Effective Chemical Process in Porous Media.

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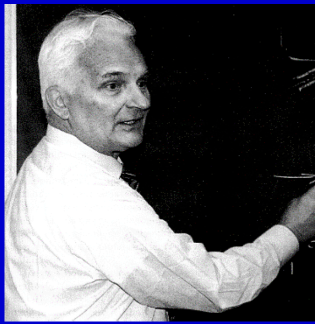
“The universal trilogy” in Applied Mathematics



J.L. Lions

“real” problem

Validation

Mathematical Modelling

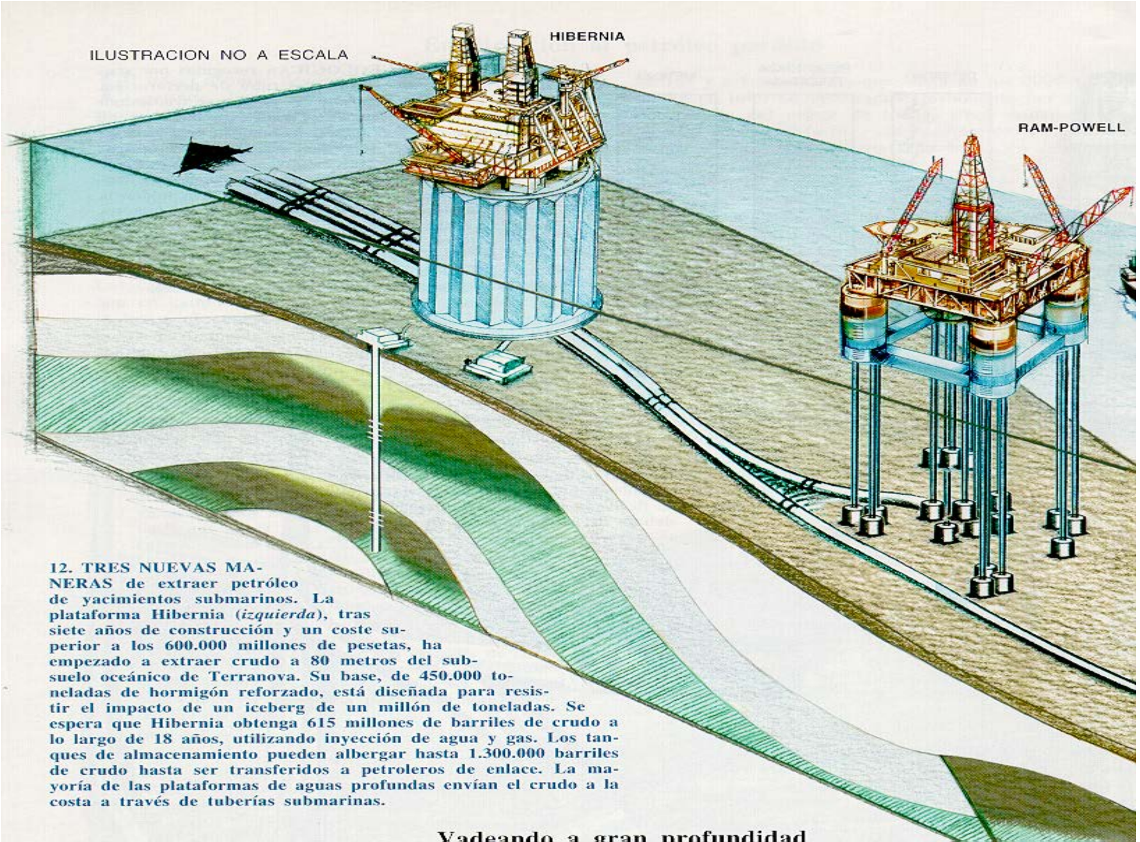
Mathematical and Numerical Analysis

Optimization and Control

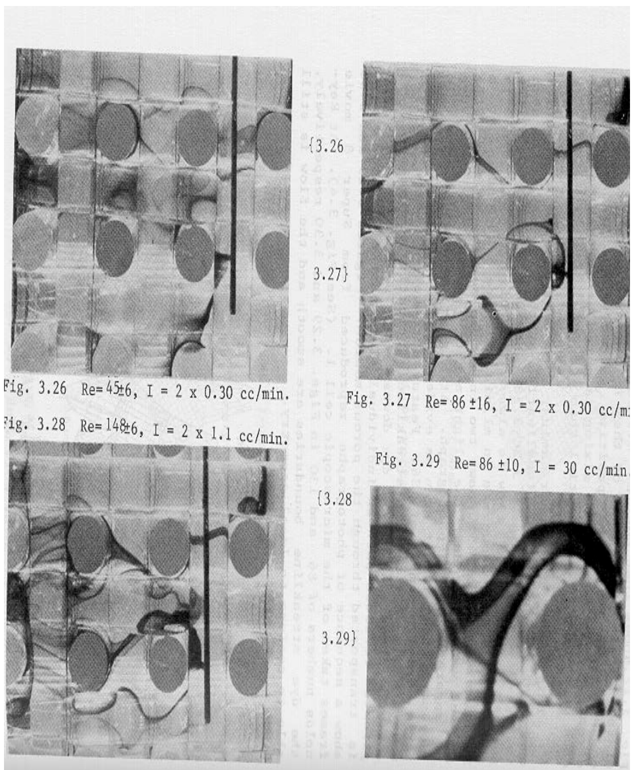
Two different problems in Fluid Mechanics in perforated domains or porous media.

A common aspect: the porous media. Some examples:

Problem 1. Compressible flow of an ideal gas through a porous media: A mathematical derivation of the Darcy's law. Of relevance in oil reservoirs, agriculture, soil infiltration, etc.



Problem 2. Incompressible flow of a fluid reacting with the exterior of many packed solid particles: Absorption and adsorption phenomena in beds or towers.   Of relevance in Chemical Engineering (separation, chemical industry, etc).



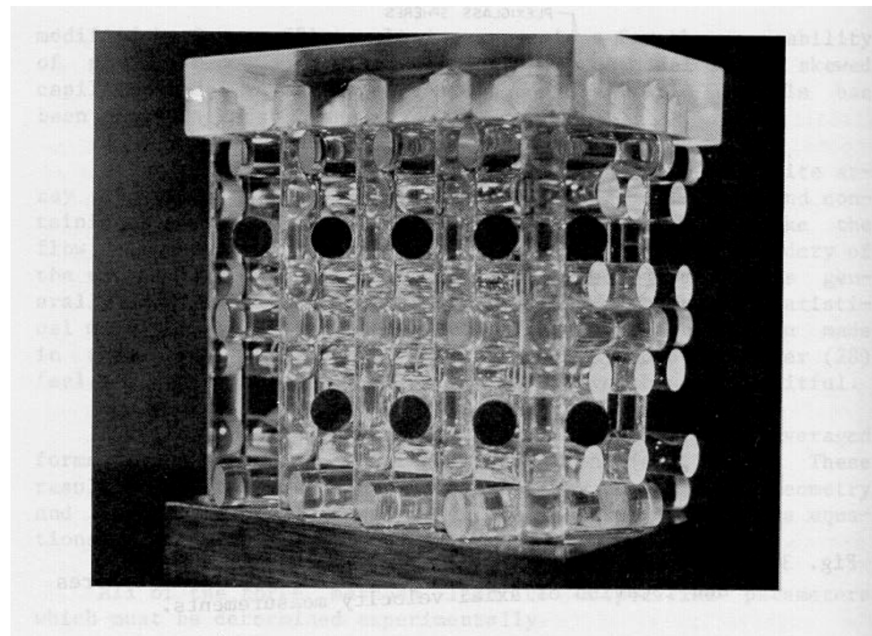
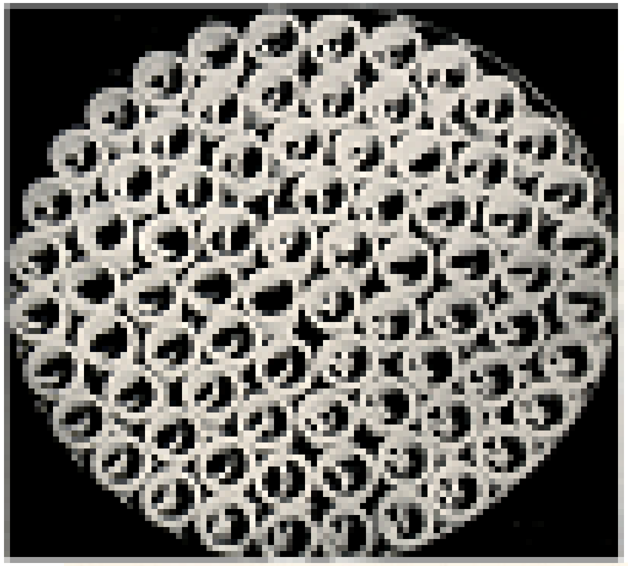


Fig. 3.2 Photograph of half-scale model of complex rod structure porous medium.



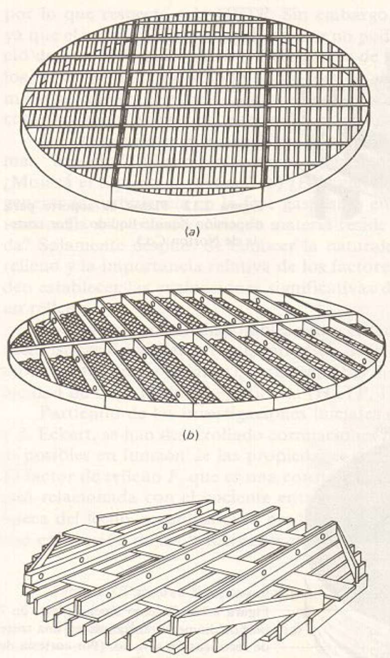


Figura 2.11 Platos de sujeción y retención. (*a*) Plato de retención. (*b*) Plato de sujeción. (*c*) Plato de sujeción. (Por cortesía de Koch Engineering Co.)

quido se crea por combinación de los efectos de penetración de superficie, burbu-

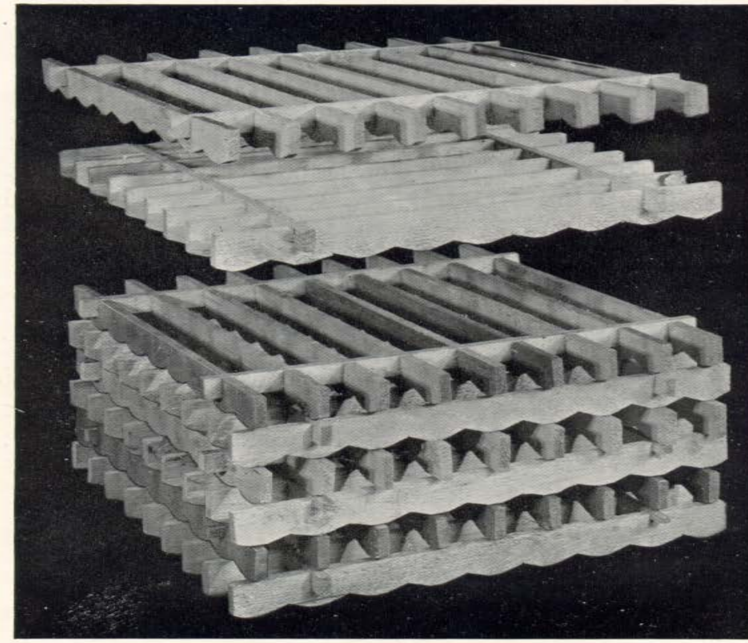
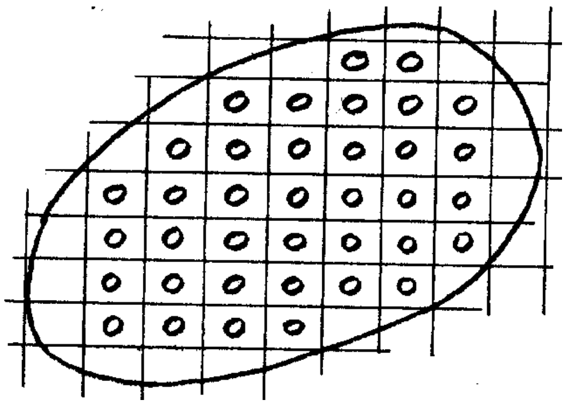
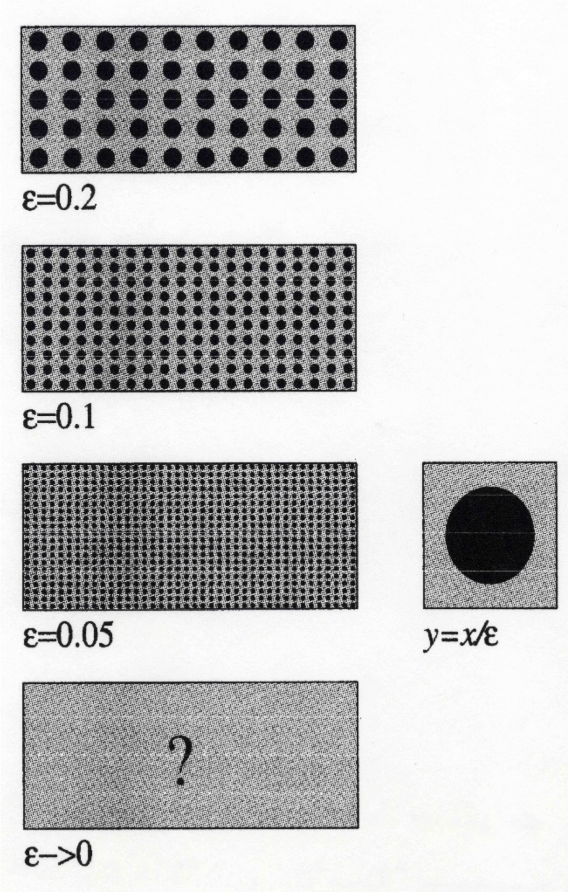


Figure 9. Serrated grid packing
These grids were also assembled in units for use in the experimental tower.

Homogenization process related to the overall modelling in presence of a double spatial scale



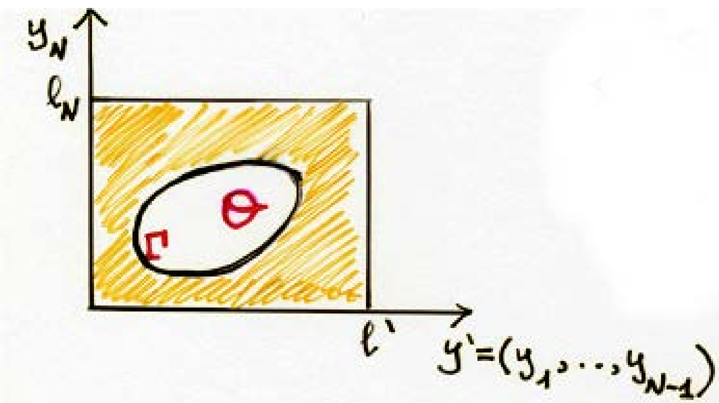


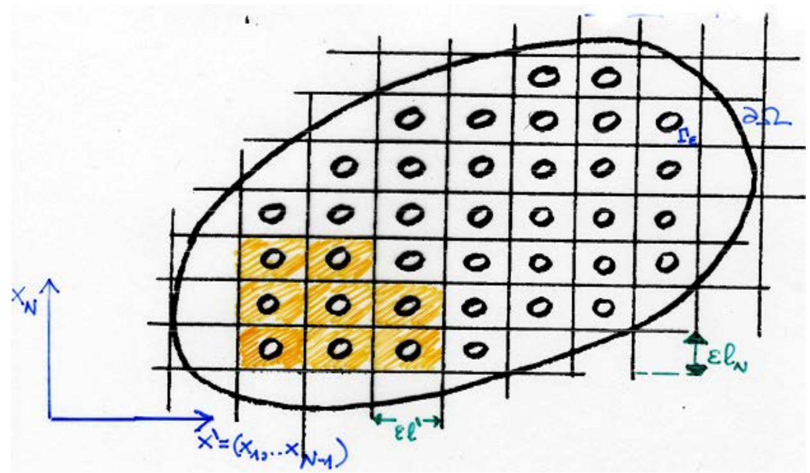
Sánchez-Palencia, Bensoussan-Lions-Papanicolau,...

Many availible methods

We shall assume some periodicity structure on the porous media. More precisely, we start by considering an open bounded set Ω of R^N, with $N = 2$ or 3, of regular boundary $\partial\Omega$. For any small $\varepsilon > 0$ we consider the perforated domain Ω_ϵ obtained by intersecting the $\epsilon-$multiple of a periodic geometry with Ω: i.e., we define $Y =]0, l_1[\times]0, l_2[\times \cdot\cdot\times]0, l_N[$, a bounded regular subset $\theta \subset Y$ with $\Gamma = \partial\theta - \partial Y$ and $Y^* = Y - \overline{\theta}$. Finally, we define

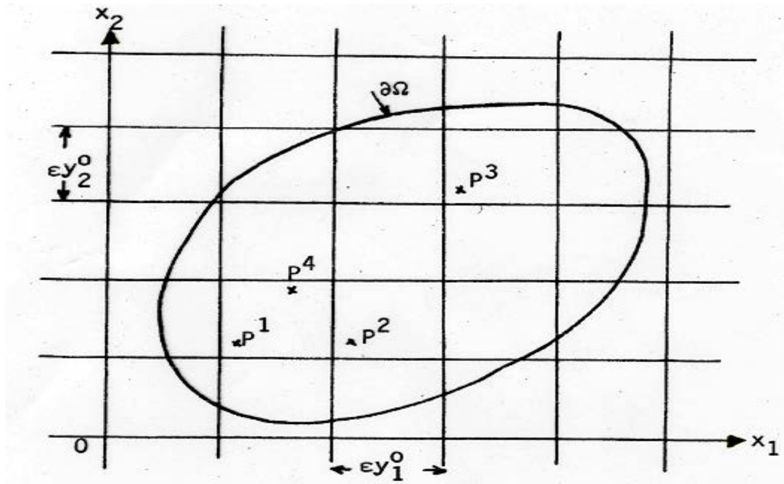
$$\Omega_\epsilon = \Omega \cap \epsilon\theta \text{ and } \Gamma_\epsilon = \Omega \cap \epsilon\Gamma.$$





In the first problem the reference open set $\theta = Y_f$ will be the exterior to the solid part Y_s and we will assume that the union of all the solid parts, $\Omega - \overline{\Omega}_\epsilon$, and all the fluid parts, Ω_ϵ, are connected (i.e. the solid and fluid parts are of one piece) which is possible when $N = 3$. In the second problem, by the contrary, we shall assume that the solid part $\Omega - \overline{\Omega_\epsilon}$ is constituted of a sequence of nonconnected obstacles, open subsets of Ω.

In general, if u is a "physical magnitud" defined on Ω_ε



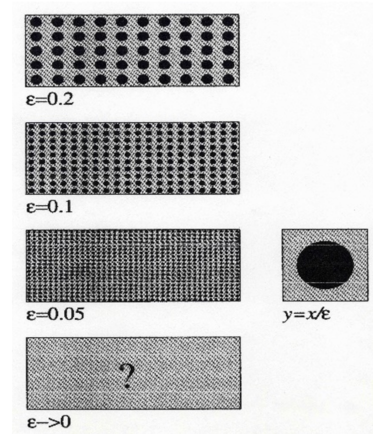
If P_1, P_2 periodically homologous and not far in the $x-$scale then $u(P_1) \approx u(P_2)$.

If P_3 is periodically homologous to P_1, P_2 but far in the $x-$scale then $u(P_1) \approx u(P_3)$ but $u(P_1) \not\approx u(P_3)$.

If P_4 is far (in the $y-$scale) from P_1 (and P_2) but near in the $x-$scale then $u(P_1) \not\approx u(P_3)$ but $u(P_1) \approx u(P_3)$.

$$u^\varepsilon = u^\varepsilon(x) = u^\varepsilon(x, y) \,|_{y=\frac{x}{\varepsilon}}$$

• **Homogeneization**: Question: $u^\varepsilon \to ?$, as $\varepsilon \to 0$ (homogeneized region Ω)



•Two different steps:

a. *Formal Asymptotic Expansion.* "Ansatz"

$$u^\varepsilon(x) = u^\varepsilon(x,y)\,|_{y=\frac{x}{\varepsilon}} = u_0(x,y) + \varepsilon u_1\ (x,y) + \varepsilon^2 u_2(x,y) +$$

with

$$u_i\ (x,y)\text{ Y-periodic with respect to the } y = \frac{x}{\varepsilon} \text{ variable.}$$

Then $u^\varepsilon \to u_0$, as $\varepsilon \to 0$.

b. *Rigurous proof.* There exists a u_0 such that $u^\varepsilon \to u_0$, in some functional space, as $\varepsilon \to 0$.

Pb1. A mathematical derivation of the Darcy's law.

H. Darcy: Les fontaines publiques de la ville de Dijon. Dalmont. Paris. 1856.

Empiric Law: "in the flow of a liquid through a porous medium the velocity is proportional to the gradient of the pressure".

$$\mathbf{u}(x) = C\nabla p(x)$$

Very useful since instead of 4 unknowns (if N=3: **u** and p) we pass to an scalar problem (we do not need the conservation of the momentum for **u**, but only the conservation of the mass ⇒ scalar second order pde for p)

First mathematical proof by L. Tartar, in 1980, by using homogeneization techniques.

Important remark: (motivation)

Tartar's proof deals merely with stationary incompressible fluids
$\Rightarrow$ the conservation of the mass is

$$div\mathbf{u} = 0$$

so Darcy's law leads to

$$\Delta p = 0$$

and so there is no mathematical derivation of the so called Porous Media Equation

$$v_t - \Delta v^m = 0, \text{ for some } m > 1,$$

(very large literature in the Theory of Nonlinear PDEs).

Even if the regime is time depending, a direct application of the Darcy's law for an incompressible fluid never ends in the Porous Media Equation (the conservation of the mass is always a stationary equation for a incompressible fluid).

Main goal of this part: " To find a mathematical derivation of the Darcy's law for the compressible flow of an ideal gas through a porous media (≡Problem 1), justifying the (PME)".

Let $\mathbf{v}_\epsilon$ be the velocity, ρ_ϵ the density and p_ε the pressure of a compressible fluid occupying the region Ω_ε. The correspondent Navier-Stokes system is formed by the *mass conservation equation*

$$\frac{\partial \rho_\epsilon}{\partial t} + div(\rho_\epsilon \mathbf{v}_\epsilon) = 0,$$

and the *momentum conservation equation*

$$\rho_\epsilon(\frac{\partial \mathbf{v}_\epsilon}{\partial t} + (\mathbf{v}_\epsilon \cdot \nabla)\mathbf{v}_\epsilon) = -\nabla p_\varepsilon + \mu \Delta \mathbf{v}_\epsilon + \lambda \nabla (div \mathbf{v}_\epsilon) + \rho_\epsilon \mathbf{f}.$$

We assume a *constitutive law* of the form

$$\rho_\epsilon = F(p_\varepsilon)$$

where

$F : R \to R$ is a strictly increasing function of class C^1.

The auxiliary conditions are formed by a boundary condition

$$\mathbf{v}_\epsilon = \mathbf{0}, \text{ on } \partial\Omega_\varepsilon \times (0, T)$$

and the initial conditions

$$\begin{aligned} \rho_\epsilon(x, 0) &= \rho_I(x), \text{ on } \Omega_\varepsilon, \\ \mathbf{v}_\epsilon(x, 0) &= \mathbf{v}_I(x), \text{ on } \Omega_\varepsilon, \end{aligned}$$

where ρ_I and $\mathbf{v}_I$ are functions defined on the whole domain Ω, $\rho_I \geq 0, \rho_I \neq 0$.

As mentioned at the introduction we assume a formal expansion in terms of powers of ϵ. In our case we introduce the variables

$$y = \frac{x}{\varepsilon} \text{ and } \tau = \varepsilon^k t$$

and assume the *ansatz*

$$\begin{aligned}
\rho_\epsilon(x,t) &= \rho_0(x,y,\tau) + \epsilon\rho_1(x,y,\tau) + \varepsilon^2\rho_2(x,y,\tau) + \cdots \big|_{y=\frac{x}{\varepsilon},\tau=\varepsilon^k t} \\
\mathbf{v}_\varepsilon(x,t) &= \varepsilon^n \left(\mathbf{v}_0(x,y,\tau) + \varepsilon\mathbf{v}_1\ (x,y,\tau) + \varepsilon^2\mathbf{v}_2(x,y,\tau) + \cdots\right|_{y=\frac{x}{\varepsilon},\tau=\varepsilon^k t} \\
p_\epsilon(x,t) &= p_0(x,y,\tau) + \varepsilon p_1(x,y,\tau) + \varepsilon^2 p_2(x,y,\tau) + \cdots \big|_{y=\frac{x}{\varepsilon},\tau=\varepsilon^k t}
\end{aligned}$$

with k and n to be determined later (

$$\begin{aligned}
\frac{\partial}{\partial t} &= \varepsilon^k \frac{\partial}{\partial \tau}, \\
\nabla &= \nabla_x + \frac{1}{\epsilon}\nabla_y, \\
div &= div_x + \frac{1}{\epsilon} div_y, \\
\Delta &= \Delta_x + \frac{2}{\epsilon}\Delta_{xy} + \frac{1}{\epsilon^2}\Delta_y \ \left(\Delta_{xy} = \sum_{i=1}^N \frac{\partial^2}{\partial x_i \partial y_i}\right).
\end{aligned}$$

By choosing

$$k = n = 2$$

the coefficients of ε^{-1} at the momentum equation leads to the condition

$$\nabla_y p_0 = 0.$$

So, "p_0 and ρ_0 are independent of y".'

The identification of the coefficients of ε at the momentum equation imply that

$$\mathbf{0} = -(\nabla_x p_0 + \nabla_y p_1) + \mu \Delta_y \mathbf{v}_0 + \lambda \nabla_y (div \mathbf{v}_0) + \rho_0 \mathbf{f}.$$

On the other hand, since $k = n$ we get, through the conservation of the mass, by identifying the coefficients of ε^n and ε^{n-1}, that

$$\frac{\partial \rho_0}{\partial t} + div_x(\rho_0 \mathbf{v}_0) + div_y(\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) = 0,$$

and

$$div_y(\rho_0 \mathbf{v}_0) = 0.$$

Then,

$$\mathbf{0} = -(\nabla_x p_0 + \nabla_y p_1) + \mu \Delta_y \mathbf{v}_0 \ + \rho_0 \mathbf{f}.$$

Since ρ_0 is independent of y and obviously

$$\rho_0(x, \tau) \neq 0$$

and as $\mathbf{v}_0$ is Y-periodic we conclude that

$$div_y \mathbf{v}_0 = 0 \text{ in } \theta \text{ (for fixed } x \text{ and } \tau).$$

So, at the local level the flow is incompressible. Now, we define the *mean operator*

$$\widetilde{\bullet} = \frac{1}{|Y|}\int_Y \bullet dy$$

and extend by zero all the functions defined on θ. The main result of this section is the following

Theorem.

$$\delta\frac{\partial \rho_0}{\partial \tau} + div_x(\rho_0 \widetilde{\mathbf{v}_0}) = 0,$$

where

$$\delta = \frac{|\theta|}{|Y|} \text{ (the porosity of the medium).}$$

Moreover, there exists a constant symmetric and positively defined matrix $\mathbf{K}$ *such that*

$$\widetilde{\mathbf{v}_0}(x,\tau) = \frac{1}{\mu}\mathbf{K}[\rho_0(x,\tau)\mathbf{f}(x,\tau) - \nabla_x p_0(x,\tau)].$$

Proof. It is clear that

$$\widetilde{\rho_0}(x,\tau) = \delta\rho_0(x,\tau)$$

$$\widetilde{div_y \mathbf{v}_1} = \frac{1}{|Y|}\int_Y div_y \mathbf{v}_1 dy = \frac{1}{|Y|}\int_{\partial Y} \mathbf{v}_1 \cdot \mathbf{n} d\sigma = 0$$

since $\mathbf{v}_1 = \mathbf{0}$ on Γ and $\mathbf{v}_1$ is Y-periodic. Moreover

$$\begin{aligned}\widetilde{\nabla_y \rho_1 \mathbf{v}_0} &= \frac{1}{|Y|}\int_Y \nabla_y \rho_1 \mathbf{v}_0 dy \\ &= \frac{1}{|Y|}[\int_Y div_y(\rho_1 \mathbf{v}_0) dy - \int_Y \rho_1 div_y \mathbf{v}_0 dy] = \frac{1}{|Y|}\int_{\partial Y} \rho_1 \mathbf{v}_0 \cdot \mathbf{n} d\sigma = 0\end{aligned}$$

and so we get equation (16) which is the *macroscopic* mass conservation of the *homogenized fluid*. In order to show (18) we point out that $\mathbf{v}_0$ solves the Stokes problem

$$\begin{cases} -\mu \Delta_y \mathbf{v}_0 = -\nabla_y p_1 + \mathbf{f}^* & \text{in } \theta \times (0,\infty), \\ div_y \mathbf{v}_0 = 0 & \text{in } \theta \times (0,\infty), \\ \mathbf{v}_0 = \mathbf{0} & \text{on } \Gamma \times (0,\infty), \\ \mathbf{v}_0 \text{ is } Y\text{-periodic.} \end{cases}$$

where $\mathbf{f}^* = \rho_0 \mathbf{f} - \nabla_x p_0$. So, $\mathbf{v}_0$ coincides with the unique *weak solution* in the sense of Leray (see, for instance [30]), i.e. $\mathbf{v}_0 \in V_\theta$ and

$$\mu \int_Y \nabla_y \mathbf{v}_0 \cdot \nabla_y \mathbf{w} dy = \int_\theta \mathbf{f}^* \cdot \mathbf{w} dy$$

for any $\mathbf{w} \in V_\theta$ where

$$V_\theta := \{\mathbf{w} \in \mathbf{H}^1(\theta) : div_y \mathbf{w} = 0, \mathbf{w} \text{ is} Y - \text{periodic and } \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}.$$

for $1 \leq i \leq N$ we define $\mathbf{v}^i\ (y)$, $\mathbf{v}^i \in V_\theta$, as

the solutions of the auxiliary problems

$$\int_Y \nabla_y \mathbf{v}^i \cdot \nabla_y \mathbf{w} dy = \int_\theta w_i dy$$

assumed $\mathbf{w} = \sum w_i \mathbf{e}_i$, then, by linearity, we get that

$$\mathbf{v}_0 = \frac{1}{\mu}(\rho_0 f_i - \frac{\partial p_0}{\partial x_i})\mathbf{v}^i.$$

Thus, applying the mean operator we get that

$$v_{0j} = \frac{K_{ij}}{\mu}(\rho_0 f_i - \frac{\partial p_0}{\partial x_i})$$

Corollary 1 *ρ_0 satisfies the quasilinear parabolic equation*

$$\delta\frac{\partial \rho_0}{\partial \tau} - div(\frac{1}{\mu}\mathbf{K}\rho_0 \nabla F^{-1}(\rho_0)) + div(\frac{1}{\mu}\mathbf{K}\rho_0^2 \mathbf{f}) = 0.$$

In particular, if $\mathbf{f} = \mathbf{0}, \boldsymbol{\delta} = 1/\mu$ *and* $\mathbf{K} = \mathbf{I}$ *(the identity matrix) then*

$$\frac{\partial \rho_0}{\partial \tau} - \Delta\varphi(\rho_0) = 0,$$

where φ is the increasing function defined as $\varphi(s) := \int_0^s \frac{\sigma}{F'(F^{-1}(\sigma))} d\sigma.$

Remark 1 *The special expansion could be replaced by a standard one (i.e. with terms in ε and ε^0 for the velocity and without a macroscopic time scale) by including physical parameters suitably scaled at the microscopic equations as, for instance*

$$\rho_\epsilon(\varepsilon^k \frac{\partial \mathbf{v}_\epsilon}{\partial t} + \varepsilon^k(\mathbf{v}_\epsilon \cdot \nabla)\mathbf{v}_\epsilon) = -\nabla p_\varepsilon + \mu\varepsilon^n \Delta \mathbf{v}_\epsilon + \lambda \varepsilon^n \nabla(div \mathbf{v}_\epsilon) + \rho_\epsilon \mathbf{f}.$$

Remark 2 *The above special expansion and the condition (12) may be justified by using Dimensional Analysis. In order to do that let us introduce some characteristics units L, t_c, T_c, p_c, ρ_c, v_c, V_c for the macroscopic length, the time in the microscopic and macroscopic flow, the pressure, the density and the velocity in the microscopic and macroscopic flow respectively. We also introduce the dimensionless variables*

$$\overline{x} = \frac{x}{L}, \overline{t} = \frac{t}{t_c}, \overline{\tau} = \frac{\tau}{T_c}, \overline{p} = \frac{p}{p_c}, \overline{\rho} = \frac{\rho}{\rho_c}, \overline{\mathbf{v}_\varepsilon} = \frac{\mathbf{v}_\varepsilon}{v_c}, \overline{\mathbf{v}_0} = \frac{\mathbf{v}_0}{V_c}.$$

Notice that the microscopic characteristic length is then given by εL. Thus we see that the microscopic momentum conservation equation becomes

$$\begin{aligned} &(\rho_c \frac{v_c}{t_c}\overline{\rho}_\epsilon)\frac{\partial \overline{\mathbf{v}}_\epsilon}{\partial \overline{t}} + (\rho_c \frac{v_c^2}{\varepsilon L})\overline{\rho}_\epsilon(\overline{\mathbf{v}}_\epsilon \cdot \nabla)\overline{\mathbf{v}}_\epsilon) \\ = \; & -(\frac{\delta_c p}{\varepsilon L})\nabla \overline{p}_\varepsilon + (\mu \frac{v_c}{\varepsilon^2 L^2})\Delta \overline{\mathbf{v}}_\epsilon + (\lambda \frac{v_c}{\varepsilon^2 L^2})\nabla(div \overline{\mathbf{v}}_\epsilon) + \rho_c \overline{\rho}_\epsilon \mathbf{f}, \end{aligned} \tag{21}$$

where $\delta_c p$ denotes the characteristic pressure changes. Since the Reynolds and Reynolds-Strouhal of the microscopic flow

$$Re = \frac{\rho_c v_c \varepsilon L}{\mu}, ReSt = \frac{\rho_c v_c \varepsilon^2 L^2}{t_c}$$

are very small (remember that $\varepsilon << 1$) the material time derivative terms of the equation (21) can be neglected and we get that

$$-(\mu\frac{v_c}{\varepsilon^2L^2})\Delta\overline{\mathbf{v}}_\epsilon-(\lambda\frac{v_c}{\varepsilon^2L^2})\nabla(div\overline{\mathbf{v}}_\epsilon)=-(\frac{\delta_c p}{\varepsilon L})\nabla\overline{p}_\varepsilon+\rho_c\overline{\rho}_\epsilon\mathbf{f}. \qquad (22)$$

Making

$$\frac{\delta_c p}{\varepsilon L}=\frac{p_c}{L}$$

we get, identifying the parameters of (22), that

$$v_c=\frac{p_cL}{(\mu+\lambda)}\varepsilon^2 \qquad (23)$$

and so the significant terms of the microscopic velocity are of order two in ε such as is implied by the special expansion and the assumption (12). On the other hand, from (23) and the expansion for $\mathbf{v}_\epsilon$ we deduce that necessarily $V_c=\frac{c}{\varepsilon^2}$ for some constant c. Then arguing as before but now for the macroscopic mass conservation equation

$$\delta\frac{\partial\rho_0}{\partial\tau}+div_x(\rho_0\widetilde{\mathbf{v}}_0)=0$$

we deduce that

$$\frac{\rho_c}{T_c}=\frac{\rho_cV_c}{L}.$$

So, we get

$$T_c=\frac{c/L}{\varepsilon^2}$$

which justifies the change of scale $\tau=\varepsilon^2t$ previously assumed.

Pb2. Effective chemical reactions

C. Conca, J.I. Díaz, A. Liñan and C. Timofte, Homogenization in Chemical Reactive Flows through Porous Media, *Electr. J. Diff. Eqns.* 2004, 1-22.

$$\begin{cases} -D_f \Delta u^\varepsilon = f \quad \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) \quad \text{on } S^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases}$$

$$g(v) = \frac{\alpha v}{1 + \beta v}, \quad \alpha, \beta > 0 \quad \text{(Langmuir kinetics)}$$

$$g(v) = |v|^{p-1} v, \quad 0 < p < 1 \quad \text{(Freundlich kinetics)}$$

$$\begin{cases} -\sum_{i,j=1}^{n} a_{ij}^{0} \frac{\partial^2 u}{\partial x_i \partial x_j} + a \frac{|T|}{|Y \setminus T|} g(u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

$$a_{ij}^{0} = \frac{1}{|Y|} \int_{Y} \left(a_{ij} + a_{ik} \frac{\partial \chi_j}{\partial y_k} \right) dy, \qquad \begin{cases} -\operatorname{div}(AD(y_j + \chi_j)) = 0 \quad \text{in } Y, \\ \chi_j - Y \text{ periodic.} \end{cases}$$

- $u_\varepsilon(x) \equiv$ concentration of a reactant in the fluid.

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ -\dfrac{\partial u_\varepsilon}{\partial n} = \underbrace{\alpha \varepsilon (u_\varepsilon)^p}_{\text{reaction}} & \text{on } \Gamma_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

Here $0 < p \leq 1$.

- THEOREM. Assume $f \in L^2(\Omega)$. Let $V_\varepsilon := \{v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}$ and let $P_\varepsilon \in \mathcal{L}(V_\varepsilon : H^1_0(\Omega))$ be a family of extension operators $\left((P_\varepsilon v)(x) = v(x) \ \forall x \in \Omega_\varepsilon \right)$.

Then $P_\varepsilon u_\varepsilon \longrightarrow u_0$ in $H^1_0(\Omega)$ (weak), as $\varepsilon \downarrow 0$.

Moreover

$$\begin{cases} -\sum_{i,j} q_{ij} \dfrac{\partial^2 u_0}{\partial x_i \partial x_j} + \underbrace{\alpha \delta (u_0)^p}_{\text{distributed reaction}} = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

here $\delta := \dfrac{\text{med}(\partial\Theta)}{\text{med}(Y)}$, $q_{ij} \equiv$ constants depending on Θ.

Idea of the proof:

- $\forall \varphi \in V_\varepsilon$

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi + \alpha \varepsilon \int_{\Gamma_3} (u_\varepsilon)^p \varphi \, ds = \int_{\Omega_\varepsilon} f \varphi \, dx$$

- Taking $\varphi = u_\varepsilon \Rightarrow \| \nabla u_\varepsilon \|_{L^2(\Omega_\varepsilon)^N} \leq C \Rightarrow \begin{array}{l} P_\varepsilon u_\varepsilon \rightharpoonup u \text{ in } L^2(\Omega) \\ P_\varepsilon \nabla u_\varepsilon \rightharpoonup \vec{\xi} \text{ in } L^2(\Omega)^N \end{array}$

- $\displaystyle\int_{\Omega_\varepsilon} f \varphi \, dx = \int_\Omega \chi_{\Omega_\varepsilon} f \varphi \, dx \longrightarrow \frac{\text{med}(Y_f)}{\text{med}(Y)} \int_\Omega f \varphi \, dx$, if $\varphi \in C_0^\infty(\Omega)$.

- $\displaystyle\varepsilon \int_{\Gamma_\varepsilon} (u_\varepsilon)^p \varphi \, ds \longrightarrow \frac{\text{med}(\partial\theta)}{\text{med}(Y)} \int_\Omega (u)^p \varphi \, dx$

- $\displaystyle\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi \not\longrightarrow \int_\Omega \nabla u \cdot \nabla \varphi$ (difficult since Ω_ε depends on ε)

$$\downarrow$$

$$\sum \int_\Omega \left(q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u \right) \varphi$$

(q_{ij} given by some "celular problem" in the y-scale)

Last remarks:

Evolution problem,

C. Conca, J.I. Díaz, C. Timofte
Effective Chemical Process in Porous Media
Mathematical Models and Methods in Applied Sciences, **13**, 2003, 1437-1462

Free boundary, ...,