On the approximate controllability of Stackelberg-Nash strategies

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1 Introduction

Let us consider a *distributed system,* i.e. a system whose state is defined by the solution of a Partial Differential Equation (PDE). We assume that we can *act* on this system by a *hierarchy of controls.* There is a "global" control v, which is the *leader*, and there are N "local" controls, denoted by $w_1, \ldots, w_N$, which are the *followers.* The followers, assuming that the leader has made a choice v of its policy, look for a *Nash equilibrium* of their cost functions (the criteria they are interested in). *Then* the leader makes its final choice for the whole system. *This is the Stackelberg–Nash strategy.*

Such situations arise in very many fields of Environment and of Engineering (and, by the way, for systems not necessarily described by PDE's). In order to explain more precisely our motivation, let us choose here an example taken from Environment: let us consider a resort lake, represented by a domain Ω of $\mathbb{R}^3$. The state of the system is denoted by $\mathbf{y}$. It is a vector function $\mathbf{y} = \{y_1, \ldots, y_N\}$, each y_i being a function of x and t, $x \in \Omega$, $t =$ time. The y_i's correspond to concentrations of various chemicals in the lake Ω or of living organisms. The y_i's are therefore given by the solution of a set of *diffusion equations.* In the resort, there are *local agents* or *local plants,* $P_1, \ldots, P_N$. Each plant P_i can decide (with some constraints) its policy w_i. There is also a general manager of the resort. He (or she) has the choice of the policy denoted by v. Therefore the *state equations* are given by

$$\frac{\partial \mathbf{y}}{\partial t} + \mathcal{A}(\mathbf{y}) = \text{sources} + \text{sinks} + \text{global control } v + \text{local control } \{w_1, \ldots, w_N\}, \quad (1)$$

where the *initial state* is supposed to be given,

$$\mathbf{y}(x, 0) = \mathbf{y}_0(x), \quad (2)$$

and where there are appropriate boundary conditions (of course this is made more precise in the next section of this paper). The general goal of the manager v is to maintain the lake as "clean" as possible. In other words, if the situation at $t = 0$ is not entirely satisfactory, he (or she) wants to "drive the system" at a chosen time horizon T *as close as possible to an ideal state,* denoted by $\mathbf{y}^T$. Each plant P_i has essentially the same goal, but of course, P_i will be particularly careful to the state $\mathbf{y}$ *near its location.* Let ρ_i be a smooth function given in $\overline{\Omega}$ such that

$$\rho_i(x) \geq 0, \ \rho_i = 1 \text{ near the location of } P_i. \tag{3}$$

Then P_i will try to choose w_i such that the state at time T, $\mathbf{y}(x,T)$, be "close" to $\rho_i \mathbf{y}^T$, and to achieve this *at minimum cost.* This leads to the introduction of

$$J_i(v; w_1, \ldots, w_N) = \frac{1}{2} |||w_i|||^2 + \frac{\alpha_i}{2} \left\| \rho_i(\mathbf{y}(.,T) - \mathbf{y}^T) \right\|^2, \tag{4}$$

where $|||w_i|||$ represents the cost of w_i, α_i is a given positive constant and $\left\|\rho_i(\mathbf{y}(.,T) - \mathbf{y}^T)\right\|$ is a measure of the "localized distance" between the actual state at time T and the desired state $\mathbf{y}^T$.

Remark 1.1 We have assumed here that the system (1), (2) (together with appropriate boundary conditions) admits a unique solution $\mathbf{y}(x,t;v;w_1,\ldots,w_N)$. In (4), $\mathbf{y}(.,T)$ denotes the function $x \mapsto \mathbf{y}(x,T;v;w_1,\ldots,w_N)$.

The "local" controls $w_1, \ldots, w_N$ assume that *the leader* has made a choice v and they try to find a *Nash equilibrium* of their cost J_i, i.e. they look for $w_1, \ldots, w_N$ (*as functions of* v) such that

$$\begin{aligned} &J_i(v; w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_N) \leq J_i(v; w_1, \ldots, w_{i-1}, \widehat{w_i}, w_{i+1}, \ldots, w_N), \\ &\text{for all } \widehat{w_i}, \text{ for } i = 1, \ldots, N. \end{aligned} \tag{5}$$

If $\mathbf{w} = \{w_1, \ldots, w_N\}$ satisfies (5), one says it is a *Nash equilibrium.*

The leader v wants now that the *global* state (i.e. the state $\mathbf{y}(.,T)$ in the whole domain Ω) to be *as close as possible to* $\mathbf{y}^T$. This will be possible, for any given function $\mathbf{y}^T$, if the problem is *approximately controllable,* i.e. if

$$\begin{aligned} &\mathbf{y}(x,t;v;w_1,\ldots,w_N) \text{ describes a dense subset of the given state} \\ &\text{space when } v \text{ spans the set of all controls available to the leader.} \end{aligned} \tag{6}$$

Remark 1.2 We emphasize again that in (6) the controls w_i are chosen so that (5) is satisfied. Therefore they are functions of v.

Remark 1.3 The above strategy is of the *Stackelberg's type.* This strategy has been introduced by Stackelberg [12] in 1934 for problems arising in Economics. It has been used in problems of distributed systems in Lions [7], without reference to controllability questions and in Lions [8] in a different setting *without using Nash equilibria.*

Remark 1.4 We have explained the family of problems we are interested in for environment questions, but problems of this type arise in many other questions, such as the control of large engineering systems.

Remark 1.5 It is clear that $\mathbf{y}^T$ is *not* going to be an *arbitrary* function in the state space. Therefore the resort could be maintained in a satisfactory state even *without* the system being approximately controllable (in the sense of (6)). But if there is a serious degradation following, for instance, an accident, then the *initial* state can be "anything" so that it is certainly preferable to live in a "controllable resort"...

Remark 1.6 Of course, the *Stackelberg's type strategy* is not the only possible! One could also replace the *Nash equilibrium* by a *Pareto equilibrium* for the followers $w_1, \ldots, w_N$ (see, for instance, Lions [9]). Here all the controls w_i agree to work in a strategy where v is the leader, and they agree to work in the context of a *Nash equilibrium.* Their personal (selfish) interests are expressed in the cost functions J_i as we shall see in the next section.

Remark 1.7 In the above context *there does not always exist a Nash equilibrium.* We prove in Section 4 some sufficient conditions for the existence and uniqueness of a *Nash equilibrium.* We also present a general counterexample showing that those conditions are, in some sense, necessary. What we (essentially) show in this paper (the first of a series) is that for *linear* systems, if there is existence and uniqueness of a Nash equilibrium for the followers, then the leader *can control the system* (in the sense of approximate controllability). The study of the case of nonlinear systems is the main subject of Díaz and Lions [2].

The content of the rest of this paper is the following: In the next section we make precise the statement of our main result by taking *one* state equation, i.e. y is a *scalar* function y instead of a vector function $\{y_1, \ldots, y_N\}$. This is just for the sake of simplicity of the exposition. It is by no means a serious restriction. But we shall make a *very strong* assumption, namely that the state equation *is linear.* The proof of the approximate controllability will be given in Section 3. The study of suitable assumptions (and their optimality) implying the existence and uniqueness of a *Nash equilibrium* is carried out in Section 4. Finally, some further remarks are presented in Section 5.

2 Statement of the approximate controllability theorem

Let A be a second order elliptic operator in Ω :

$$A\varphi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial \varphi}{\partial x_j}\right) + \sum_{i=1}^{N} a_i(x)\frac{\partial \varphi}{\partial x_i} + a_0(x)\varphi, \tag{7}$$

where all coefficients are smooth enough and where

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{N} \xi_i^2, \ \alpha > 0, \ x \in \overline{\Omega}. \tag{8}$$

We assume that the state equation is given by

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^{N} w_i\chi_i \tag{9}$$

where

$$\left.\begin{aligned} &\chi \text{ is the characteristic function of } \mathcal{O} \subset \Omega, \text{ and} \\ &\chi_i \text{ is the characteristic function of } \mathcal{O}_i \subset \Omega. \end{aligned}\right. \tag{10}$$

Remark 2.1 The control function $v(x,t)$ of the leader is distributed in $\mathcal{O}$ and the control function $w_i(x,t)$ of the follower "i" is distributed in $\mathcal{O}_i$.

Remark 2.2 All the results to follow are also valid for *boundary controls.* The case of distributed controls permits to avoid some difficulties of a purely technical type.

We assume that the *initial state* is

$$y(x,0) = 0, \; x \in \Omega. \tag{11}$$

Remark 2.3 Since the system is *linear*, there is no restriction in assuming the initial state to be zero, in the same way as there is no restriction in assuming in (9) that sources + sinks are zero (compare to (1)).

We assume that the *boundary conditions* are

$$y = 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{12}$$

Remark 2.4 Again (12) is not at all a serious restriction. We could consider as well y to be nonzero and that the following results apply for other boundary conditions.

We introduce now functions ρ_i such that

$$\left.\begin{aligned} &\rho_i \in L^\infty(\Omega), \; \rho_i \geq 0, \\ &\rho_i = 1 \; \text{ in a domain } \mathcal{G}_i \subset \Omega, \end{aligned}\right\} \tag{13}$$

and we define the cost function J_i (compare to (4))

$$J_i(v; w_1, \ldots, w_N) = \frac{1}{2}\int_0^T \int_{O_i} w_i^2 \, dx dt + \frac{\alpha_i}{2} \left\| \rho_i y(T; v, \mathbf{w}) - \rho_i y^{\mathbf{T}} \right\|^2, \tag{14}$$

where $\|\cdot\|$ is the norm in $L^2(\Omega)$.

Remark 2.5 In the case of the example presented in the Introduction, $\mathcal{G}_i$ is the region of the lake the plant P_i is particularly interested in (the place near P_i for instance!). If P_i is selfish, then $\rho_i = 0$ outside $\mathcal{G}_i$.

Remark 2.6 From a mathematical view point, the only hypothesis needed on ρ_i is that $\rho_i \in L^\infty(\Omega)$ (one could even take ρ_i in a suitable $L^p(\Omega)$ space, but this is irrelevant here).

Remark 2.7 We assume that

$$v \in L^2(\mathcal{O} \times (0,T)), \; w_i \in L^2(\mathcal{O}_i \times (0,T))$$

and that $y(x,t;v,\mathbf{w})$ is *the* solution of (9), (11), (12).

Given $v \in L^2(\mathcal{O} \times (0,T))$, we now define (cf. (5))

$$\left.\begin{array}{c} \mathbf{w} = \{w_1,\ldots,w_N\}, \text{ a } \textit{Nash equilibrium} \text{ for the cost,} \\ \text{and functions } J_1,\ldots,J_N \text{ given by (14).} \end{array}\right\} \quad (15)$$

We will show in Section 3 how (under hypotheses which are presented in Section 4) that this *Nash equilibrium* can be defined as a function of v:

$$\mathbf{w} = \mathbf{w}(v) \text{ or } w_i = w_i(v), \; i = 1,\ldots,N. \quad (16)$$

We then replace in (9) w_i by $w_i(v)$:

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^{N} w_i(v)\chi_i \quad (17)$$

subject to (11) and (12). The system (17), (11) and (12) admits a unique solution $y(x,t;v,\mathbf{w}(v))$. In Section 3 we prove the following result.

Theorem 2.1 *Assume that*

the set of inequalities (5) *admits a unique solution (a Nash equilibrium).* (18)

Then, when v spans $L^2(\mathcal{O} \times (0,T))$, the functions $y(.,T;v,\mathbf{w}(v))$ describe a dense subset of $L^2(\Omega)$. In other words,

there is approximate controllability of the system
when a strategy of the Stackelberg–Nash type is followed. (19)

3 Proof of the main theorem

3.1 Nash equilibrium

We have (5) iff

$$\int_0^T \int_{O_i} w_i \widehat{w_i}\, dxdt + \alpha_i \int_\Omega \rho_i^2 (y(T;v,\mathbf{w}) - y^T)\, \widehat{y_i}(T)\, dx = 0, \; \forall \widehat{w_i}, \quad (20)$$

where $\widehat{y_i}$ is defined by

$$\left.\begin{aligned} &\frac{\partial \widehat{y_i}}{\partial t} + A\widehat{y_i} = \widehat{w_i}\chi_i, \\ &\widehat{y_i}(0) = 0 \text{ in } \Omega,\ \widehat{y_i} = 0 \text{ in } \partial\Omega \times (0,T). \end{aligned}\right\} \tag{21}$$

In order to express (20) in a convenient form, we introduce the adjoint state p_i defined by

$$\left.\begin{aligned} &-\frac{\partial p_i}{\partial t} + A^* p_i = 0 \text{ in } \Omega \times (0,T), \\ &p_i(x,T) = \rho_i^2(x)(y(x,T;v,\mathbf{w}) - y^T(x)) \text{ in } \Omega, \\ &p_i = 0 \text{ in } \partial\Omega \times (0,T), \end{aligned}\right\} \tag{22}$$

where A^* stands for the adjoint of A. If we multiply (22) by $\widehat{y_i}$ and if we integrate by parts, we find

$$\int_\Omega \rho_i^2 (y(T;v,\mathbf{w}) - y^T)\, \widehat{y_i}(T)\, dx = \int_0^T \int_\Omega p_i \widehat{w_i}\chi_i \, dxdt,$$

so that (20) becomes

$$\int_0^T \int_{O_i} (w_i + \alpha_i p_i)\widehat{w_i}\, dxdt = 0,\ \forall \widehat{w_i},$$

i.e.

$$w_i + \alpha_i p_i \chi_i = 0. \tag{23}$$

Then, if $\mathbf{w} = \{w_1, \ldots, w_N\}$ is a *Nash equilibrium*, we have

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} + Ay + \sum_{i=1}^N \alpha_i p_i \chi_i = v\chi, \\ &-\frac{\partial p_i}{\partial t} + A^* p_i = 0,\ i = 1, \ldots, N, \\ &y(0) = 0,\ p_i(x,T) = \rho_i^2(x)(y(x,T;v,\mathbf{w}) - y^T(x)) \text{ in } \Omega, \\ &y = 0,\ p_i = 0 \text{ in } \partial\Omega \times (0,T). \end{aligned}\right\} \tag{24}$$

We recall that here we are assuming the existence and uniqueness of a *Nash equilibrium* (hypothesis (18)). We return to that in Section 4.

3.2 Approximate controllability: Proof of Theorem 2.1

We want to show that the set described by $y(.,T;v)$ is dense in $L^2(\Omega)$, where y is the solution given by (24) and when v spans $L^2(\mathcal{O} \times (0,T))$. We do not restrict the problem by assuming that

$$y^T \equiv 0$$

(it suffices to use a translation argument). Let f be given in $L^2(\Omega)$ and let us assume that

$$(y(.,T;v), f) = 0,\ \forall v \in L^2(\Omega). \tag{25}$$

We want to show that $f \equiv 0$. Let us introduce the solution $\{\varphi, \psi_1, \ldots, \psi_N\}$ of the adjoint system

$$\left.\begin{aligned} &-\frac{\partial\varphi}{\partial t} + A^*\varphi = 0, \\ &\frac{\partial\psi_i}{\partial t} + A\psi_i = -\alpha_i\varphi\chi_i, \\ &\varphi(T) = f + \textstyle\sum_i \psi_i(T)\rho_i^2, \\ &\psi_i(0) = 0, \\ &\varphi = 0,\ \psi_i = 0 \text{ in } \partial\Omega \times (0,T). \end{aligned}\right\} \tag{26}$$

We multiply the first (resp. the second) equation in (26) by y (resp. p_i). We obtain

$$\left.\begin{aligned} &-(f + \sum_i \psi_i(T)\rho_i^2, y(T)) + \int_0^T\int_\Omega \varphi\left(\frac{\partial y}{\partial t} + Ay\right) dxdt + \\ &\sum_i (\psi_i(T), p_i(T)) + \\ &+\sum_i \int_0^T\int_\Omega \psi_i\left(-\frac{\partial p_i}{\partial t} + A^* p_i\right) dxdt = -\sum_i \alpha_i \int_0^T\int_\Omega \varphi p_i\chi_i\, dxdt. \end{aligned}\right\} \tag{27}$$

Using (24) (where $y^T \equiv 0$), (27) reduces to

$$-(f, y(T)) + \int_0^T\int_\Omega \varphi v\chi\, dxdt = 0. \tag{28}$$

Therefore, if (25) holds, then

$$\varphi = 0 \text{ on } \mathcal{O} \times (0,T). \tag{29}$$

Using Mizohata's Uniqueness Theorem (see Mizohata [5] or Saut and Scheurer [10]) —this is the only place where some smoothness on the coefficients of A is needed— it follows from $(26)_1$ and (29) that

$$\varphi = 0 \text{ on } \Omega\times(0,T). \tag{30}$$

Then $(26)_2$, $(26)_4$ and $\psi_i = 0$ in $\partial\Omega \times (0,T)$ imply that

$$\psi_i = 0 \text{ in } \Omega \times (0,T),\ i = 1, \ldots, N, \tag{31}$$

so that $(26)_3$ gives $f \equiv 0$.

4 On the existence and uniqueness of Nash equilibrium

4.1 A criterion of existence and uniqueness

We consider the functionals (14). Let us define

$$\left.\begin{aligned} &\mathcal{H}_i = L^2(\mathcal{O}_i \times (0,T)), \\ &\mathcal{H} = \textstyle\prod_{i=1}^N \mathcal{H}_i, \\ &L_i\widehat{w_i} = \widehat{y_i}(T) \text{ (cf. (21)), which defines } L_i \in L(\mathcal{H}_i; L^2(\Omega)). \end{aligned}\right\} \tag{32}$$

Since v is fixed, one can write

$$y(T; v, \mathbf{w}) = \sum_{i=1}^{N} L_i w_i + z^T, \ z^T \text{ fixed.} \tag{33}$$

With these notations (14) can be rewritten

$$J_i(v; \mathbf{w}) = \frac{1}{2} \|w_i\|_{\mathcal{H}_i}^2 + \frac{\alpha_i}{2} \left\| \rho_i \left(\sum_j L_j w_j - \eta^T \right) \right\|^2 \tag{34}$$

where $\eta^T = y^T - z^T$. Then $\mathbf{w} \in \mathcal{H}$ is a *Nash equilibrium* iff

$$(w_i, \widehat{w_i})_{\mathcal{H}_i} + \alpha_i \left(\rho_i \left(\sum_j L_j w_j - \eta^T \right), \ \rho_i L_i \widehat{w_i} \right) = 0, \ i = 1, \dots, N, \ \forall \widehat{w_i}. \tag{35}$$

or

$$w_i + \alpha_i L_i^* \left(\rho_i^2 \sum_{j=1}^{N} L_j w_j \right) = \alpha_i L_i^* \left(\rho_i^2 \eta^T \right), \ i = 1, \dots, N \tag{36}$$

(where $L_i^* \in \mathcal{L}(L^2(\Omega); \mathcal{H}_i)$ is the adjoint of L_i), or equivalently

$$\left.\begin{aligned} &\mathbf{L}\mathbf{w} = \text{given in } \mathcal{H}, \\ &\mathbf{L} \in \mathcal{L}(\mathcal{H}; \mathcal{H}), \\ &(\mathbf{L}\mathbf{w})_i = w_i + \alpha_i L_i^* \left(\rho_i^2 \textstyle\sum_{j=1}^{N} L_j w_j \right). \end{aligned}\right\} \tag{37}$$

Then we have

Proposition 4.1 *Assume that*

$$\alpha_i = \alpha, \text{ for all } i, \tag{38}$$

and that

$$\alpha \left\| \rho_i - \rho_j \right\|_{L^\infty(\Omega)} \ \|\rho_i\|_{L^\infty(\Omega)} \text{ is small enough, for any } i, j = 1, \dots, N. \tag{39}$$

Then $\mathbf{L}$ *is invertible. In particular there is a unique Nash equilibrium of* (14).

Remark 4.1 Of course, if $N = 1$ the situation is much simpler. In that case,

$$(\mathbf{L}w, w) = \|w_1\|^2 + \alpha_1 \|\rho_1 L_1 w_1\|^2,$$

hence $\mathbf{L}$ is *coercive* and so the existence and uniqueness of a *minimum* w of $J_1(v; w)$, when v is fixed, is a classical result.

Proof of Proposition 4.1: In the general case $N > 1$, one has

$$(\mathbf{Lw}, \mathbf{w}) = \sum_i \|w_i\|^2_{\mathcal{H}_i} + \sum_j \alpha_i \left(\rho_i \sum_j L_j w_j, \, \rho_i L_i w_i \right). \quad (40)$$

Then one can write

$$(\mathbf{Lw}, \mathbf{w}) = \sum_{i=1}^N \|w_i\|^2_{\mathcal{H}_i} + \alpha \left\| \sum_{i=1}^N \rho_i L_i w_i \right\|^2 + \alpha \sum_{i,j=1}^N (\rho_i - \rho_j)^2 (L_j w_j, \, \rho_i L_i w_i). \quad (41)$$

Applying Young's inequality, it follows that, under hypothesis (39), $\mathbf{L}$ is coercive, i.e.

$$(\mathbf{Lw}, \mathbf{w}) \geq \gamma \|\mathbf{w}\|^2_{\mathcal{H}}, \text{ for some } \gamma > 0. \quad (42)$$

The conclusion is now a consequence of the Lax–Milgram theorem.

Remark 4.2 The hypothesis (39) is certainly satisfied if $\rho_i = \rho$ for all i, in which case there is *only* one function $J_i = J_1$ for all i, and we are back to Remark 4.1 (with $\mathbf{w} = \{w_1, \ldots, w_N\}$).

4.2 Some non-existence and non-uniqueness results

We begin this subsection by some general considerations on the existence, or non-existence, of Nash equilibrium solutions.

Let $\mathcal{H}_i$, $\mathcal{K}_j$ be two families of N real Hilbert spaces ($i, j = 1, \ldots, N$), the scalar product (or norm) in a space $\mathcal{H}$ being denoted by $(\, , \,)_{\mathcal{H}}$ (or $\| \, \|_{\mathcal{H}}$).

We consider linear continuous operators $a_{i,j}$

$$a_{i,j} \in \mathcal{L}(\mathcal{H}_j, \mathcal{K}_i), \; \forall i, j, \quad (43)$$

and we assume that

$$a_{i,j} \text{ is compact}, \; \forall i, j. \quad (44)$$

We define $\mathbf{w} = \{w_1, \ldots, w_N\}$, $\mathbf{w} \in \mathcal{H} = \prod_{i=1}^N \mathcal{H}_i = \prod_{i=1}^N \mathcal{K}_i$,

$$J_i(\mathbf{w}) = \frac{1}{2} \|w_i\|^2_{\mathcal{H}_i} \, dxdt + \frac{\alpha_i}{2} \left\| \sum_{j=1}^N a_{i,j} w_j - \eta_i \right\|^2_{\mathcal{K}_i} \quad (45)$$

where α_i is a positive given constant, and where

$$\eta = \{\eta_1, \ldots, \eta_N\} \text{ is given in } \prod_{i=1}^N \mathcal{K}_i. \quad (46)$$

We are looking for the Nash equilibrium points of the functionals $J_1, \ldots, J_N$. We are going to show that "in general" with respect to $\alpha = \{\alpha_i\} \in \mathbb{R}^N_+$, there *exists a unique Nash equilibrium for the functionals J_i. When α is "exceptional" in $\mathbb{R}_+^N$, then "in general" with respect to $\eta = \{\eta_i\} \in \prod_{i=1}^N \mathcal{K}_i$, there is no solution. When α and η are "exceptional", there is a finite dimensional subspace of solutions in $\prod_{i=1}^N \mathcal{K}_i$.*

Of course, this "result" has to be made precise. An element $\mathbf{w} = \{w_1, \ldots, w_N\}$ is a Nash equilibrium iff

$$(w_i, \widehat{w_i})_{\mathcal{H}_i} + \alpha_i \left(\sum_j a_{ij} w_j - \eta_i, a_{ii} \widehat{w_i} \right)_{\mathcal{K}_i} = 0, \; i = 1, \ldots, N, \; \forall \widehat{w_i} \in \mathcal{K}_i$$

i.e.

$$a_{ii}^* \sum_{j=1}^N a_{ij} w_j + \frac{1}{\alpha_i} w_i = a_{ii}^* \eta_i, \; i = 1, \ldots, N, \tag{47}$$

where $a_{ij}^* \in \mathcal{L}(\mathcal{K}_i, \mathcal{H}_j)$ denotes the adjoint of a_{ij}.

Let us define

$$\left. \begin{aligned} &\mathcal{A} \in \mathcal{L}\left(\textstyle\prod_{i=1}^N \mathcal{H}_i, \prod_{i=1}^N \mathcal{H}_i\right), \\ &\mathcal{A}\mathbf{w} = \{a_{ii}^* \textstyle\sum_{j=1}^N a_{ij} w_j\}, \end{aligned} \right\} \tag{48}$$

$$\left(\frac{1}{\boldsymbol{\alpha}}\right) = \text{diagonal operator } \{w_i\} \mapsto \left\{ \frac{1}{\alpha_i} w_i \right\}, \tag{49}$$

$$\zeta_i = a_{ii} \eta_i, \; \boldsymbol{\zeta} = \{\zeta_i\}. \tag{50}$$

Then (47) is equivalent to

$$\mathcal{A}\mathbf{w} + \left(\frac{1}{\boldsymbol{\alpha}}\right) \mathbf{w} = \boldsymbol{\zeta}, \text{ in } \mathcal{H} = \prod_{i=1}^N \mathcal{H}_i, \tag{51}$$

where, by virtue of (44), $\mathcal{A}$ is compact in $\mathcal{L}(\mathcal{H}, \mathcal{H})$. Then the "result" stated above is a trivial consequence of the classical Fredholm alternative. Indeed, let us consider the $\boldsymbol{\alpha}$'s such that

$$\frac{1}{\alpha_i} = \gamma_i \lambda, \; \gamma_i \text{ fixed}, \tag{52}$$

all these numbers being positive. Then, according to the Fredholm alternative, (51) and (52) admits a *unique solution except for a countable set of λ's*. This makes precise the fact that there is, "in general" with respect to $\boldsymbol{\alpha}$, a unique solution. If λ belongs to the spectrum of $\mathcal{A} + \gamma\lambda$, then there is a solution iff $\boldsymbol{\zeta}$ is orthogonal to the null space of $\mathcal{A}^* + \gamma$, a conclusion which is "in general" not satisfied by $\boldsymbol{\zeta}$, i.e. by $\boldsymbol{\eta} = \{\eta_i\}$. If it is satisfied, then there is a finite dimensional space of solutions.

Remark 4.3 Of course, the formula (51) does *not* use the hypothesis (44). *Therefore, one has that without the hypotesis* (44) *there exists a unique Nash equilibrium if*

$$\|\boldsymbol{\alpha}\mathcal{A}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} < 1 \tag{53}$$

(where $(\boldsymbol{\alpha}\mathcal{A})\mathbf{w} = \left\{\alpha_i a_{ii}^* \sum_j a_{ij} w_j\right\}$).

All the above remarks apply to (32), (33) if we take

$$a_{ij} = \rho_i L_j, \quad \eta_i = \rho_i \eta^T, \quad \mathcal{K}_i = L^2(\Omega),\ \forall i \tag{54}$$

(then (53) amounts to $\alpha \left\|\rho_i - \rho_j\right\|_{L^\infty(\Omega)} \|\rho_i\|_{L^\infty(\Omega)}$ being small enough) if one verifies that L_j, as defined by

$$L_i w_i = y_i(T),\ y_i \text{ solution of (20) (with } \widehat{w_i} \text{ replaced by } w_i), \tag{55}$$

is compact from $L^2(\mathcal{O}_i \times (0,T)) = \mathcal{H}_i$ into $L^2(\Omega)$.

If the coefficients of the operator A are smooth enough, then the solution y_i of (20) satisfies

$$y_i \in L^2(0,T : H^2(\Omega) \cap H_0^1(\Omega)),\ \frac{\partial y_i}{\partial t} \in L^2(0,T : L^2(\Omega))$$

(recall that $y_i(0) = 0$), so that $L_i \in \mathcal{L}(\mathcal{H}_i; H_0^1(\Omega))$, hence L_i is *compact* from $\mathcal{H}_i$ into $L^2(\Omega)$ (since the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact when Ω is bounded).

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