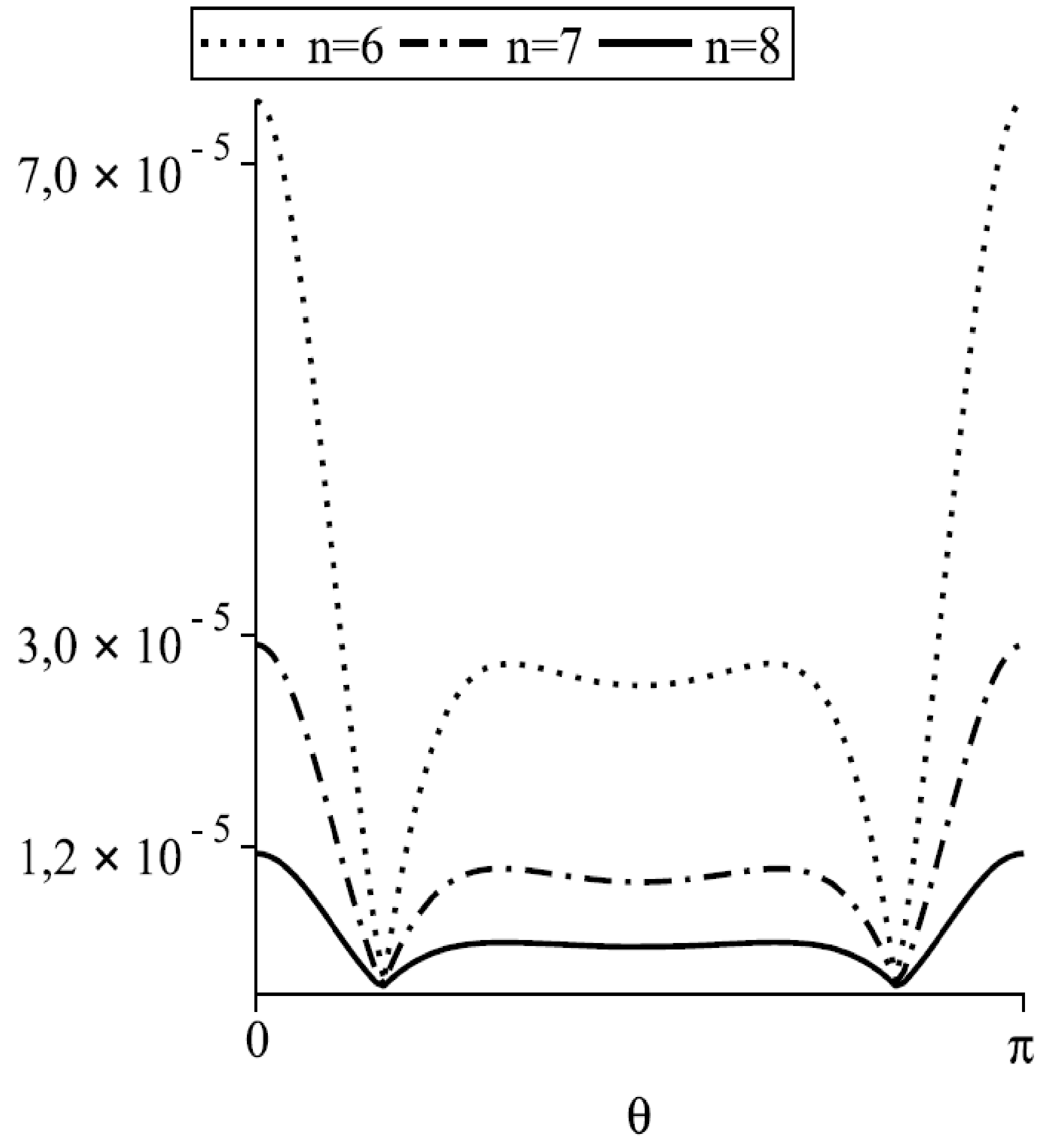
Example

$u = z/r^3 \in \mathcal{H}^1(\bar{\Omega}) \to g^2 = 1 + 3\cos^2\theta$

- Sequence μ_n

n	μ_n
1	0.541667
2	0.540808
3	0.542183
4	0.541950
5	0.542064
6	0.542030
7	0.542043
8	0.542038
9	0.542040

Absolute value of the successive differences $U_{n+1} - U_n$



α-norm of the radial derivatives

Theorem

Let $g \in C^{\alpha}(S)$ *be such that* $(4\pi)^{-1} \int_S g^2 \, ds = 1$. *Let* $k = \|g^2\|_\alpha (1+c) \|1 - g^2\|_\alpha$ *where* $\|g^2\|_\alpha \geq 1$ *and* **c** *is a positive constant . If* $k \leq 1$, *then*

$$\left\| \frac{\partial v_n}{\partial r} \right\|_\alpha \leq \varepsilon \, , \tag{1}$$

for all $n \geq 1$, *where*

$$\varepsilon = \frac{1}{\|g^2\|_\alpha (1+c)} \left(1 - \sqrt{1-k}\right) < 1 \, . \tag{2}$$

Moreover,

$$\| |\nabla v_n| \|_\alpha \leq \varepsilon^2 (1+c) \ \text{ and } \ \mu_n \leq 1 + 2\varepsilon^2 \, . \tag{3}$$

Existence theorem in $\mathcal{H}_0^-(\Omega)$

The constant **c** is the positive constant of the following fundamental inequality for harmonic functions in the exterior of the sphere:

$$\| \, |\nabla_t u| \, \|_\alpha \leq c^{1/2} \left\| \frac{\partial u}{\partial r} \right\|_\alpha$$

Theorem

Under the assumptions of the previous theorem, the sequence $\{v_n\}$ contains a subsequence converging in $C^1(\bar{\Omega})$, i.e. there is a function $v \in C^1(\bar{\Omega})$ such that $\|v_n - v\|_{1;\bar{\Omega}} \to 0$ as $n \to \infty$. In addition, the function $u = \mu^{-1/2}(1/r + v)$, where $\mu = \lim_{n\to\infty} \mu(v_n)$, is the maximal solution of BP.

Conjecture

The convergence theorem is valid for functions f close enough to constant functions in C^α. However, from the example, and others that we have worked out, **we conjecture that the sequence of successive approximations converge to the maximal solution for whatever given f > 0**.

Conclusions

- We have tried to formulate correctly the Backus problem in geophysics resulting in a new non linear degenerate oblique derivative problem

1. $u \in \mathcal{H}^{\pm}(\Omega)$,
2. $|\nabla u| = g$ on S,
3. $Z_u = \{z \in S : u'_n(z) = 0\}$ is a simple closed curve (a priori unknown),
4. $\nabla u(z)$, $z \in S$, points to the part of S where ∇u is exterior with respect to Ω, i.e. where $u'_n < 0$.

Thanks

for your

attention