On the free boundary for quenching type parabolic problems via local energy methods

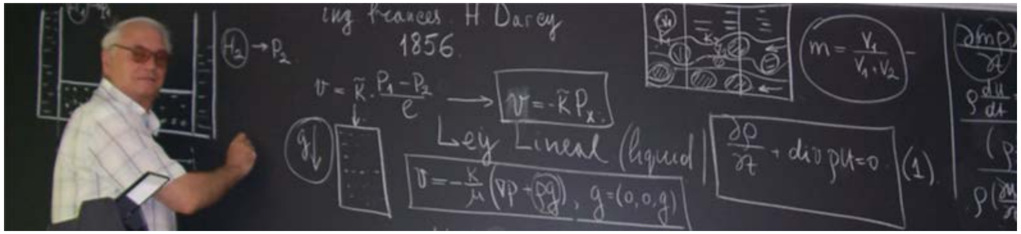
J.I.Díaz

The Energy of Mathematics:

Two Days in the Occasion of the

70th Anniversary of S.N. Antontsev

Madrid, UCM, November 11-12, 2013



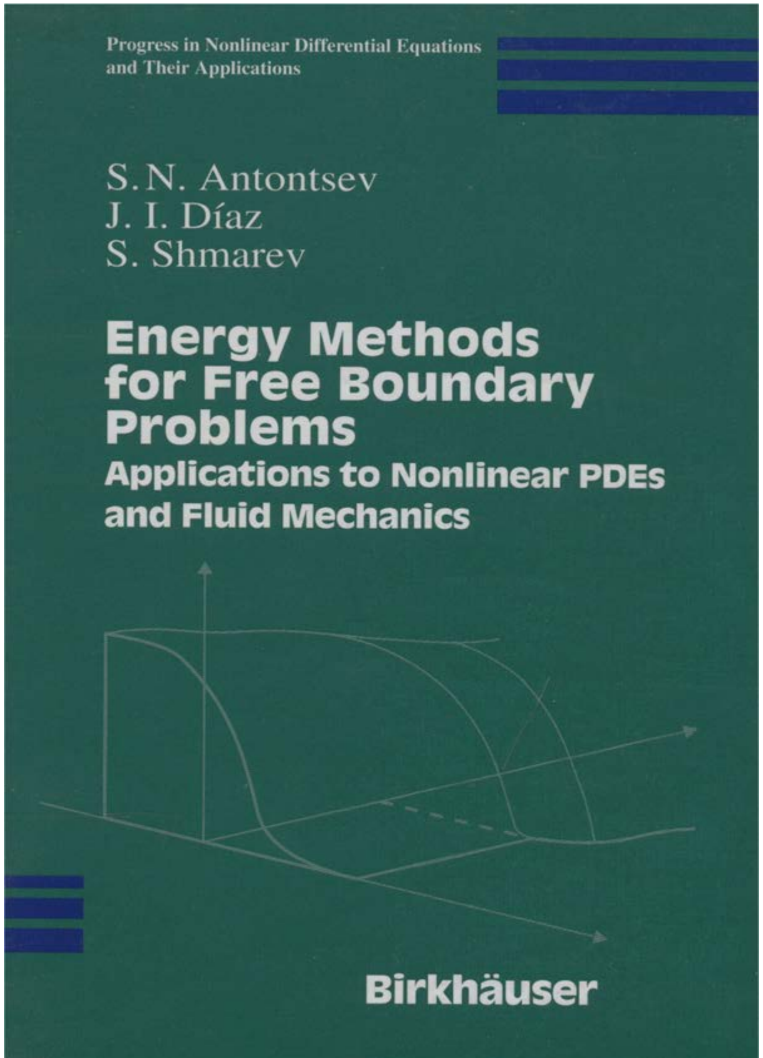
1. Introduction

In his pioneering paper,

"Some problems with free boundaries for the degenerating equations of gas dynamics"
Dinamika Sploshnoi Sredy, Novosibirsk, 1973, Vyp. 13, pp. 5-17 (Russian).

Stanislav N. Antontsev proposed the idea of a general method to prove the existence and location of free boundaries for quasilinear parabolic equations of degenerate type.

Since then, many other paper developed this clever idea giving rise to a general methodology which today is applicable to nonlinear partial differential equations on any type (elliptic, parabolic and hyperbolic) and of any order (higher order too) and systems


Progress in Nonlinear Differential Equations
and Their Applications
S.N. Antontsev
J. I. Díaz
S. Shmarev
Energy Methods
for Free Boundary
Problems
Applications to Nonlinear PDEs
and Fluid Mechanics
Birkhäuser

Some “posterior” papers on energy methods for free boundary problems:

F.Andreu, V. Caselles, J. I. Díaz, J. M. Mazón. Some Qualitative properties for the Total Variation Flow, Journal of Functional Analysis, 188, 516-547, 2002

Dirichlet problem

$$P_D \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{Du}{|Du|}\right) & \text{in} \quad Q = (0, \infty) \times \Omega \\ u(t, x) = 0 & \text{on} \quad S = (0, \infty) \times \partial\Omega \\ u(0, x) = u_0(x) & \text{in} \quad x \in \Omega \end{cases}$$

and of the Neumann problem

$$P_N \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{Du}{|Du|}\right) & \text{in} \quad Q = (0, \infty) \times \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad S = (0, \infty) \times \partial\Omega \\ u(0, x) = u_0(x) & \text{in} \quad x \in \Omega \end{cases}$$

S. N. Antontsev, J. I. Díaz, H. B. de Oliveira. On the confinement of a viscous fluid by means of a feedback external field, C.R. Mecanique, 330, 797-802, 2002

S. N. Antontsev, J. I. Díaz, H. B. de Oliveira. Stopping a viscous fluid by a feedback dissipative external field: I. The stationary Stokes problem. Rend. Mat. Acc. Lincei, s.9, 15, 257-270, 2004.

planar stationary flow of an incompressible viscous fluid

$$\Omega = (0,\infty)\times(0,L), \qquad \mathbf{u}(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x})) \quad \mathbf{x} = (x,y)\in\Omega,$$

$\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} - \nabla p \quad \text{in } \Omega,$$

$$\text{div}\mathbf{u} = 0 \quad \text{in } \Omega,$$

$$\mathbf{u}(0,y) = \mathbf{u}_*(y),\ y\in(0,L)$$

$$\mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0},\ x\in(0,\infty).$$

$$|\mathbf{u}(x,y)| \to 0,\ \text{as } x\to\infty \text{ and } \ y\in(0,L).$$

Problem: *can we find an external localized forces field* $\mathbf{f}$ *stopping the fluid at a finite distance, i.e., such that*

$$\mathbf{u}(x,y) = \mathbf{0} \ \text{ for } x > x_{\mathbf{u}} \text{ and } y\in(0,L), \ \textbf{?}$$

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \mathbf{0} \ \text{ for } x \geq x_{\mathbf{f}} \ \text{ and } y\in(0,L),$$

P. Bégout, J. I. Díaz. On a nonlinear Schrdinger equation with a localizing effect. Comptes Rendus Acad. Sci. Paris, t. 342, Série I, 2006, 459-463

P. Bégout and J. I. Díaz, Localizing Estimates of the Support of Solutions of Some Nonlinear Schrödinger Equations: The Stationary Case. Annales de l'Institut Henri Poincare (C) Non Linear Analysis, 29, (2012), 35-58

P. Bégout and J. I. Díaz, A sharper energy method for the localization of the support to some stationary Schrodinger equations with a singular nonlinearity. Discrete and Continuous Dynamical Systems

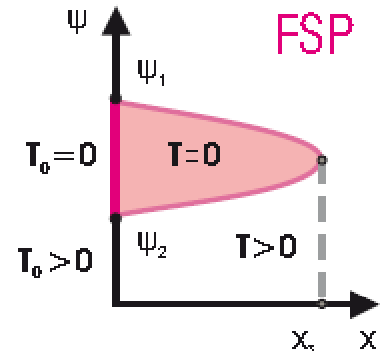
P. Bégout and J. I. Díaz, Self-similar solutions with compactly supported profile of some nonlinear Schrödinger equations. Submitted

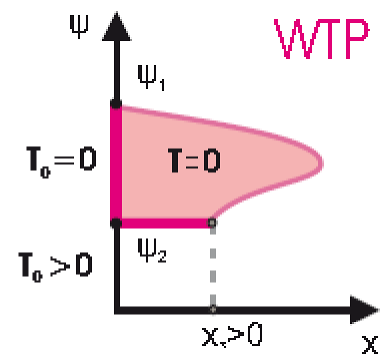
$$i\frac{\partial u}{\partial t}+\Delta u=a|u|^{-(1-m)}u+f(t,x),$$

S. Antontsev, J. I. Díaz, Mathematical analysis of the discharge of a laminar hot gas in a colder atmosphee, Rev. R. Acad. Cien.Serie A Matem, 101, 2007, 235-24

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi}\left(T^{\sigma-1} u \frac{\partial u}{\partial \psi}\right) + \frac{G}{u} T \left(1 - \frac{\varepsilon}{T}\right),$$

$$\frac{\partial T}{\partial x} = \frac{1}{Pr}\frac{\partial}{\partial \psi}\left(T^{\sigma-1} u \frac{\partial T}{\partial \psi}\right).$$





J. I. Díaz, Estimates on the location of the free boundary for the obstacle and Stefan problems by means of some energy methods , *Georgian Mathematical Journal* (special issue dedicated to the memory of J.L. Lions on the ocassion of his 80th birthday). 15, 2008, nº 3, 455-484.

$$\frac{\partial\psi(u)}{\partial t} - \operatorname{div}\mathbf{A}(x,t,u,Du) + B(x,t,u,Du) + C(x,t,u) + \beta(u) \ni f(x,t), \qquad (1)$$

where $\beta(u)$ is the maximal monotone graph given by $\beta(u) = \{0\}$ if $u \leq 0$ and $\beta(u) = \phi$ (the empty set) if $u < 0$. The general structural assumptions we shall made are the following

$$|\mathbf{A}(x,t,r,\mathbf{q})| \leq C_1|\mathbf{q}|^{p-1}, C_2\,|\mathbf{q}|^p \leq \mathbf{A}(x,t,r,\mathbf{q})\cdot\mathbf{q}, \qquad (2a)$$

$$|B(x,t,r,\mathbf{q})| \leq C_3\,|r|^{\alpha}\,|\mathbf{q}|^{\beta},\ 0 \leq C(x,t,r)\,r, \qquad (2b)$$

$$C_6\,|r|^{\gamma+1} \leq G(r) \leq C_5\,|r|^{\gamma+1},\ \text{where} \qquad (2c)$$

$$G(r) = \psi(r)\,r - \int_0^r \psi(\tau)\,d\tau.$$

Here $C_1 - C_6$, p, α, β, σ, γ, k are positive constants which will be specified later on. We shall also consider the Stefan problem

$$\frac{\partial\psi(u)}{\partial t} - \operatorname{div}\mathbf{A}(x,t,u,Du) + B(x,t,u,Du) + C(x,t,u) \ni f(x,t), \qquad (3)$$

where now $\psi(u)$ is the maximal monotone graph $\psi(u) = k_+u + L$ if $u > 0$, $\psi(u) = k_-u$ if $u < 0$, $\psi(0) = [0,L]$, with k_+, k_- and L positive constants. In both cases, we shall deal with weak solutions satisfying the initial condition

$$u(x,0) = u_0(x) \qquad x \in \Omega. \qquad (4)$$

J. I. Díaz, R. Glowinski, G. Guidoboni, T. Kim, Qualitative properties and approximation of solutions of Bingham flows: on the stabilization for large time and the geometry of the support. Rev. R. Acad. Cien. Serie A. Mat RACSAM 104 (1), 2010, 157–200

$$\varrho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = \nabla\cdot\sigma + \mathbf{f}(t,x) \quad \text{in } (0,T)\times\tilde{\Omega},$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,T)\times\tilde{\Omega},$$

$$\sigma = -p\mathrm{I} + \sqrt{2}g\frac{\mathrm{D}(\mathbf{u})}{|\mathrm{D}(\mathbf{u})|} + 2\mu\mathrm{D}(\mathbf{u}),$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad (\text{with } \nabla\cdot\mathbf{u}_0 = 0).$$

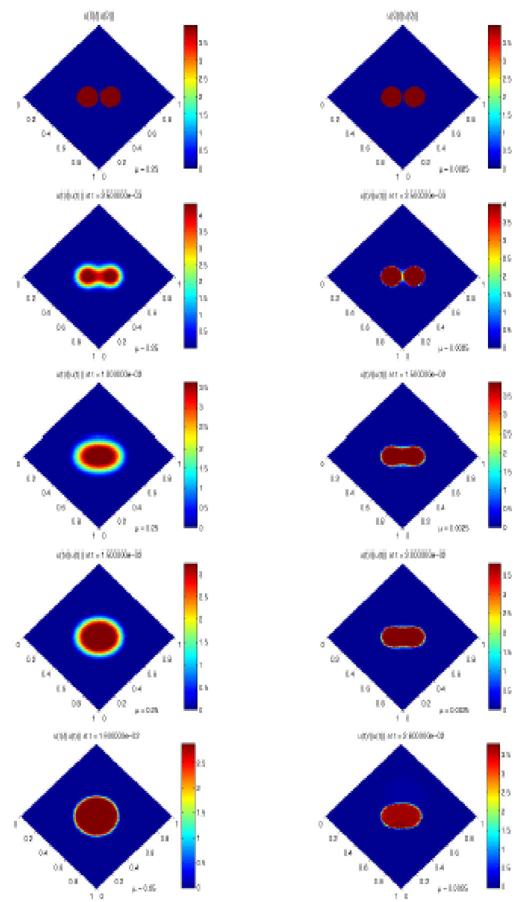


Figure 17. Case IV - *On the left:* Snapshots of the normalized solution $u(t,x)/||u||_{L^2(\Omega)}(t)$ obtained with $\mu = 0.25$ at $t = 0, 0.0025, 0.01, 0.015, 0.019$ seconds; *On the right:* Snapshots of the normalized solution $u(t,x)/||u||_{L^2(\Omega)}(t)$ obtained with $\mu = 0.0025$ at $t = 0, 0.0025, 0.015, 0.02, 0.028$ seconds.

Y. Belaud, J.I. Díaz, Abstract results on the finite extinction time property: application to a singular parabolic equation. Journal of Convex Analysis 17 (2010), No. 3&4, 827-860.

$$\begin{cases} u_t + Au \ni 0,\ t > 0 \text{ in } H, \\ u(0) = u_0, \end{cases}$$

$$\lambda_1(h) = \inf_{u \in H,\ ||u||^2 \geq h} (A^\circ u, u),$$

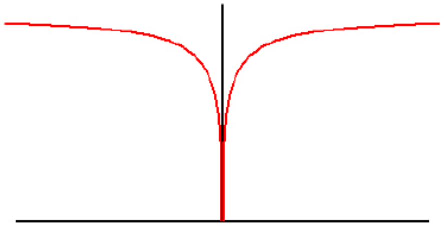
$$\int_0^1 \frac{dh}{\lambda_1(h)} < +\infty.$$

$$\begin{cases} u_t - \Delta u + a(x) u^q \chi_{u>0} = 0 & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{on} \quad \Omega, \end{cases}$$

$$-1 < q < 1.$$

a is measurable, positive a.e. and bounded on Ω *such that*

$$\left| \ln \frac{1}{a} \right|^s \in L^1(\Omega), \qquad \textit{for some } s > N/2.$$



A. N. Dao and J.I. Díaz, The existence of renormalized solutions with right-hand side data integrable with respect to the distance to the boundary, and application of the energy method for free boundary problems to some solutions out of the energy space. To appear.

$$-\Delta u + F(u) = f, \quad \text{in } \Omega, \quad F(u) = |u|^{\sigma-1}.u, \ \sigma \in (0,1)$$

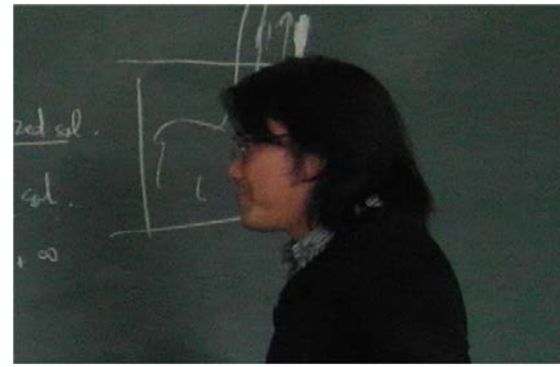
$f \in L^1_\delta(\Omega)$, with $\delta(x) = dist(x, \partial\Omega)$

u is called is a renormalized solution of (1) if $u \in L^1_{loc}(\Omega)$ and $\nabla T_k(u) \in L^2_\delta(\Omega)$, for any $k \geq 1$, and u fulfills the following equation

$$\int_\Omega < \nabla u, \nabla H(u) > v.J_\delta + < \nabla u, \nabla v > H(u).J_\delta$$

$$+ < \nabla u, \nabla J_\delta > H(u).v + |u|^{\sigma-1} u H(u) v J_\delta = \int_\Omega f.J_\delta H(u).v, \quad (2)$$

for any $H \in C^1_c(\mathbb{R})$; and $J_\delta \in C^1(\Omega) \cap C(\overline{\Omega})$ behaves as $\delta(x)$ near the boundary of Ω; and $v \in \mathcal{V}^\delta_0 \cap L^\infty(\Omega)$.



$\mathcal{V}^\delta_0$ is the subset of $\{g \in L^1_{loc}(\Omega), \nabla g \in L^2_\delta(\Omega)\}$ such that there exists a sequence $\{g_n\} \in C^\infty_c(\Omega)$ such that $\nabla g_n \to \nabla g$ in $L^2_\delta(\Omega)$, and $g_n \to g$ in $L^1_{loc}(\Omega)$.

Theorem

Let u be a renormalized solution of (1). Given an arbitrary point $x_0 \notin Supp(f)$,

$$u(x) = 0, \quad \text{a.e in } B(x_0, \varrho) \cap \Omega,$$

with

$$\varrho^\nu = (\varrho_0^\nu - \psi(E(\varrho_0), b(\varrho_0)))_+;$$

$\varrho_0 = min\{dist(x_0, Supp(f)), \frac{1}{2} dist(x_0, \partial\Omega)\}$; $\nu = \frac{\sigma+1}{N(1-\sigma)+2(1+\sigma)}$;
ψ is a positive function of $E(\varrho_0), b(\varrho_0)$;

Electron. J. Probab. **17** (2012), no. 10, 1–11.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-1768

Localization of solutions to stochastic porous media equations: finite speed of propagation*

Viorel Barbu† Michael Röckner‡

Consider the stochastic porous media equation

$$\begin{cases} dX - \Delta(|X|^{m-1}X)dt = \sigma(X)dW_t, & t \geq 0, \\ X = 0 & \text{on } \partial\mathcal{O}, \\ X(0) = x & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where $m \geq 1$, W_t is a Wiener process in $L^2(\mathcal{O})$ of the form

$$W_t = \sum_{k=1}^{N} \beta_k(t) e_k. \tag{1.2}$$

$\{\beta_k\}_{k=1}^N$ is a sequence of independent Brownian motions on a filtered probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ while $\{e_k\}_{k \in N}$ is an orthonormal system in $L^2(\mathcal{O})$ and

$$\sigma(X)W_t = \sum_{k=1}^{N} \mu_k X e_k \beta_k(t), \tag{1.3}$$

[1] S.N. Antontsev, On the localization of solutions of nonlinear degenerate elliptic and parabolic equations, *Dokl. Akad. Nauk SSSR*, 260 (1981), 1289-1293; English translation *Soviet Math. Dokl.*, 24 (1981), 420-424. MR-0636152

[2] S.N. Antontsev, J.I. Diaz, On space or time localization of solutions of nonlinear elliptic or parabolic equations via energy methods, in "Recent Advances in nonlinear Elliptic and Parabolic Problems", Ph. Benilan et al. (eds.), Pitman Research Notes in Mathematics, Longman, 1983, 2-14.

[3] S.N. Antontsev, J.I. Diaz, S. Shmarev, *Energy Methods for Free Boundary Problems*, Birkhäuser, Basel, 2002. MR-1858749

[4] S.N. Antontsev, S.I. Shmarev, A model porous medium equation with variable exponent nonlinearity: existence, uniqueness and localization properties of solutions, *Nonlinear Analysis*, 60 (2005), 515-545. MR-2103951

2. Local energy methods to the quenching problem

J.I. Díaz, On the free boundary for quenching type parabolic problems via local energy methods.
Invited paper to the journal
Communications on Pure and Applied Analysis,
Special issue in Memory of M.I. Vishik.

Professor Vishik visited Spain, mainly Barcelona and Madrid, from October 1, 1994 to January 31, 1995.

The main goal of this lecture is to present the application of this kind of energy methods to the case of equations involving terms with negative exponents (the so called **quenching problems**).

(1)
$$\begin{cases} u_t - \Delta u + u^{-k}\chi_{\{u>0\}} = \lambda u^q \chi_{\{u>0\}} + g(t,x) & \text{in } (0,T)\times\Omega, \\ u = \varphi & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{on } \Omega, \end{cases}$$

$k \in (0,1)$ $\quad$ $q \in \mathbb{R}$

$$w_t - \Delta_p(|w|^{m-1}w) + \frac{\chi_{\{w\neq 0\}}}{w^{\widehat{k}}} = \lambda w^{\widehat{q}}\chi_{\{w\neq 0\}} + g(t,x)$$

(2)

$$\frac{\partial \psi(u)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,u,D\,u) + C(x,t,u) = f(x,t,u),$$

$$|\mathbf{A}(x,t,r,\mathbf{q})| \leq C|\mathbf{q}|^{p-1}, C|\mathbf{q}|^p \leq \mathbf{A}(x,t,r,\mathbf{q})\cdot\mathbf{q},$$

$$C|r|^{\gamma+1} \leq G(r) \leq C^*|r|^{\gamma+1},$$

$$G(r) = \psi(r)\,r - \int_0^r \psi(\tau)\,d\tau,$$

$$C|r|^{\alpha} \leq C(x,t,r)\,r,$$

$$f(x,t,r)r \leq \lambda|r|^{q+1} + g(x,t)r,$$

$$p > 1, q \in \mathbb{R}$$

$$\gamma \in (0, p-1],$$

$$\alpha \in (0, \min(1, \frac{\gamma p}{(p-1)}))$$

Other formulations

$$\begin{cases} w_t - \Delta w = \frac{\chi_{\{u>0\}}}{(1-w)^k} & \text{in } (0,T_0)\times\Omega, \\ w = 1 & \text{on } (0,T_0)\times\partial\Omega, \\ w(0,.) = w_0, & \text{on } \Omega, \quad 0 \le w_0(x) \le 1 \end{cases}$$

H. Kawarada, On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1-u)$, *Publ. Res. Inst. Math. Sci.*, **10** (1974/75), 729–736.

D. Phillips, Existence of solutions of quenching problems, *Appl. Anal.*, **24** (1987), 253–264.

H. A. Levine, Quenching and beyond: a survey of recent results, in *Nonlinear mathematical problems in industry, II* (Iwaki, 1992), vol. 2 of GAKUTO Internat. Ser. Math. Sci. Appl., Gakkōtosho, Tokyo, 1993, 501–512.

B. Kawohl, Remarks on Quenching. *Doc. Math., J. DMV* **1**, (1996) 199-208.

M. Fila A. H. Levine, and J. L. Vázquez, *Stabilization of solutions of weakly singular quenching problems*, Proc. Amer. Math. Soc., 119 (1993), 555–559.

J. Dávila and M. Montenegro, Existence and asymptotic behavior for a singular parabolic equation, *Transactions of the AMS*, **357** (2004) 1801–1828.

The uniqueness of solution fails

M. Winkler, Nonuniqueness in the quenching problem, *Math. Ann.* **339** (2007), 559–597.

This is one of the reasons why it looks difficult to apply, directly, super and subsolutions methods to study such a free boundary. Our alternative is the application of local energy methods

For a global energy method (the complete quenching)

J. Giacomoni, P. Sauvy and S. Shmarev, Complete quenching for a quasilinear parabolic equation. *J. Math. Anal. Appl.* **410** (2014), 607–624.

For some very recent continuous dependence

A. N. Dao, J.I. Díaz and P. Sauvy, Quenching phenomenon of singular parabolic problems with L^1 initial data. To appear.

Notations (for the simpler formulation (1))

B_ρ denotes the open ball of radius ρ of $\mathbb{R}^N$

$Q_{\rho,T} := (0,T) \times B_\rho$ and $\Sigma_{\rho,T} := (0,T) \times \partial B_\rho$

We introduce the local energies

$$E(\rho,T) = \int_{Q_{\rho,T}} |Du|^2 \, dxdt \quad b(\rho,T) = \frac{1}{2} ess \sup_{0 \le t \le T} \int_{B_\rho} |u(x,t)|^2 \, dx,$$

$$c(\rho,T) = \int_{Q_{\rho,T}} |u|^\alpha \, dxdt. \quad \alpha = 1-k.$$

Remark.

$$\int_P |u|^\alpha \, dxdt$$

is not a norm but merely a seminorm (in fact it arises the so called reversed Minkowski inequality) and so the usual "interpolation-trace inequality" (such as it is being formulated in the previous literature (see, ADS) cannot be directly applied. But we shall show that a systematic use of the Hölder interpolation inequality

$$\|u\|_{L^s} \leq \|u\|_{L^\alpha}^{d} \, \|u\|_{L^p}^{1-d} \quad \frac{1}{s} = \frac{d}{\alpha} + \frac{1-d}{p},$$

which is valid for any d∈[0,1], even for 0<α<1.

L. Nirenberg, An extended interpolation inequality, *Ann. Scuola Norm. Sup. Pisa*, **3** (1966), 733-737.

The notion of local solution we need for the application of the following local energy method does not need to be specified: we shall only require that u is any function such that the above local energies are finite, for almost all $\rho \in (0, \rho_0)$, for some ρ_0, and the "local integration by parts inequality" holds

$$\begin{cases} b(\rho,T)+E(\rho,T)+c(\rho,T)\le \int_{\Sigma_{\rho,T}} |Du|\,u\,dx\,dt, \text{ a.e. } \rho\in(0,\rho_0), \\ \text{assumed } g(t,x)=0 \text{ and } u_0(x)=0 \ a.e. \text{ respectively on } Q_{\rho_0,T} \text{ and } B_{\rho_0} \end{cases}$$

The following result shows the finite speed of propagation property.

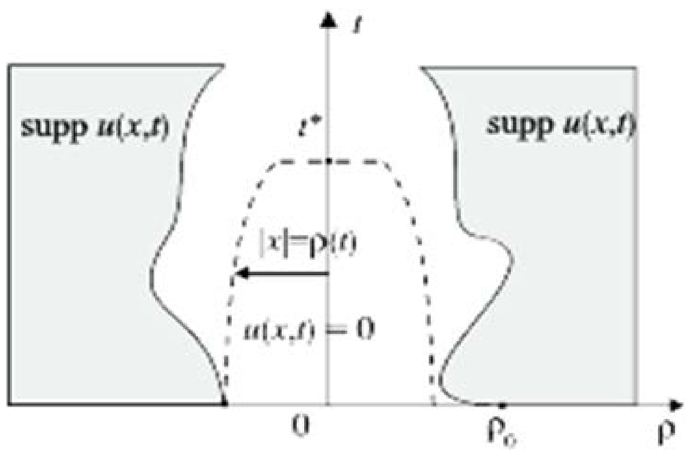
Theorem 2.1. *Let $B_{\rho_0}\subset\Omega$ be such that $u_0=0$ on B_{ρ_0} and $g=0$ on $Q_{\rho_0,T}$. Let u satisfying (10). Then $u=0$ on $Q_{\rho_1,T}$ with ρ_1 defined by*

$$\rho_1^{1+2\beta}=\rho_0^{1+2\beta}-CK(\rho_0,T)\frac{(1+2\beta)}{(1-\xi)}E(\rho_0,T)^{1-\xi} \tag{11}$$

where

$$\beta:=\frac{N(2-\alpha)+2}{4},$$

$$\xi=\frac{N(2-\alpha)+2}{N(2-\alpha)+4},$$



and

$$K(\rho,T)=\max\{\rho^{2\beta},b(\rho,T)^{\theta(2-\alpha)/2}\}. \tag{14}$$

For the proof we shall need a suitable interpolation-trace result.
We start by recalling a well-known version of such results:

J. I. Díaz, L. Veron. Local vanishing properties of solutions of elliptic and parabolic quasilinear equations. *Trans.of Am. Math. Soc.*, **290** 2, (1985), 787-814.

Lemma 3.1. *([15]). Assume $u \in H^1(B_\rho)$ and $1 \leq s \leq 2$. Then*

$$\|u\|_{L^2(\partial B_\rho)} \leq C\left(\|Du\|^{\theta}_{L^2(B_\rho)} \|u\|^{1-\theta}_{L^s(B_\rho)} + \rho^{-\beta}\|u\|_{L^s(B_\rho)}\right) \tag{36}$$

where the constant C depends only on N and s, and

$$\beta := \frac{N(2-s)+s}{2s},\ \theta = \frac{N(2-s)+s}{N(2-s)+2s}. \tag{37}$$

Remark 2. Although we are going to consider terms with 0<α<1, Lemma 1 will be applied in now with s>1. We postpone for the moment a generalization which will be used in problem **(2)**.

The main interpolation-trace result used in the proof of Theorem 1 is the following one:

Lemma 3.2. *Let $0 < \alpha \leq 2$. Assume that $Du \in L^2(Q_{\rho,T})$ and $u \in L^\infty(0,T : L^2(B_\rho))$. Then*

$$\frac{1}{C}\int_{\Sigma_{\rho,T}} |u|^2 \leq E(\rho,T)^{\theta} c(\rho,T))^{1-\theta} b(\rho,T)^{(1-\theta)(2-\alpha)/2} + \rho^{-2\beta} c(\rho,T)) b(\rho,T)^{(2-\alpha)/2} \tag{38}$$

where the positive constant C depends only on N and α, and

$$\beta := \frac{N(2-\alpha)+2}{4},\ \theta = \frac{N(2-\alpha)+2}{N(2-\alpha)+4}. \tag{39}$$

Proof. Applying Hölder interpolation inequality (8) for $0 < \alpha < 1$ and choosing $d = \alpha/2$ (in order to obtain C independent of T) we get

$$\|u\|_{L^s}^2 \leq \|u\|_{L^\alpha}^\alpha \|u\|_{L^2}^{2-\alpha} \text{ where } s = 4/(4-\alpha). \tag{40}$$

This choice gives $1 < s \leq 2$ (since $0 < \alpha \leq 2$) and (12) follows from (37). From (40) and (36) we obtain for almost all $t \in (0, T)$

$$\begin{aligned} \frac{1}{C}\int_{\partial B_\rho} |u|^2 \quad &\leq \left(\int_{B_\rho} |Du|^2\right)^\theta \left(\int_{B_\rho} |u|^\alpha\right)^{1-\theta} \left(\int_{B_\rho} |u|^2\right)^{(1-\theta)(2-\alpha)/2} \\ &+\rho^{-2\beta}\left(\int_{B_\rho} |u|^\alpha\right)\left(\int_{B_\rho} |u|^2\right)^{(2-\alpha)/2} \quad . \end{aligned} \tag{41}$$

We estimate $\int_{B_\rho} |u|^2$ by $2b(\rho, T)$. Then (38) follows integrating in t between 0 and T and applying Hölder inequality. □

Remark 3. The hypotheses of Lemma 3.2 imply easily that $u \in L^2((0,T) \times \partial B_\rho)$ and $u \in L^\alpha(Q_{\rho,T})$. The main feature of the lemma is that the constant C is independent of ρ and T, while T does not appear as a separate factor. A similar (but slightly different) result was given in [15, Lemma 3.2] . This new statement was inspired on Lemma I.2 of [3].

A sharper estimate, for T small, can be also obtained:

Theorem 2.2. *Let $B_{\rho_0} \subset \Omega$ be such that $u_0 = 0$ on B_{ρ_0} and $g = 0$ on $Q_{\rho_0,T}$. Let u satisfying (10). Then $u = 0$ on $Q_{\rho_2,T}$ with ρ_2 defined by*

$$\rho_2^{1+2\beta} = \rho_0^{1+2\beta} - F(T, E(\rho_0, T)), \tag{21}$$

where

$$F(T,s) := C\frac{(1+2\beta)}{(1-\xi)}A(T,\rho_0)\sqrt{T}\log(1+\frac{K(\rho_0,T)}{A(T,\rho_0)\sqrt{T}}s^{1-\xi}), \tag{22}$$

$$A(T,\rho_0) := \rho_0^{2\beta-1}K_1(\rho_0,T), \tag{23}$$

and

$$K_1(\rho,T) := \max\{\rho,\sqrt{T}\}. \tag{24}$$

Remark 1. Since $\log(1+x) \le x$, (21) implies (11) for some constant C. But (22) gives more information as $T \to 0$. Indeed, since for $x > 0$

$$\log(1+x) < \log x + \frac{1}{x}$$

(22) behaves as $(constant)\cdot\sqrt{T}(\log T)$ as $T \to 0$ (for fixed a, A and s).

The proof of Theorem 2.2 requires some sharper interpolation inequalities:

Lemma 3.3. *Assume that $Du \in L^2(Q_{\rho,T})$ and $u \in L^\infty(0,T : L^2(B_\rho))$. Then*

$$\frac{1}{C}\int_{\Sigma_{\rho,T}} |u|^2 \leq \sqrt{T}E(\rho,T)^{1/2}b(\rho,T)^{1/2} + \rho^{-1}Tb(\rho,T). \tag{45}$$

where the positive constant C depends only on N.

Proof. We apply Lemma 3.1 with $s = 2$ and take squares. Then we bound $\int_{B_\rho} |u|^2$ by $2b(\rho,T)$, integrate in t between 0 and T and applying Hölder inequality. □

Corollary 3. *Under the hypotheses of Lemma 3*

$$\frac{1}{C}\int_{\Sigma_{\rho,T}} |u|^2 \leq \rho^{-1}\sqrt{T}K_1(\rho,T)\left(E(\rho,T) + b(\rho,T)\right), \tag{46}$$

where the positive constant C depends only on N and $K_1(\rho,T)$ is given by (24).

Proof. It follows from Lemma 3.3 and the inequality $E^{1/2} + b^{1/2} \leq C\,(E+b)^{1/2}$. □

Coming back to Theorem2.2:

Proof. Taking squares in (15) and applying Corollary 3 we obtain

$$(b(\rho,T) + E(\rho,T) + c(\rho,T))^2 \leq C\rho^{-1}\sqrt{T}K_1(\rho,T)(b(\rho,T)+E(\rho,T))\,\|Du\|^2_{L^2(\Sigma_{\rho,T})}. \tag{25}$$

From (24) $K_1(\rho,T) \leq K_1(\rho_0,T)$. Recalling (19) we obtain

$$\rho E(\rho,T) \leq C\sqrt{T}K_1(\rho_0,T)\frac{\partial E}{\partial \rho}(\rho,T). \tag{26}$$

This differential inequality does not imply vanishing properties, but combining (20) and (26) gives

$$\frac{\rho^{2\beta}}{K(\rho_0,T)}E(\rho,T)^{\xi}+\frac{\rho}{\sqrt{T}K_1(\rho_0,T)}E(\rho,T)\le C\frac{\partial E}{\partial\rho}(\rho,T). \tag{27}$$

Noting that $2\beta>1$, we set

$$\rho=\frac{\rho^{2\beta}}{\rho^{2\beta-1}}\ge\frac{\rho^{2\beta}}{\rho_0^{2\beta-1}}$$

in order to obtain the following explicitly integrable differential inequality

$$\rho^{2\beta}\left(\frac{E(\rho,T)^{\xi}}{a}+\frac{E(\rho,T)}{A\sqrt{T}}\right)\le C\frac{\partial E}{\partial\rho}(\rho,T), \tag{28}$$

with $a(T,\rho_0):=\rho_0^{2\beta-1}K$, and A given by (23). If $E(\rho,T)\neq 0$, an integration of (28) yields

$$\rho_0^{1+2\beta}-\rho^{1+2\beta}\le F(T,E(\rho_0,T))-F(T,E(\rho,T))\le F(T,E(\rho_0,T)) \tag{29}$$

with F given by (22). Thus we arrive to estimate (21) provided that the right hand side of (21) is positive. $\square$

Corollary 1. *Assume u as in Theorem 2.1 and*

$$\rho_0 - \rho \geq \left(CK(\rho_0, T) \frac{(1+2\beta)}{(1-\xi)} E(\rho_0, T)^{1-\xi} \right)^{1/(1+2\beta)} . \tag{30}$$

Then $u = 0$ on $Q_{\rho,T}$. In particular, if $N = 1$, this implies that the free boundary is Hölder continuous where it is monotone.

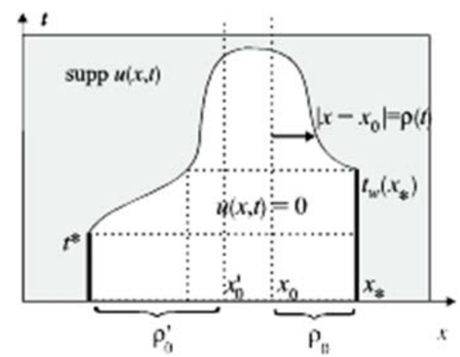
Concerning the behaviour for small time we can prove a first result showing the **local waiting time** or, what we can call perhaps more properly as the **non dilatation of the initial support**: the free boundary cannot invade the subset where the initial datum is nonzero.

Theorem 2.3. *Let $B_{\rho_0} \subset \Omega$ be such that $g = 0$ on $Q_{\rho_0,T}$ and assume u as in Theorem 2.1. We also assume, in addition, that*

$$b(\rho, 0) \leq \varepsilon \left[\rho_0 - \rho\right]^{\omega} \quad a.e. \ \rho \in [0, \rho_1)$$

with

$$\omega = \frac{2N(2-\alpha)+8}{(2-\alpha)}$$



for some ε small enough and $\rho_1 > \rho_0$ large enough. Then there exists a $t^* \leq T$ such that $u(x,t) = 0$ a.e. $x \in B_{\rho_0}$ and for any $t \in [0, t^*]$.

Non cylindrical local energy subsets technique: case of the quasilinear equation (2)

(2) $$\frac{\partial \psi(u)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,u,D\,u) + C(x,t,u) = f(x,t,u),$$

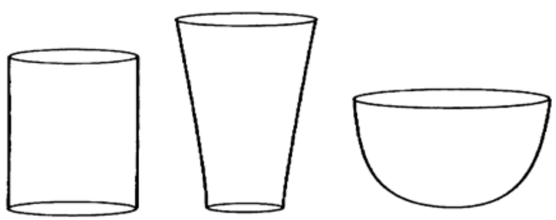
Free boundary formation even in the case of strictly positive initial data: **instantaneous shrinking of the support property**.

Energy functions defined on local domains of a special form

S. N. Antontsev, J. I. Díaz and S.Shmarev. The support shrinking properties for solutions of quasilinear parabolic equations with strong absorption terms. *Annales de la Faculté des Sciences de Toulouse*, **IV** 1 (1995),5-30.

$$P(t) \equiv P(t;\vartheta,\upsilon) = \{(x,s) \in B_{\rho(s)}(x_0) \times (t,T) : |x - x_0| < \rho(s) := \vartheta(s-t)^{\upsilon}\}$$

Notice that the shape of P(t), the local energy set, is determined by the choice of the parameters ϑ and υ. Here we shall take $\vartheta>0$, $0<\upsilon<1$ and so P(t) becomes a paraboloid.



$$E(P(t)) := \int_{P(t)} |Du(x,\tau)|^p \, dx d\tau, \quad C(P(t)) := \int_{P(t)} |u(x,\tau)|^{\alpha} \, dx d\tau,$$

$$b(P(t)) := ess \sup_{s\in(t,T)} \int_{|x-x_0|<\vartheta(s-t)^{\upsilon}} |u(x,s)|^{\gamma+1} \, dx.$$

Although our results have a local nature (as already said they are independent of the boundary conditions) our statements become easier under the additional global information on the boundedness of the *global energy function*

$$D(u,t^*,T) := ess \sup_{s\in(t^*,T)} \int_\Omega |u(x,s)|^{\gamma+1}\,dx + \int_{\Omega\times(t^*,T)} (|Du|^p + |u|^\alpha)\,dxdt.$$

The key new ingredient, is the following interpolation-trace result which extends Corollary 2.1 of DV in the sense that $0<s<1$ and that the interpolation inequality involves a seminorm:

Lemma 4.1. *Assume $u \in W^{1,p}(B_\rho)$, $p \geq 1$ and $0 < s \leq p$. Then for any $r \in [s,p]$*

$$\|u\|_{L^p(\partial B_\rho)} \leq C(\|Du\|_{L^p(B_\rho)} + \rho^{-\beta}\|u\|_{L^s(B_\rho)})^\theta \|u\|^{1-\theta}_{L^r(B_\rho)} \qquad (48)$$

where the constant C depends only on N and s,

$$\theta = \frac{N(p-r)+r}{N(p-r)+pr} \text{ and } \beta := (\frac{N(p-s)+ps}{ps}). \qquad (49)$$

We shall only require that u is any function such that the local energies let finite, for almost all $\rho\in(0,\rho_0)$, for some ρ_0, and satisfies the "local integration by parts inequality" on the paraboloid $P=P(t;\vartheta,\upsilon)$

$$\begin{cases} \displaystyle\int_{P\cap\{t=T\}} G(u(x,t))dx + \int_P \mathbf{A}\cdot Du\, dxd\theta + \int_P Cudxd\theta \\ \displaystyle\leq \int_{\partial_l P} \mathbf{n}_x \cdot \mathbf{A}\, u\, d\Gamma d\theta + \int_{\partial_l P} n_\tau G(u(x,t))d\Gamma d\theta + \lambda \int_P |u|^{q+1} dxd\theta \\ \text{assumed } g(t,x)=0 \text{ a.e. on } P, \end{cases}$$

were $\partial_l P$ denotes the lateral boundary of P i.e. $\partial_l P = \{(x,s) : |x-x_0| = \vartheta(s-t)^v,\ s\in(t,T)\}$, $d\Gamma$ is the differential form on the hypersurface $\partial_l P\cap\{t=const\}$, $\mathbf{n}_x$ and $\mathbf{n}_\tau$ are the components of the unit normal vector to $\partial_l P$. Let us mention that $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_\tau) = \frac{1}{(\vartheta^2 v^2 + (\theta-t)^{2(1-v)})^{1/2}}((\theta-t)^{1-v}\mathrm{e}_x - v\mathrm{e}_\tau)$ with $\mathrm{e}_x, \mathrm{e}_\tau$ orthogonal unit vectors to the hyperplane $t=0$ and the axis t, respectively, where we used the notation $\mathbf{n}_\tau = n_\tau \mathrm{e}_\tau$. Notice that P does not touch the initial plane $\{t=0\}$ and that $P\subset B_{\rho(T)}(x_0)\times[0,T]$, and that we assume $B_{\rho(T)}(x_0)\subset\Omega$.

$$\zeta(x,\theta) := \psi_\varepsilon(|x-x_0|,\theta)\,\xi_k(\theta)\frac{1}{h}\int_\theta^{\theta+h} T_m(u(x,s))\,ds, \quad h>0,$$

$$\xi_k(\theta) := \begin{cases} 1 & \text{if } \theta\in[t, T-\frac{1}{k}], \\ k(T-\theta) & \text{for } \theta\in[T-\frac{1}{k}, T], \\ 0 & \text{otherwise}, \quad k\in\mathbb{N}, \end{cases} \qquad \psi_\varepsilon(|x-x_0|,\theta) := \begin{cases} 1 & \text{if } d>\varepsilon, \\ \frac{1}{\varepsilon}d & \text{if } d<\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \quad d = dist((x,\theta),\partial_l P(t))$$

Theorem 4.2. *Assume (5) and (6). Let u satisfying (53) on any paraboloid of the form $P = P(t;\vartheta,\upsilon)$ and assume $\lambda \leq 0$. Then there exists some positive constants M, t^*, and $\mu \in (0,1)$ such that if $t^* \leq T$ and*

$$D(u,t^*,T) \leq M \tag{54}$$

we have

$$u(x,t) = 0 \text{ in the paraboloid } \{(x,t) : |x - x_0| < (t - t^*)^{\mu}, t \in (t^*,T)\},$$

independently either u_0 vanishes or not. Moreover, if $\lambda > 0$ the above conclusion remains true (with the same $t^* \leq T$) under one of the following conditions: either

$$q \geq \gamma \quad \text{and} \quad \|u\|^{q-\gamma}_{L^\infty(P(t^*))} < \infty \tag{55}$$

or

$$q + 1 \geq \alpha \quad \text{and} \quad \lambda \|u\|^{q+1-\alpha}_{L^\infty(P(t^*))} < 1. \tag{56}$$

Remark 4. The assumption (54) is, in some sense, optimal. Indeed, it is clear that any solution u_∞ of the stationary problem associated to a global formulation, as for instance (1) with $g = \varphi = 0$, is a solution of the parabolic problem for $u_0 = u_\infty$. In the special case of $N = 1$ it is possible to obtain the exact multiplicity diagram (see [11]) showing that the part of the branch of (stable) corresponding to the maximal solution $\overline{u}_\infty$ is strictly positive (for any $\lambda > \lambda_0$ for some $\lambda_0 > 0$). Nevertheless, for λ large enough the part of the branch corresponding to the minimal solution $\underline{u}_\infty$ satisfies that $\lambda \|\underline{u}_\infty\|^{q+1-\alpha}_{L^\infty(\Omega)}$ is small and $\underline{u}_\infty$ vanishes locally near the boundary of Ω. See also, in this context, the nonuniqueness results mentioned in the paper [30].

Remark 5. Notice that assumptions (55) and (56) are perfectly compatible with the existence of a global blow-up time T_∞ (satisfying, obviously that $T_\infty \geq t^*$). This is the case of equation (9) when $\widehat{q} > \max(p, 2)$ (see [16]).

Remark 6. Assumptions (54), (55) and (56) are also perfectly compatible with possible initial datum outside the natural energy space when some $L^1 - L^\infty$ regularizing effects holds (see [5] and [6]).

Remark 7. Theorem 4.2 can be extended, under suitable modifications, to the case in which $g(x, t) \neq 0$. This is the case, for instance of the associated obstacle problem (see [9] for the application of this local energy method to a similar class of obstacle problems).

Thanks for your attention

Thanks,
Stanislav,
for so many years of
collaboration ¡¡¡