On the coupling between the deep ocean and an atmospheric balanced climate model.

March 9, 2009

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Abstract. After a general introduction about the convenience of to study the coupling of atmospheric energy balance climate models with the deep ocean temperature, we consider the model proposed by Watts and Morantine, in 1990, about such a coupling leading to a nonlinear dynamic problem in which it appear a dynamic and diffusive boundary condition. We show that for the special case of discontinuous coalbedo functions (as proposed by Budyko in 1969) there is not uniqueness of solution of this climate model, at least for some special data.

1 Introduction.

The scientific study of the Global Change is one of the motivations of very large time scale models to study the evolution of the Earth locally averaged temperature: the climate. The pioneering global climate energy balance mathematical models (EBMs) were introduced by M. Budyko and W. Sellers in 1969, independently. EBMs are diagnostic models for the surface atmospheric temperature. Several aspects of the mathematical treatment of different versions of climate EBMs have been studied by many authors, among them, [5], [10], [16], [22], [12], [7] and [3], among many others.

Nevertheless, in this class of global climate models the averaging process is taken place only at the surface atmospheric level and, in particular, deep ocean effect was not directly included but only as some special values of the constitutive coefficients of the superficial model. A growing set of evidence, however, indicates that variations in the climate may be strongly connected to variations in the tropical ocean. An obvious case in point is the existence of climatic anomalies in various parts of the world following an El Niño event. If the ocean were a stagnant body of water, the bottom of the equatorial ocean would be warm. However, it has been known, for a long time, that the deep ocean is quite cold.

The important role played by oceans was not correctly taken into account and, for instance, rapid climate changes during Glacial-Holocene transition could have been the results of variations in the rate of deep water formation (see Berger et al [2]). The coupling among atmosphere and ocean have been considered by many other purposes, as, for instance, the elevation of the ocean levels,

Many different mathematical models coupling atmosphere and ocean are available today in the literature (see, e.g., the important series of papers by Lions, Temam and Wang [19], [18] and [20]). But the great complexity of such systems of nonlinear partial differential equations made almost impossible to use them for the study planetary scale phenomena as, the evolution of ice sheets, the sensitivity with respect the Solar incoming energy, etc.

To simplify the understanding of this coupling, different models were proposed in the literature: see, e.g. [24], [13], [14] and the many references of the books [15], [23] and [27], to only mention a few of them.

In this paper we shall follows the simplified model proposed by Watts and Morantine [31] (and then used by many other authors, as, for instance, [29], [21], [30] and [28]). Such a model consists of a parabolic equation in a global ocean with a dynamic and diffusive nonlinear boundary condition. This boundary condition is obtained through a global energy balance for the atmosphere surface temperature. A previous detailed mathematical study of the model was carried out by the authors in [8], [10] and [11]. The goal of this work is to show that under the discontinuous coalbedo function , proposed by M. Budyko in 1969, there is lack of the uniqueness of solutions. This is constrast with what it is usually assumed in many studies on this kind of coupling (as, for instance, in the numerical treatment of such coupling models: see, e.g. [24]). Here we construct an example of nonuniqueness solutions of this evolution system under suitable conditions on the model.

This completes the work by the authors [9] in which such a result was merely announced (without proof).

2 The model.

The global model studied is the simplification of a climate model with deep ocean effect where the surface is the sphere of radius R and the temperature is constant over each parallel. The resulting spatial variables x, z, represent the sine of the latitude and minus the depth, respectively. The spatial domain is $\Omega = (-1,1) \times (-H,0)$ and its boundary, $\Gamma_H \cup \Gamma_0 \cup \Gamma_1$, where $\Gamma_H = \{(x,z) \in \overline{\Omega} : \; z = -H\}$, $\Gamma_0 = \{(x,z) \in \overline{\Omega} : \; z = 0\}$, $\Gamma_1 = \{(x,z) \in \overline{\Omega} : x = \pm 1\}$.

The boundary condition on Γ_0 is based on a global energy balance over the Earth surface. The unknown U on Γ_0 is the mean temperature over each parallel.

We are concerned with the following model

$$\begin{aligned}
\frac{\partial U}{\partial t} - \frac{K_H}{R^2}\frac{\partial}{\partial x}((1-x^2)\frac{\partial U}{\partial x}) - K_v\frac{\partial^2 U}{\partial z^2} + w\frac{\partial U}{\partial z} = 0 \quad \text{in } (0,T)\times\Omega,\\
wx\frac{\partial U}{\partial x} + K_V\frac{\partial U}{\partial z} = 0 \quad \text{in } (0,T)\times\Gamma_H,\\
D\frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{p/2}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right) + K_V\frac{\partial U}{\partial n}\\
+wx\frac{\partial U}{\partial x} + \mathcal{G}(U) \in \frac{1}{\rho c}QS(x)\beta(x,U) \quad \text{in } (0,T)\times\Gamma_0,\\
(1-x^2)^{p/2}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} = 0 \quad \text{in } (0,T)\times\Gamma_1,\\
U(0,x,z) = U_0(x,z) \quad \text{in } \Omega,\\
U(0,x,0) = u_0(x) \quad \text{in } \Gamma_0.
\end{aligned}$$

The physical description and the numerical approximation if $p = 2$ and where β depends only on x is in Watts Morantine [31]. The proof of existence of bounded weak solution to this 2-D model is in Díaz and Tello [8]. The number of steady states of (P_{2D}) was studied in [11].

$BU+C$ is the emitted energy from the atmosphere layer (by cooling). The planetary coalbedo β is eventually discontinuous on u. K_V and K_H are the vertical and horizontal diffusivity in the inner ocean, K_{H_0} is the horizontal diffusivity in the mixed layer (atmosphere-ocean). w is the vertical velocity, R is the Earth radius and Q the solar constant.

We study the uniqueness / nonuniqueness of solutions to (P) under the following structural hypotheses,

(H_β) β is a bounded maximal monotone graph, that is, $|v| \leq M \quad \forall v \in \beta(s)$, $\forall s \in D(\beta) = I\!R$,

($\mathrm{H}_{B,C,p}$) B and C are positive constants and $p \geq 2$,

(H_S) $S : (-1,1) \to I\!R$, $s_1 \geq S(x) \geq s_0 > 0$ a.e. $x \in (-1,1)$,

(H_w) $w \in C^1(\overline{\Omega})$.

We say that $(U,u) \in L^2(0,T : W^{1,2}(\Omega) \times W^{1,p}(\Gamma_0)) \cap W^{1,2}(0,T : L^2(\Omega) \times L^p(\Gamma_0))$ is a solution of (P) if $U_{|\Gamma_0} = u$ and $\forall(\psi,\zeta)$ test functions in $L^2(0,T : W^{1,2}(\Omega) \times W^{1,p}(\Gamma_0)) \cap W^{1,2}(0,T : L^2(\Omega) \times L^p(\Gamma_0))$ we have

$$\int_0^T \int_\Omega \frac{\partial U}{\partial t}\psi dAdt + \int_0^T \int_\Omega \frac{K_H}{R^2}(1-x^2)\frac{\partial U}{\partial x}\frac{\partial \psi}{\partial x}dAdt + \int_0^T \int_\Omega K_v \frac{\partial U}{\partial z}\frac{\partial \psi}{\partial z}dAdt + $$
$$+ \int_0^T \int_\Omega w\frac{\partial U}{\partial z}\psi dAdt$$

$$-\int_0^T \int_{-1}^1 wx\frac{\partial U}{\partial x}(x,-H)\psi(x,-H)dxdt - \int_0^T \int_{-1}^1 K_v \frac{\partial U}{\partial z}(x,0)\psi(x,0)dxdt = 0,$$

$$\int_0^T \int_{-1}^1 D\frac{\partial u}{\partial t}\zeta dxdt + \int_0^T \int_{-1}^1 \frac{DK_{H_0}}{R^2}(1-x^2)^{\frac{p}{2}}\left|\frac{\partial u}{\partial x}\right|^{\frac{p}{2}}\frac{\partial u}{\partial x}\frac{\partial \zeta}{\partial x} +$$
$$+ \int_0^T \int_{-1}^1 K_v\frac{\partial u}{\partial z}(x,0)\zeta dxdt + \int_0^T \int_{-1}^1 wx\frac{\partial u}{\partial x}\zeta dxdt + \int_0^T \int_{-1}^1 (Bu+C)\zeta dxdt$$
$$= \int_0^T \int_{-1}^1 \frac{1}{\rho c}QS(x)h\zeta dxdt.$$

for some $h \in L^\infty(0,T : L^\infty(\Gamma_0))$, $h \in \beta(\cdot,u)$.

Existence of solutions was proved in Díaz-Tello [8] by fixed point techniques. Moreover, when $\beta(u)$ is Lipschitz continuous (as proposed by Sellers) it is not difficult to adapt the arguments of [5] to prove the uniqueness of a solution of the above system.

3 A nonuniqueness result.

In this section we pay an special attention to the case of discontinuous (or multivalued) coalbedo functions (as proposed by Budyko). We give here a counterexample to the uniqueness for the problem (P_{2D}) under the following hypotheses:

(H_1) The coalbedo function is

$$\beta(u) = \begin{cases} [m, M] \text{ if } u = -10, \\ m \text{ if } u < -10, \\ \beta(u) = M \text{ if } u > -10, \text{ with } 0 < m < M. \end{cases} \tag{1}$$

(H_2) $B + C$ are positive constants, and

$$-10B + C > \frac{Qs_1 m}{\rho c}. \tag{2}$$

(H_3) We also assume

$$w(x) \leq 0 \text{ for all } x \in (-1, 1).$$

(H_4) The initial data (U_0, u_0) satisfy

$$\begin{cases} (U_0, u_0) \in C^\infty(\Omega) \times C^\infty(\Gamma_0), u_0(x) = u_0(-x) \text{ for all } x \in [-1, 1], \\ \dfrac{du_0}{dx}(0) = \dfrac{d^2 u_0}{dx^2}(0) = 0, \ u_0(0) = -10, \\ \dfrac{du_0}{dx}(x) < 0 \text{ if } x \in (0, 1), \dfrac{du_0}{dx}(1) = 0, \\ \dfrac{\partial U_0}{\partial z}(x, 0) > 0, \ U_0(x, 0) = u_0(x), \text{ if } x \in (0, 1). \end{cases}$$

Theorem 1 *Under the above conditions, Problem (P_{2D}) has at least two bounded weak solutions.*

Idea of the proof.

Step 1. First, we consider the problem (P_m)

$$\begin{cases} \dfrac{\partial U}{\partial t} - \dfrac{K_H}{R^2}\dfrac{\partial}{\partial x}((1-x^2)\dfrac{\partial U}{\partial x}) - K_v\dfrac{\partial^2 U}{\partial z^2} + w\dfrac{\partial U}{\partial z} & = 0 & (0,T)\times\Omega, \\ wx\dfrac{\partial U}{\partial x} + K_V\dfrac{\partial U}{\partial z} & = 0 & (0,T)\times\Gamma_H \\ D\frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right) + & & \\ + K_V\frac{\partial U}{\partial n} + wx\frac{\partial U}{\partial x} + Bu + C & = \frac{1}{\rho c}QS(x)m & \text{on } (0,T)\times\Gamma_0 \\ (1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} & = 0 & \Gamma_1\cup\Gamma_{-1} \\ U(0,x,z) = U_0(x,z) & & \Omega, \\ U(0,x,0) = u_0(x) & & (-1,1), \end{cases}$$

Denote (U^m, u^m) to the solution of (P_m).

We notice that if $u^m \leq -10$ then (U^m, u^m) is also a solution of (P) because $h(x,t) \equiv m \in \beta(u_m)$.

Now, by changing $U^* = -10 - U^m$ and $u^* = -10 - u^m$, we have that u^* verifies

$$Du_t^* - \frac{DK_{H_0}}{R^2}((1-x^2)|u_x^*|^{p-2}u_x^*)_x + Bu^* = -\frac{QSm}{\rho c} - 10B + C - K_V\frac{\partial U^*}{\partial n} - wx\frac{\partial u^*}{\partial x}.$$

From hypotheses (H_0) and (2), there exists $T_0 > 0$ s.t. if $t < T_0$ then the right hand side term is positive. Consequently $u^* = -10 - u^m$ is positive and $u^m < -10$.

Notice that $K_V\frac{\partial U}{\partial n} + wx\frac{\partial U}{\partial x} \leq 0$ in $(0,T_0)\times\Gamma_0)$.

Step 2. Now, we prove that there exist a solution which takes values bigger than -10 in a subset of Γ_0. To see the existence of this second solution, we shall construct a family of auxiliary functions U^λ (and the restrictions $U^\lambda_{|\Gamma_0} = u^\lambda$), as follows:

$\Omega\times[0,\lambda] = \mathcal{Q}_1^\lambda \cup \mathcal{Q}_2^\lambda \cup \Sigma^\lambda$, where

$$\mathcal{Q}_1^\lambda = \{(x,z,t)\in\Omega\times[0,\lambda] : \; x^2+z^2 > \frac{t^2}{\lambda^2}\},$$

$$\mathcal{Q}_2^\lambda = \{(x,z,t)\in\Omega\times[0,\lambda] : \; x^2+z^2 < \frac{t^2}{\lambda^2}\},$$

$$\Sigma^\lambda = \{(x,z,t) \in \Omega \times [0,\lambda] : \; x^2+z^2 = \frac{t^2}{\lambda^2}\}.$$

$\triangleright$ In the region $\mathcal{Q}_1^\lambda$. Let (U^λ, u^λ) be a solution of problem $(P_{\mathcal{Q}_1^\lambda})$

$$\begin{cases} \dfrac{\partial U}{\partial t} - \dfrac{K_H}{R^2}\dfrac{\partial}{\partial x}((1-x^2)\dfrac{\partial U}{\partial x}) - K_v\dfrac{\partial^2 U}{\partial z^2} + w\dfrac{\partial U}{\partial z} & = 0 & \mathcal{Q}_1^\lambda \\ wx\dfrac{\partial U}{\partial x} + K_V\dfrac{\partial U}{\partial z} & = 0 & \mathcal{Q}_1^\lambda \cap (0,T)\times\Gamma_H \\ D\frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right) + & & \\ + K_V\frac{\partial U}{\partial n} + wx\frac{\partial U}{\partial x} + Bu + C & = \frac{1}{\rho c}QS(x)m & \mathcal{Q}_1^\lambda \cap (0,T)\times\Gamma_0 \\ (1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} & = 0 & \text{on } \Gamma_1\cup\Gamma_{-1} \\ U(0,x,z) = U_0(x,z), U(0,x,0) = u_0(x) & & \\ U^\lambda = -10 & & \Sigma^\lambda \end{cases}$$

$\triangleright$ At the region $\mathcal{Q}_2^\lambda$, we define

$$U^\lambda = -10 - C^\lambda(t)(x^2+z^2-\frac{t^2}{\lambda^2}).$$

Notice that if $C^\lambda > 0$ then $U^\lambda > -10$ in $\mathcal{Q}_2^\lambda$.

$\triangleright$ At the region $\mathcal{Q}_2^\lambda$, we define

$$U^\lambda = -10 - C^\lambda(t)(x^2+z^2-\frac{t^2}{\lambda^2}).$$

Notice that if $C^\lambda > 0$ then $U^\lambda > -10$ in $\mathcal{Q}_2^\lambda$.

Is easy to see that (U^λ, u^λ) is a solution of Problem (P_λ),

$$\begin{cases} \dfrac{\partial U}{\partial t} - \dfrac{K_H}{R^2}\dfrac{\partial}{\partial x}((1-x^2)\dfrac{\partial U}{\partial x}) - K_v\dfrac{\partial^2 U}{\partial z^2} + w\dfrac{\partial U}{\partial z} & = H^\lambda & \text{in } (0,T)\times\Omega, \\ wx\dfrac{\partial U}{\partial x} + K_V\dfrac{\partial U}{\partial z} & = g^\lambda & \text{in } (0,T)\times\Gamma_H \\ D\frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right) + & & \\ + K_V\frac{\partial U}{\partial n} + wx\frac{\partial U}{\partial x} + Bu + C & = h^\lambda & \text{in } (0,T)\times\Gamma_0 \\ (1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} & = 0 & \text{in } (0,T)\times(\Gamma_1\cup\Gamma_{-1}) \\ U(0,x,z) = U_0(x,z) \text{in } \Omega, & & U(0,x,0) = u_0(x) \text{in } (-1,1), \end{cases}$$

where

$$H^\lambda = -(C^\lambda)'(t)(x^2+z^2-\frac{t^2}{\lambda^2}) - C^\lambda(t)[(\frac{-2t}{\lambda^2}) - \frac{2K_H}{R^2}(1-3x^2) - 2K_v + 2wz], \quad (3)$$

for $(t,x,z) \in \mathcal{Q}_2^\lambda$

$$\begin{aligned} h^\lambda &= -D(C^\lambda)'(t)(x^2-\frac{t^2}{\lambda^2}) - C^\lambda(t)[-\frac{2Dt}{\lambda^2} + 2wx^2 + B(x^2-\frac{t^2}{\lambda^2}) \\ &\quad -2^{p-1}\frac{DK_{H_0}}{R^2}|C^\lambda(t)|^{p-2}(-p(1-x^2)^{\frac{p-2}{2}}|x|^p + \\ &\quad +(p-1)(1-x^2)^{\frac{p}{2}}|x|^{p-2}] - 10B + C), \end{aligned}$$

$$g^\lambda = -2C^\lambda(t)(x^2w - K_vH) \geq 0$$

.

There exist $\lambda > 0$ and $C^\lambda : [0,T_0] \to I\!R$ such that (U^λ, u^λ) is a lower solution of Problem (P).

Then, by upper and lower solution method we deduce that there exists a solution (V,v) of (P) satisfying $u^\lambda < v$. Consequently $v > -10$ in some subset of positive measure. (V,v) is different than the solution of step 1.

Finally, we get two different solution of (P_{2D}) for an initial data satisfying (H_4).

Remark 1 *We have proved here that, for some special initial data, there exist more than one time dependent solution of the coupled system. The behaviour of this type of systems leads us to the conjecture that it could be exist, in fact, a continuum of solutions, and not only two, in a similar way to the results in the literature concerning the EBM (without any coupling with the deep ocean). .*

References

[1]

[2] W.H. Berger, S. Burker, E. Vincent. Glacial-Holocene transition: Climate Pulsations and Sporadic Shutdown of NADW production, in *Abrupt Climatic Change - Evidence and Implications*, (eds. W.H. Berger, L.D. Labeyrie), Reidel Publishing Co. Dordrecht Holland (1987).

[3] R. Bermejo, J. Carpio, J.I. Díaz, L. Tello. 'Mathematical and Numerical Analysis of a Nonlinear Diffusive Climate Energy Balance Model'.*Mathematical and Computer Modelling,* 49 (2009) 1180-1210.

[4] M.I. Budyko. The effects of solar radiation variations on the climate of the Earth, *Tellus* 21 (1969) 611-619.

[5] J. I. Diaz. Mathematical analysis of some diffusive energy balance climate models, in the book *Mathematics, Climate and Environment*, (J. I. Díaz and J. L. Lions, eds.) Masson, Paris, 28-56 (1993).

[6] J. I. Díaz, J. Hernandez, L. Tello. On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in Climatology, *J. Math. Anal. Appl.* **216** (1997), 593-613.

[7] J.I. Díaz, G. Hetzer, L. Tello. 'An energy balance climate model with hysteresis'. *Nonlinear Analysis.* 64, (2006) pp. 2053-2074.

[8] J. I. Díaz, L. Tello, *Sobre un modelo climatico de balance de energia superficial acoplado con un oceano profundo*, Actas del XVII CEDYA/ VI CMA, (2001).

[9] J. I. Díaz, L. Tello, On a parabolic problem with diffusión on the boundary arising in Climatology. In

International Conference on Differential Equations. World Scientific, New Jersey 2005. 1056-1058.

[10] J. I. Díaz, L. Tello. A 2D climate energy balance model coupled with a 3D deep ocean model *Electronic Journal of Differential Equations*, Conf. 16, (2007).

[11] J.I. Díaz and L. Tello. On a climate model with a dynamic nonlinear diffusive boundary condition. *Discrete and Continuous Dynamical Systems* series S, Vol 1, N. 2, (2008), 253-262.

[12] P.G.Drazin and D.H. Griffel , On the branching structure of diffusive climatological models, *J. Atmos. Sci.,* 1977; **34**: 1969-1706.

[13] L.D.D. Harvey and S.H. Schneider, Transient Climate Response to External Forcing on 100–104 Year Time Scales Part 1: Experiments With Globally Averaged, Coupled, Atmosphere and Ocean Energy Balance Models, *J. Geophys. Res.,* 90(D1), (1985) 2191–2205.

[14] L.D.D. Harvey and S.H. Schneider, Transient Climate Response to External Forcing on 100–104 Year Time Scales Part 1: Experiments With Globally Averaged, Coupled, Atmosphere and Ocean Energy Balance Models, *J. Geophys. Res.*, 90(D1), (1985), 2207-2222.

[15] A. Henderson and K.M. McGuffie, *A Climate Modelling Primer*, John Wiley and Sons, Chicheter, UK, 1987.

[16] G. Hetzer, The structure of the principal component for semilinear diffusion equations from energy balance climate models. *Houston Journal of Math.* **16** (1990), 203-216.

[17] K.-Y. Kim, G.R. North and J.Huang, On the Transient Response of a Simple Coupled Climate System, *Journal of Geophysical Research*, vol. 97, NO. D9 (1992), 10.069-10.081.

[18] J.-L. Lions, R.Temam, and S.Wang, New formulations of the primitive equations of atmosphere and applications, *Nonlinearity,* **5**, 1992, 237-288.

[19] J.-L. Lions, R.Temam, and S.Wang, On the Equations of the Large-scale Ocean, New formulations of the primitive equations of atmosphere and applications, *Nonlinearity,* **5**, 1992, 1007-1053.

[20] J.-L. Lions, R.Temam, and S.Wang, Models for the coupled atmosphere and ocean, *Computational Mechanics Advances*, **1,** 1993.

[21] M. C. Morantine, and R. G. Watts, Time scales in energy balance climate models 2: The intermediate time solutions, *J. Geophys. Res.*, 99(D2), (1994) 3643–3653.

[22] G.R. North, Introduction to simple climate models, in the book *Mathematics, Climate and Environment* (J.I. Díaz and J.L. Lions, eds.) Masson, Paris, (1993), 139-159.

[23] J. Peixoto and A.H. Oort, *Physics of Climate*, American Institute of Physics, New York, 1992.

[24] S.H. Schneider and T. Gal-Chen, Numerical Experiments in Climate Stability. *Journal of Geophysical Research*, vol. 78, NO. 27(1973), 6182-6194.

[25] W.D. Sellers. A global climatic model based on the energy balance of the earth-atmosphere system, *J. Appl. Meteorol.* 8 (1969) 392-400.

[26] P.H. Stone. A simplified radiative - dynamical model for the static stability of rotating atmospheres, *Journal of the Atmospheric Sciences,* **29**, No. 3, 405-418 (1972).

[27] K.E. Trenberth ed., *Climate System Modeling*, Cambridge University Press, Cambridge, 1992.

[28] E.J.M. Veling and M.E. Wit, A simple two-dimensional climate model with ocean and athmosphere coupling. In *Predictability and Nonlinear Sciences and Economics*, J. Grasman and G. van Straten eds, Kluwer, Acad. Publish. 1994, 95-112.

[29] R.G. Watts, The Mathematics in Climate Change, In *Mathematics Applied to Science: In Memoriam Edward D. Conway*, J. Goldstein *et al.* eds. 1988, Academic Press, Boston, 263-309.

[30] RG Watts, *Engineering response to global climate change: planning a research and development agenda*, CRC Press, Boca Raton, 1997.

[31] R.G. Watts and M. Morantine. Rapid climatic change and the deep ocean, *Climatic Change* , **16**, (1990) 83-97.

[32] R. G.Watts, M. C. Morantine and K. A. Rao, Timescales in energy balance climate models 1. The limiting case solutions, *J. Geophys. Res.*, 99(D2), (1994) 3631–3641.