Partially flat solutions of a nonlinear elliptic equation with a singular absorption term proposed on 1903 for electron beams modeling

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CURSO DE DOCTORADO IMEIO

Modelos No Lineales en Ingeniería Matemática

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0. *Plan* of the lecture

1. The pioneering modeling of electron beams: the auxiliary formulation on a bounded domain.
2. The one-dimensional Child-Langmuir law revisited.
3. The 2-d parallel-plate geometry.
    1. Statements of the existence of partially flat solutions (via super and subsolutions method).
    2. On the construction of a partially flat subsolution (I).
    3. Bifurcation diagram for a singular eigenvalue ODE problem.
    4. On the construction of a partially flat subsolution (II).
    5. On the construction of a partially flat supersolution.
    6. Proof of the uniqueness of partially flat solutions.

1. The pioneering modeling of electron beams: the auxiliary formulation on a bounded domain.

Given $a > 0$ the main goal of the lecture is to find sufficient conditions on a function $j : \mathbb{R} \to [0, +\infty)$, with

$$\begin{cases} j(x) > 0 \quad \text{if } x \in (-a, a), \; j \in L^1_{loc}(-a, a), \\ j(x) = 0 \qquad\qquad\qquad\qquad \text{if } x \notin [-a, a], \end{cases} \tag{1}$$

in order to get the solvability of the singular nonlinear boundary value problem

$$P_{\infty,a,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-\infty, +\infty), \; y \in (0,1), \\ u(x, 0) = 0 & x \in (-\infty, +\infty), \\ u(x, 1) = 1 & x \in (-\infty, +\infty), \\ \lim_{|x| \to +\infty} u(x, y) = y & y \in (0, 1), \end{cases} \tag{2}$$

with the additional conditions

$$AC_{\infty,a} = \begin{cases} \frac{\partial u}{\partial y}(x, 0) = 0 & x \in (-a, a), \\ u(x, y) > 0 & x \in (-\infty, +\infty), \; y \in (0, 1). \end{cases} \tag{3}$$

The study of the overdetermined problem (2), (3) was initiated, in the one-dimensional case (formally corresponding to the case $a = +\infty$), in the early part of the last century (by C.D. Child on 1903, and by I. Langmuir in a series of papers starting on 1904).

Remember: Cathode ray tube, J.J. Thompson (1897) [Prix Nobel in Physics, 1906]: the existence and charge of the electron.

The onedimensional Child-Langmuir law: find $j > 0$ in order to get a function $u \in W^{2,1}(0,1)$ solving the boundary problem

$$\begin{cases} -u''(x) + \dfrac{j}{\sqrt{u(x)}} = 0 & x \in (0,1), \ u \geq 0, \\ u(0) = 0 \quad u(1) = 1, & \end{cases} \tag{4}$$

$$j \equiv \frac{4}{9}, \ u(x) = x^{4/3}$$

and such that $u'(0) = 0$.

In contrast with the Ohm's law (1823) $[u(x) = Cx]$

The PDE (2), (3) was proposed in A. Rokhlenko and J.L. Lebowitz, Phys. Rev. Lett. (2003). Notice that the additional conditions imply a failure of the "unique continuation property".

Haïm Brezis …

Université Paris VI ,
May 2005

Stationary Maxwell system of equations for the electric and magnetic fields $(\mathbf{E}, \mathbf{B})$
$\widetilde{\Omega} \subset \mathbb{R}^3$, $\widetilde{\Omega} = \mathbb{R} \times (0, D) \times \mathbb{R}$, with $D > 0$ given, separating two conducting electrodes placed on the planes $Y = 0$ (cathode) and $Y = D$ (anode) [with $\partial\widetilde{\Omega} = \Gamma_0 \cup \Gamma_1$]:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = \mathbf{0} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \end{cases}$$

$\rho(X, Y, Z)$ is the charge (electron) density,
ϵ_0 the free space permittivity and
$\mathbf{J}(X, Y, Z)$ denotes the current density.

ρ stationary implies $div\mathbf{J} = 0$ in $\widetilde{\Omega}$

Now we assume that the **cathode** is in the (X, Z) plane $Y = 0$, it have a width $2A$, and that there is a **very strong magnetic field B**, which is perpendicular to the electrodes ($\mathbf{B}$ is in the Y-direction), inhibiting the transversal components of the electron velocities $v(X, Y)$, and and $\rho(X, Y, Z) = \rho(X, Y)\chi_{\{|X| \leq A\}}(X, Y)$, where $\chi_{\{|X| \leq A\}}(X, Y)$ is the characteristic function of the set

$$\chi_{\{|X| \leq A\}}(X, Y) = \begin{cases} 1 & \text{if } |X| \leq A \\ 0 & \text{otherwise.} \end{cases}$$

Due to the assumption on $\mathbf{B}$, we know that **the potential** U of **the electric field** $(\mathbf{E} = -\nabla V)$ is Z-independent $U = U(X, Y)$.

We also assume that **the emitted electrons leave the cathode with zero velocity** (and thus, if we take $U = U(X, Y) = 0$ in the cathode) the total mechanical energy is $E_0 = 0$.

The electric potential *U(X,Y)* is *a priori* **unknown** !!!

Pollack-Stump
Electromagnetism (2002), pag 242

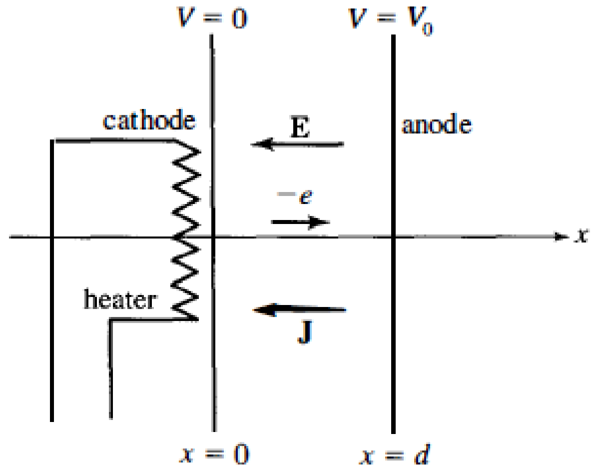


FIGURE 7.10 Cathode and anode of a vacuum tube diode. Electrons boil off the hot cathode and accelerate in **E** to the anode.

Then, if e and m represents the charge and the mass of the electron, the **conservation of the mechanical energy** leads to the equation

$$\frac{m}{2}v^2(X,Y) = eU(X,Y).$$

$$v(X,Y) = \sqrt{\frac{2eU(X,Y)}{m}}$$

Remember that the mechanical force (by a negative charge) is given by $\mathbf{F} = (-e)\mathbf{E} = -e(-\nabla U) = e\nabla U$ and thus the potential energy is $-eU$.

From this we deduce that the current density is only dependent on X, $\mathbf{J}(X,Y,Z) = J(X)\chi_{\{|X|\leq A\}}(X,Y)\mathbf{e}_2$ and determine the velocity of electrons.

We also recall that

$$\mathbf{J} = -\rho v\mathbf{e}_2 := J(X)\chi_{\{|X|\leq A\}}(X,Y)\mathbf{e}_2$$

(see the figure). Then

$$\rho(X,Y) = -\frac{J(X)\chi_{\{|X|\leq A\}}(X,Y)}{\sqrt{\frac{2eU(X,Y)}{m}}}.$$

We introduce now the **dimensionless variables**

$x = \frac{X}{D}, y = \frac{Y}{D}$ and $a = \frac{A}{D}$ (the cathode is now the subset $[-a, a] \times \{0\}$)

and the **dimensionless functions**

$$u(x,y) = \frac{U(X,Y)}{V}, \; j(x) = \frac{9}{4}\sqrt{\frac{m}{2e}}\frac{J(X)}{\epsilon_0 V^{3/2}}.$$

The first equation of the Maxwell system and **the conservation of the mechanical energy** leads to the nonlinear Poisson equation

$$\Delta u(x,y) = -4\pi\rho(x,y) = \frac{j(x)}{\sqrt{u(x,y)}}$$

with $\rho(x,y) = 0$ (i.e. $j(x) = 0$) if $|x| > a$.

$$P_{\infty,a,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-\infty, +\infty), \; y \in (0,1), \\ u(x,0) = 0 & x \in (-\infty, +\infty), \\ u(x,1) = 1 & x \in (-\infty, +\infty), \\ \lim_{|x| \to +\infty} u(x,y) = y & y \in (0,1), \end{cases}$$

$$AC_{\infty,a} = \begin{cases} \frac{\partial u}{\partial y}(x,0) = 0 & x \in (-a,a), \\ u(x,y) > 0 & x \in (-\infty,+\infty), \; y \in (0,1). \end{cases}$$

The additional condition $\frac{\partial u}{\partial y}(x,0) = 0$ for $x \in (-a,a)$ represents the vanishing of the electric field on the cathode.

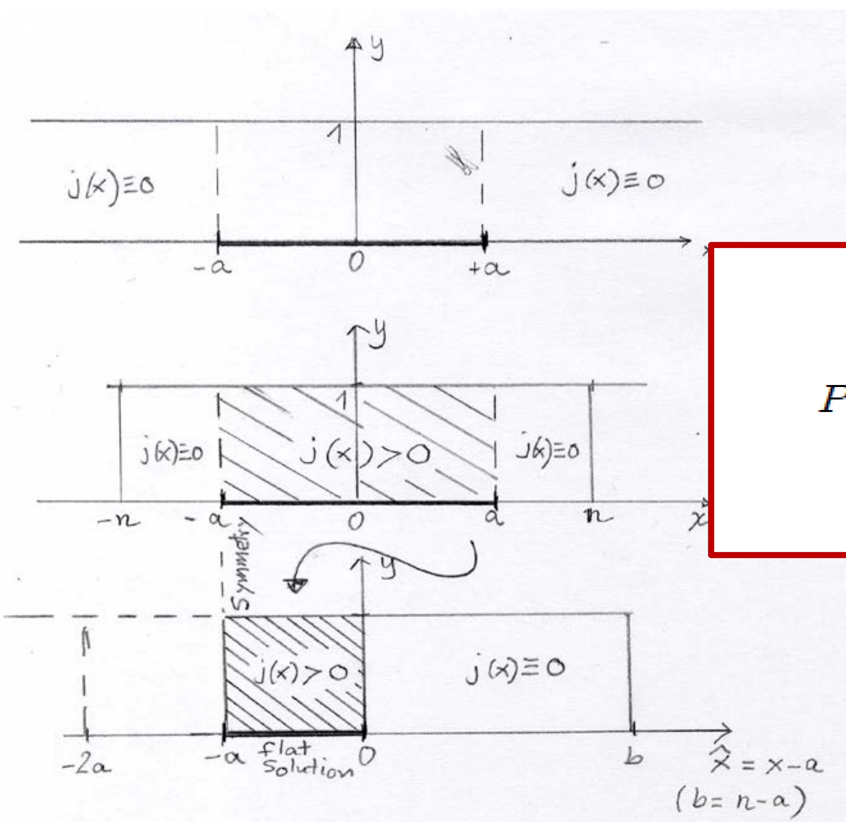
Remark. Singular PDEs of this type arises in Chemical Engineering

W. Fulks and J.S. Maybee. A singular non-linear equation. Osaka. Math. J. 12 (1960), 1–19

In this other framework solutions may vanish in some parts of the spatial domain (*dead cores*) and then the PDE is reformulated as

$$\Delta u = \frac{j}{\sqrt{u}} \chi_{\{u>0\}}$$

A spatial domain simplification keeping the main difficulties of the problem



$$P_{n,a,j} \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-n,n), \ y \in (0,1), \\ u(x,0) = 0 & x \in (-n,n), \\ u(x,1) = 1 & x \in (-n,n), \\ u(\pm n, y) = y & y \in (0,1), \end{cases}$$

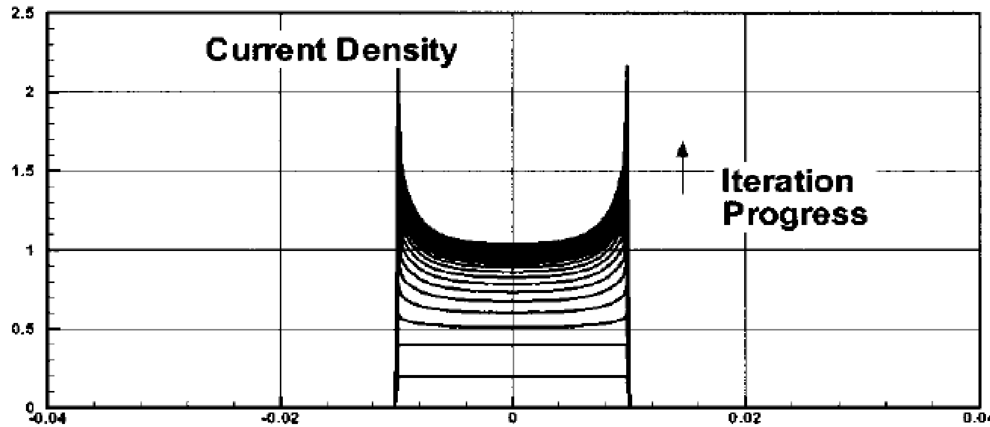
$$P_{a,b,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-a,b), \ y \in (0,1), \\ u(x,0) = 0 & x \in (-a,b), \\ u(x,1) = 1 & x \in (-a,b), \\ u(x,-a) = y^{4/3} & y \in (0,1), \\ u(x,b) = y & y \in (0,1), \end{cases}$$

$$AC_{a,b} = \begin{cases} \frac{\partial u}{\partial y}(x,0) = 0 & x \in (-a,0), \\ u(x,y) > 0 & x \in (-a,b), \ y \in (0,1). \end{cases}$$

$$\begin{cases} j(x) > 0 & \text{if } x \in (-a,0), \ j \in L^1_{loc}(-a,0), \\ j(x) = 0 & \text{if } x \in (0,b). \end{cases}$$

In the lecture we will prove, and make precise, a conjecture by A. Rokhlenko raised on *Journal of Applied Physics* (2006): 6 pages after using asymptotic and numerical methods:

If $j(x)$ behaves (near the boundary of the cathode, $x = \pm a$) as $A/|x \pm a|^{\beta}$, for $\beta \in [0, \beta_0)$, with $\beta_0 < 1/2$ and $A > 0$, then there exists a solution $u(x,y)$ of (7), satisfying (8), and $u(x,y)$ behaves (near the cathode $[-a,a] \times \{0\}$) as y^{α}, for some $\alpha \in (1, 4/3]$ with the law $\alpha = 4/3 - 2\beta$.



PHYSICS OF PLASMAS VOLUME 8, NUMBER 9 SEPTEMBER 2001

Current and current density of a finite-width, space-charge-limited electron beam in two-dimensional, parallel-plate geometry

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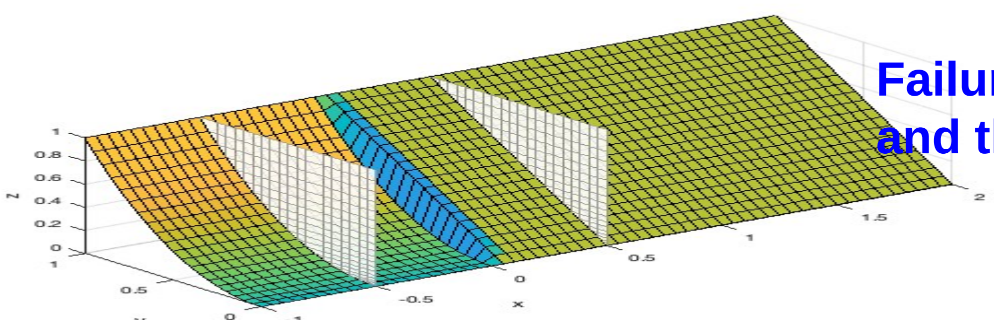


Failure of the Strong Maximum Principle and the Unique Continuation Principle !!!

2.The one-dimensional Child-Langmuir law revisited.

$$\begin{cases} -u''(x) + \dfrac{j}{\sqrt{u(x)}} = 0 \qquad x \in (0,1),\ u \geq 0, \\ u(0) = 0 \quad u(1) = 1, \end{cases}$$

$$\boxed{u'(0) = 0.}$$

$$K = \{u \in H^1(0,1) \text{ such that } u(0) = 0, u(1) = 1 \text{ and } u \geq 0 \text{ on } (0,1)\}$$

$$J(u) = \int_0^1 \left(\frac{1}{2} (u'(x))^2 + 2j\sqrt{u(x)} \right) dx \qquad -u''(x) + \frac{j}{\sqrt{u(x)}} \chi_{\{u>0\}} = 0 \quad x \in (0,1),$$

Theorem 8 *(H. Brezis: Personal communication 2004) There exists a unique variational solution. Moreover if we define*

$$j^* = \frac{4}{9},$$

a) if $j = j^*$ *the variational solution is a flat solution and it is given by* $u(x) = x^{4/3}$.

b) if $j > j^*$ *the variational solution is a free boundary solution with* $\xi = 1 - \dfrac{1}{\sqrt{\frac{9}{4}j}}$,

$$u(x) = A(x-\xi)_+^{4/3} \qquad A = (\frac{9}{4}j)^{2/3}.$$

c) if $0 < j < j^*$ *then the variational solution is such that* $u > 0$ *on* $(0,1]$ *and*

$$u'(0) = K_0 > 0, \quad \textit{for some } K_0 = K_0(j).$$

Remark. Optimal regularity: in cases a) and b): $u \in W^{2,p}(\xi,1) \quad \forall p \in [1,\frac{3}{2})$, and in case c) $u \in W^{2,p}(0,1) \quad \forall p \in [1,2)$.
In fact, sharp gradient estimate

$$|u'(x)| \leq Cu^{1/4}(x), \textit{ for some } C>0, \textit{ for any } x \in (0,1).$$

Estimates of this nature play an important role in the study of the existence of solutions of the associate parabolic problem (the so called "quenching problem"): see, e.g. Phillips (1987), Dao-Díaz (2016),...

The distributed one-dimensional current density

The main goal of this subsection is to consider a similar problem for some $j = j(x)$ with $j \in L^1_{loc}(0,1)$, $j \geq 0$. The problem under consideration is

$$\begin{cases} -u''(x) + \dfrac{j(x)}{\sqrt{u(x)}}\chi_{\{u>0\}} = 0 \quad x \in (0,1), u \geq 0 \\ \\ u \geq 0 \quad x \in (0,1) \\ \\ u(0) = 0 \quad u(1) = 1, \end{cases}$$

We can consider some special cases of $j(x)$ which allow to get some results in the line of the above Theorem

$$j(x) = \lambda x^q \text{ with } q \in (-\frac{1}{2}, 1), \ \lambda > 0. \tag{41}$$

Note that, obviously, $q = 0$ corresponds to the case treated before and that our study will consider cases in which $j(0) = 0$ ($q \in (0,1)$) as well as cases in which $j(0) = +\infty$ ($q \in (-\frac{1}{2}, 0)$).

Theorem 14 *(D-2022) Assume (41). Then if we define*

$$\lambda_q^* = \frac{2(1+2q)(2+q)}{9}$$

then:
a) if $\lambda = \lambda_q^*$ *the function*

$$u(x) = x^{(4+2q)/3}$$

is a flat solution of the problem and

$$\frac{j(x)}{\sqrt{u(x)}} = \lambda_q^* x^{-\frac{(2-2q)}{3}} \in L^1(0,1).$$

b) if $\lambda > \lambda_q^*$ *the function*

$$u(x) = A(x-\xi)_+^{(4+2q)/3}$$

is a free boundary solution with

$$\xi = 1 - \frac{1}{\left[\frac{\lambda}{\lambda_q^*}\right]^{1/(2+q)}},$$

and

$$A = (\frac{\lambda}{\lambda_q^*})^{2/3}.$$

Moreover

$$\frac{j(x)}{\sqrt{u(x)}}\chi_{\{u>0\}} = \frac{\lambda}{\sqrt{A}}(x-\xi)^{-\frac{(2-2q)}{3}}\chi_{\{x>\xi\}} \in L^1(0,1).$$

c) if $0 < \lambda < \lambda_q^*$, *then the unique solution* u *is such that* $u > 0$ *on* $(0,1]$ *and*

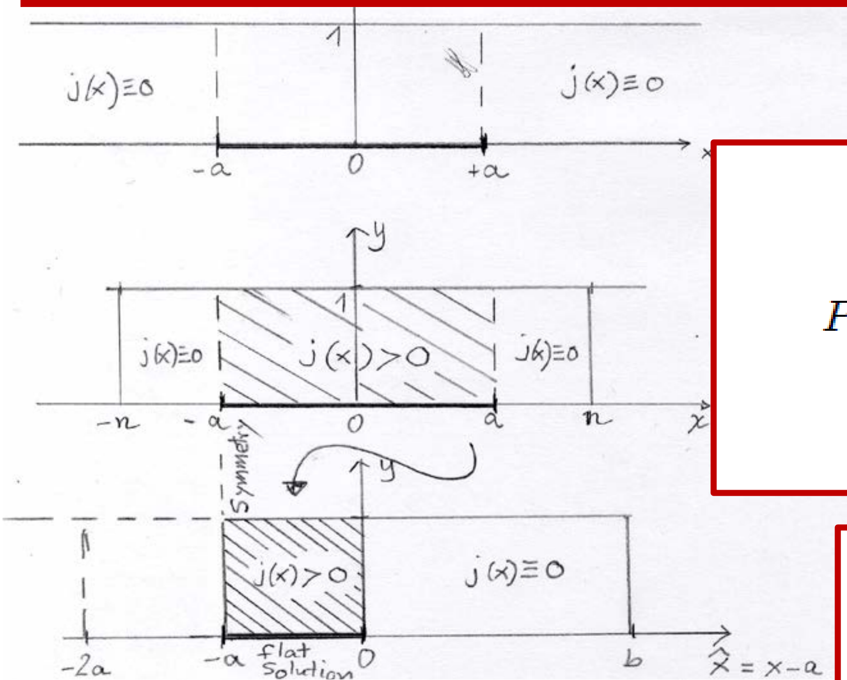
$$u'(0) = K_0 > 0, \quad \text{for some } K_0 = K_0(\lambda).$$

Remark. *If we use a different notation,* $\beta = -q$ *and* $\alpha = (4-2\beta)/3$, *then we get the existence of solutions of the form* $u(x) = x^\alpha$, *with* $\alpha \in (1, 4/3)$, *once we assume* $\beta \in (0, 1/2)$. *Curiosly enough, the constraint* $3\alpha/2 + \beta = 2$ *arises also in the Rokhlenko (2006) approach to the two-dimensional problem.*

Remark. *The gradient estimate mentioned when j is a constant is no longer valid when $j(x) = \lambda x^{-\beta}$ with $\beta \in (0, \frac{1}{2})$. If $\lambda \geq \lambda_\beta$, for some $\lambda_\beta > 0$*

$$|u'(x)| \leq C u^{\frac{1+2\beta}{4+2\beta}}(x), \text{ for some } C > 0, \text{ for any } x \in (0,1).$$

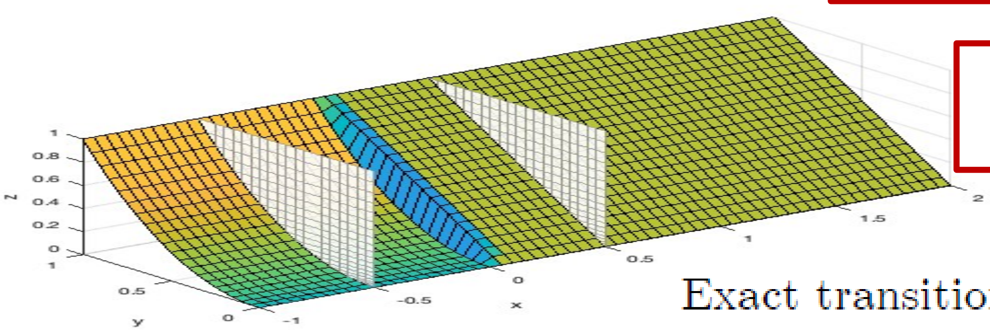
3. The 2-d parallel-plate geometry.
3.1. Statements of the existence of partially flat solutions (via super and subsolutions method).



$$P_{a,b,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-a,b),\ y \in (0,1), \\ u(x,0) = 0 & x \in (-a,b), \\ u(x,1) = 1 & x \in (-a,b), \\ u(x,-a) = y^{4/3} & y \in (0,1), \\ u(x,b) = y & y \in (0,1), \end{cases}$$

$$GAC_{a,b} = \begin{cases} Cd(x, \partial\Omega)^\alpha \leq u(x,y) \\ \text{for a.e. } (x,y) \in (-a,b) \times (0,1). \end{cases}$$

$$\begin{cases} j(x) > 0 & \text{if } x \in (-a,0),\ j \in L^1_{loc}(-a,0), \\ j(x) = 0 & \text{if } x \in (0,b). \end{cases}$$



Exact transition, at $x = 0$, between flat and linear profiles

Definition 19 *A function $u^0 \in W^{1,p}(\Omega)$, is said a $p-$positive supersolution of $P_{a,b,j}$ if $u^0 \geq 0$, $\frac{j}{\sqrt{u^0}} \in L^p(\Omega)$, for some $p \geq 1$, and verifies that*

$$\begin{cases} -\Delta u^0 \geq -\frac{j(x)}{\sqrt{u^0}} & \text{in } \Omega, \\ u^0(x,0) \geq 0 & x \in (-a,b), \\ u^0(x,1) \geq 1 & x \in (-a,b), \\ u^0(-a,y) \geq y^{4/3} & y \in (0,1), \\ u^0(b,y) \geq y & y \in (0,1). \end{cases}$$

The notion of $p-$positive subsolution u_0 is introduced similarly.

Theorem 20 *Assume that*

$$\begin{cases} \text{there exists } p > 1, \text{ a } p-\text{positive supersolution } u^0 \text{ and a } p-\text{positive subsolution } u_0 \\ \text{of } P_{a,b,j} \text{ such that } 0 < u_0 \leq u^0 \text{ a.e. in } \Omega. \end{cases} \tag{63}$$

Then problem $P_{a,b,j}$ possesses a minimal and maximal solutions u_* and u^* in the interval $[u_0, u^0]$, i.e.,

$$u_0 \leq u_* \leq u^* \leq u^0 \text{a.e.in} \quad \Omega.$$

Idea of the proof. We define the iterative schemes (starting with u^0 and u_0)

$$\begin{cases} -\Delta u^n = -\frac{j(x)}{\sqrt{u^{n-1}}} & \text{in } \Omega, \\ u^n(x,0) = 0 & x \in (-a,b), \\ u^n(x,1) = 1 & x \in (-a,b), \\ u^n(-a,y) = y^{4/3} & y \in (0,1), \\ u^n(b,y) = y & y \in (0,1). \end{cases} \qquad \begin{cases} -\Delta u_n = -\frac{j(x)}{\sqrt{u_{n-1}}} & \text{in } \Omega, \\ u_n(x,0) = 0 & x \in (-a,b), \\ u_n(x,1) = 1 & x \in (-a,b), \\ u_n(-a,y) = y^{4/3} & y \in (0,1), \\ u_n(b,y) = y & y \in (0,1). \end{cases}$$

By using the comparison principle for the Laplace operator we get that

$$0 < u_0 \leq u_1 \leq ... \leq u_n \leq ... \leq u^n \leq ... \leq u^1 \leq u^0 \text{ a.e. in } \Omega,$$

and so the sequences $\{u^n\}, \{u_n\}$ converge (monotonically) in $L^p(\Omega)$ to some functions u_* *and* u^* and the sequences $\{\frac{j}{\sqrt{u^{n-1}}}\}$, $\{\frac{j}{\sqrt{u_{n-1}}}\}$ are bounded in $L^p(\Omega)$ and converge also (monotonically) in $L^p(\Omega)$.

Remark. With some minor modifications, the above result holds when the super and subsolutions are in the weighted space $\frac{j}{\sqrt{u^0}} \in L^p(\Omega, \delta)$, for some $p \geq 1$, with $\delta = d((x,y), \partial\Omega)$. See, e.g.

* H. Brézis, T. Cazenave, Y. Martel and A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, *Adv. Differential Equations*, **1**(1) (1996), 73–90,

* J.I. Díaz and J.M. Rakotoson, On very weak solutions of semilinear elliptic equations with right hand side data integrable with respect to the distance to the boundary, *Discrete and Continuum Dynamical Systems*, **27** 3 (2010) 1037-1058.

Remark. By some standard approximating arguments, it is well-known that the above notion of $p-$super and subsolutions of $\widehat{P}_{a,b,\widehat{j}}$ can be extended to the case in which the diffusion term generate an additional distribution over a simple curve Γ separating Ω in two different parts (matching without a $W^{1,p}$-contact. A.M. Ii'in, A.S. Kalashnikov and O.A. Oleinik. Linear equations of the second order of parabolic type. Russian Math. Surveys 17 3 (1962), 1-143.
H. Berestycki and P. L. Lions, Some applications of the method of sub- and supersolutions, in Bifurcation and Nonlinear Eigenvalue Problems (Bardos, Lasry and Schatzman, eds.), Math. No.782, Springer, New York,1979.
Then the subsolution $\underline{u}$ of problem $P_{a,b,j}$ satisfies (for some $c \geq 0$)

$$-\Delta u_0 + \frac{j(x)}{\sqrt{u_0}} \leq -c\delta_\Gamma \text{ in } \mathcal{D}'(\Omega).$$

The existence of a (not-flat) supersolution u^0 can be easily proved.

Lemma 22 *Let $j(x)$ satisfying*

$$\begin{cases} j(x) > 0 & \text{if } x \in (-a,0),\ j \in L^1_{loc}(-a,0), \\ j(x) = 0 & \text{if } x \in (0,b). \end{cases}$$

Then the function $u^0(x,y) = y$ is a positive $p-$supersolution of $P_{a,b,j}$ for any $p \in [1,2)$.

Proof. It is a trivial fact since

$$-\Delta u^0 = 0 \geq -\frac{j(x)}{\sqrt{u^0}} \quad x \in (-a,b),\ y \in (0,1),$$

and

$$\begin{cases} u^0(x,0) = 0 & x \in (-a,b), \\ u^0(x,1) = 1 & x \in (-a,b), \\ u^0(-a,y) \geq y^{4/3} & y \in (0,1), \\ u^0(b,y) = y & y \in (0,1). \end{cases}$$

The construction of a positive subsolution is a very delicate task which will be presented in the following Section.

One of the main results of this lecture is the following existence result. We define

$$\delta(x,y) := \operatorname{dist}((x,y),\partial\Omega)$$

(which sometimes we shall denote simply as δ).

Theorem A *Assume that $j(x)$ satisfies*

$$j(x) \leq \frac{A}{(-x)^{\beta}},\ \textit{for } x \in (-a,0) \textit{ and } j(x) = 0 \textit{ if } x \in (0,b),$$

with

$$0 \leq \beta < 1/2 \textit{ and } A > 0 \textit{ small enough.} \tag{65}$$

Then there exists a weak solution $u \in L^2(\Omega;\delta)$ of

$$P_{a,b,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-a,b),\ y \in (0,1), \\ u(x,0) = 0 & x \in (-a,b), \\ u(x,1) = 1 & x \in (-a,b), \\ u(-a,y) = y^{4/3} & y \in (0,1), \\ u(b,y) = y & y \in (0,1), \end{cases} \tag{66}$$

such that

$$C\delta(x,y)^{\alpha} \leq u(x,y) \leq y \text{ a.e. } (x,y) \in (-a,b)\times(0,1),$$

for some $C > 0$, with $1 < \alpha \in (1, \frac{4}{3}]$, given by

$$\alpha = \frac{2}{3}(2-\beta).$$

It is possible to show the uniqueness of solutions in the class of flat (or *nondegenerate*) solutions by applying the techniques introduced in the recent paper
J.I. Díaz and J. Giacomoni, Monotone continuous dependence of solutions of singular quenching parabolic problems, *Rendiconti del Circolo Matematico di Palermo Series 2 (2022).*

Let

$$\nu \in \left(0, \frac{4}{3}\right] \tag{66}$$

and define the class of functions

$$\mathcal{M}(\nu) := \left\{ u \in L^2(\Omega;\delta) \mid \text{such that } u(x,y) \geq C\delta(x,y)^{\nu} \quad \text{in } \Omega, \text{ for some } C > 0 \right\}. \tag{67}$$

Theorem B *Assume $j(x)$ as in the precedent Theorem. Then, there exists at most a solution $u_j \in \mathcal{M}(\nu)$ of $P_{a,b,j}$.*

The proof will be obtained through some smoothing estimates for the associate parabolic problem (see the last section).

Theorem C. *There exists* $A_0, b_0 > 0$ *and* $\beta_0 \in (0, \frac{1}{2})$ *such that, if*

$$b \geq b_0 > 0,$$

and if we assume

$$\begin{cases} j(x) = \frac{A}{(-x)^\beta}, & \text{for } x \in (-a, 0), \\ j(x) = 0 & \text{for } x \in (0, b), \end{cases}$$

with

$$0 \leq \beta < \beta_0 \text{ and } A \in (0, A_0),$$

then there exists a partially flat supersolution $\overline{u}(x,y)$ *of problem* $P_{a,b,j}$*, i.e., such that* $\overline{u} \in L^2(\Omega; \delta)$,

$$\begin{cases} -\Delta\overline{u} + \frac{j(x)}{\sqrt{\overline{u}}} \geq 0 & x \in (-a,b), \ y \in (0,1), \\ \overline{u}(x,0) \geq 0 & x \in (-a,b), \\ \overline{u}(x,1) \geq 1 & x \in (-a,b), \\ \overline{u}(-a,y) \geq y^{4/3} & y \in (0,1), \\ \overline{u}(b,y) \geq y & y \in (0,1), \end{cases}$$

and

$$0 < \overline{u}(x,y) \leq C\delta(x,y)^\alpha \text{ a.e. } (x,y) \in (-a,0) \times (0,1),$$

for some $C > 0$, *with* $\alpha \in (\alpha_0, \frac{4}{3}]$ *given by*

$$\alpha = \frac{2}{3}(2-\beta), \text{ where } \alpha_0 = \frac{2}{3}(2-\beta_0).$$

In particular, under the above assumptions, there exists a unique partially flat solution of the problem $P_{a,b,j}$.■

3.2. On the construction of a partially flat subsolution (I).

To complete the study of the 2-d problem $P_{a,b,j}$ we will construct a **flat positive subsolution.**

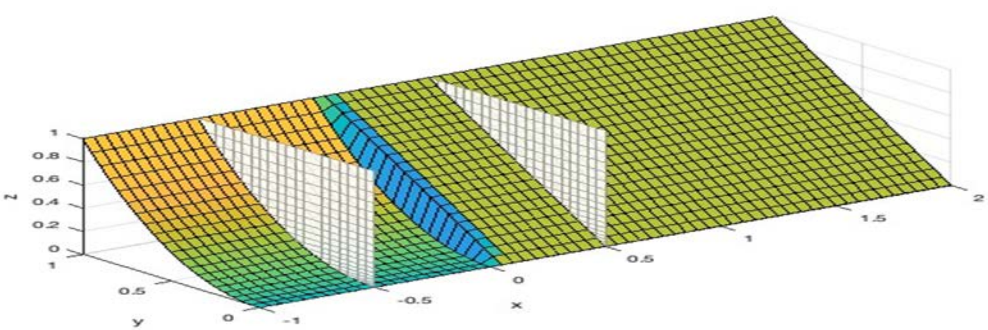
We start by reducing the difficulty of this task by splitting the domain in two well differentiate problems. We define the subsets

$$\Omega_- := \{(x,y) \in \Omega : x \in (-a,0)\}, \quad \Omega_+ := \{(x,y) \in \Omega : x \in (0,b)\}.$$

Once again, we need to introduce an artificial boundary condition on the points $(0,y)$, $y \in (0,1)$, corresponding to the boundary of the cathode. We will prove something already suggested in Rokhlenko (2006) in its study made by asymptotics techniques and numerical analysis.

It is an important conclusion which modifies what it was expected in many previous papers in the literature on space charge problems: the correct behaviour in the border of the cathod is higher than the profile $y^{4/3}$ corresponding to the Child-Langmuir law. The precise **artificial boundary condition** we will introduce is of Dirichlet type

$$u(0,y) = hy^\alpha, \quad y \in (0,1), \text{for some } h \in (0,1] \text{ and } \alpha \in (1,4/3).$$



The following result simplify the task since it allows to pass from the study of a discontinuous absorption coefficient $j(x)$ to a problem with a strictly positive one.

Proposition 25 *Let $j(x)$ satisfying*

$$\begin{cases} j(x) > 0 & \text{if } x \in (-a,0),\ j \in L^1_{loc}(-a,0), \\ j(x) = 0 & \text{if } x \in (0,b). \end{cases}$$

Let $\alpha \in (1, 4/3)$. Given $h \in (0,1]$, consider the problem on Ω_-

$$P_{a,0,j} = \begin{cases} -\triangle u_- + \dfrac{j(x)}{\sqrt{u_-}} = 0 & \text{in } \Omega_-, \\ u_-(-a,y) = y^{4/3} & y \in (0,1), \\ u_-(0,y) = hy^{\alpha} & y \in (0,1), \\ u_-(x,0) = 0 & x \in (-a,0), \\ u_-(x,1) = 1 & x \in (-a,0). \end{cases}$$

Assume that there exists $u_{0,-}(x,y)$, a p_0-subsolution of problem $P_{a,0,j}$, for some $p_0 \geq 1$, such that

$$\frac{\partial u_{0,-}}{\partial x}(0,y) \leq 0 \text{ for } y \in (0,1), \tag{68}$$

satisfying also the additional conditions

$$AC_{a,0} = \begin{cases} \frac{\partial u_{0,-}}{\partial y}(x,0) = 0 & x \in (-a,0), \\ u_{0,-}(x,y) > 0 & x \in (-a,0),\ y \in (0,1). \end{cases} \tag{69}$$

Then problem $P_{a,b,j}$ has a p_0-subsolution $\underline{u}$ satisfying the additional conditions $AC_{a,b}$.

Proof. Let $u_+(x,y)$ be the unique classical solution $u_+ \in C^2(\Omega_+) \cap C^0(\overline{\Omega_+})$ of the linear problem

$$P_{0,b,0} = \begin{cases} -\triangle u_+ = 0 & \text{in } \Omega_+, \\ u_+(0,y) = hy^\alpha & y \in (0,1), \\ u_+(b,y) = y & y \in (0,1), \\ u_+(x,0) = 0 & x \in (0,b), \\ u_+(x,1) = 1 & x \in (0,b). \end{cases}$$

Define the function

$$\underline{u}(x,y) = \begin{cases} u_-(x,y) & \text{if } (x,y) \in \Omega_-, \\ u_+(x,y) & \text{if } (x,y) \in \Omega_+. \end{cases}$$

It is clear that $\underline{u}$ is a continuous function $\underline{u} \in C^0(\Omega)$ but its gradient has a discontinuity in the segment $x = 0, y \in (0,1)$, since $j(x)$ is discontinuous in that segment. All the boundary conditions of $P_{a,b,j}$ are fulfilled and also the additional conditions $AC_{a,b}$. In order to check that $\underline{u}(x,y)$ is a p_0−subsolution of problem $P_{a,0,j}$ we will apply Corollary I.1 of H. Berestycki and P. L. Lions (1979). To do this, if we define the segment $\Gamma = \{(0,y),\ y \in (0,1)\}$ then it suffices to check that

$$\frac{\partial u_-}{\partial \mathbf{n}} \leq \frac{\partial u_+}{\partial \mathbf{n}} \quad \text{on } \Gamma, \tag{71}$$

where $\mathbf{n}$ is the unit exterior normal vector to Ω_-. In our case, $\mathbf{n} = \mathbf{e_1}$ and then condition (71) is expressed as

$$\frac{\partial u_-}{\partial x}(0,y) \leq \frac{\partial u_+}{\partial x}(0,y) \qquad y \in (0,1).$$

By assumption (68) it suffices to check that

$$\frac{\partial u_+}{\partial x}(0,y) \geq 0 \text{ for } y \in (0,1).$$

To do that, let us consider the function $U_+(x,y) = u_+(x,y) - y$. Then

$$P_{0,b,0} = \begin{cases} -\triangle u_+ = 0 & \text{in } \Omega_+, \\ u_+(0,y) = hy^\alpha & y \in (0,1), \\ u_+(b,y) = y & y \in (0,1), \\ u_+(x,0) = 0 & x \in (0,b), \\ u_+(x,1) = 1 & x \in (0,b). \end{cases}$$

We can define, now, the auxiliary function,

$$\underline{U}(x,y) = (b-x)(hy^\alpha - y),\ (x,y) \in \Omega_+$$

and then we get that

$$\begin{cases} -\triangle \underline{U} \le 0 & \text{in } \Omega_+, \\ \underline{U}(0,y) = \omega(y) = hy^\alpha - y & y \in (0,1), \\ \underline{U}(b,y) = 0 & y \in (0,1), \\ \underline{U}(x,0) = 0 & x \in (0,b), \\ \underline{U}(x,1) = 0 & x \in (0,b). \end{cases}$$

Thus, by the maximum principle we get that $\underline{U}(x,y) \le U_+(x,y)$ on Ω_+. But since $\underline{U}(0,y) = U_+(0,y)$ and we have that $\frac{\partial \underline{U}}{\partial x}(0,y) \ge 0$ for $y \in (0,1)$, we deduce that necessarily, $\frac{\partial U_+}{\partial x}(0,y) \ge 0$ for $y \in (0,1)$, which leads to the required inequality. ■

Remark. For the construction of the supersolution we will use a different matching argument (see Section 3.5).

To finish with the construction of the global subsolution u_0, according the previous Proposition, we must justify the existence of a function $u_{0,-}(x,y)$ solution of the nonlinear problem $P_{a,0,j}$, raised on Ω_-, and to check that $u_{0,-}(x,y)$ satisfies the additional conditions (69) and (68).

Remark. For the special case of a bounded current

$$j(x) \leq \frac{4}{9}$$

we can take $u_{0,-}(x,y) = y^{4/3}$ (which corresponds to take $\beta = 0$ and $\alpha = 4/3$). Notice that, obviously, $\frac{\partial u_{0,-}}{\partial x}(0,y) = 0$ for $y \in (0,1)$ and thus the above matching argument to extend it to the whole domain applies.

In order to construct the subsolution for the case of $j(x) \leq \frac{A}{(-x)^{\beta}}$ on $(-a,0)$ we will use some ideas coming from the study of Fluid Mechanics in the consideration of spatial domains with corners (see, e.g., G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967).

We will try to find the subsolution $u_{0,-}(x,y)$ in the form

$$u_{0,-}(x,y) = \phi(r,\theta) = kr^{\alpha}U(\theta), \text{ for some } k > 0,\ \alpha > 1,$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$. Then, the partial differential equation becomes

$$-\Delta\phi + \frac{j(r\cos\theta)}{\sqrt{\phi}} = -\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) - \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{j(r\cos\theta)}{\sqrt{\phi}} \leq 0. \tag{77}$$

We will assume $r \in [0, R)$, for a suitable $R > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$. The additional conditions (69) will require $\phi(r,\theta) > 0$ if $r > 0$ and

$$\phi(r,\pi) = \frac{\partial\phi}{\partial\theta}(r,\pi) = 0 \text{ for } r \in [0, R).$$

We make now the **structural condition on** $j(x)$

$$j(r\cos\theta) \leq \frac{A}{(-r\cos\theta)^{\beta}}, \text{ if } \theta \in (\frac{\pi}{2}, \pi), \text{ for some } A > 0 \text{ and } \beta \in (0,1),$$

the partial differential equation (77) leads to the ordinary differential equation

$$-U''(\theta) + \frac{V(\theta)}{\sqrt{U(\theta)}} = \lambda U(\theta) \quad \theta \in (\tfrac{\pi}{2}, \pi),$$

once we assume the constraint

$$\alpha + 2\beta = 4/3,$$

and then

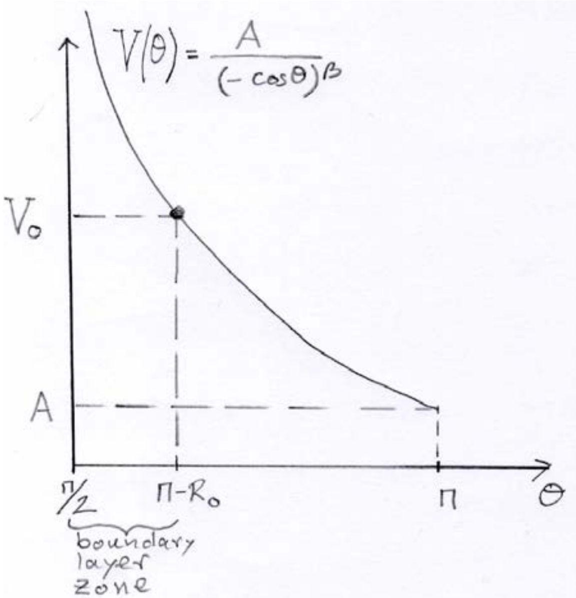
$$V(\theta) = \frac{A}{(-\cos\theta)^{\beta}} \text{ and } \lambda = \alpha^2.$$

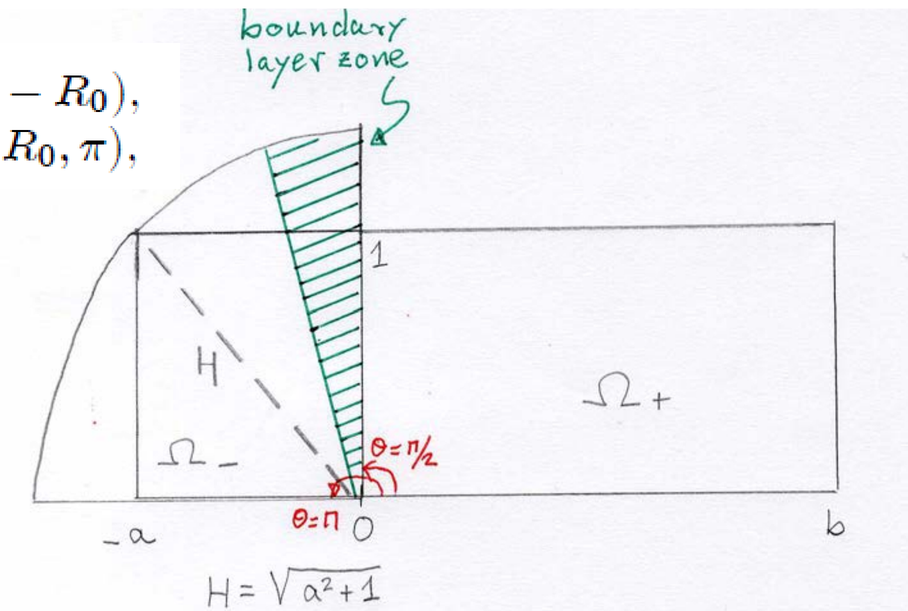
The complemmentary conditions become now

$$\begin{cases} U(\theta) > 0 & \theta \in (\frac{\pi}{2}, \pi), \\ U(\pi) = U'(\pi) = 0. \end{cases}$$

Notice that the potential $V(\theta)$ is singular only for $\theta = \frac{\pi}{2}$. On the other hand, we will need to match this subsolution with other function which is positive for $\theta = \frac{\pi}{2}$. Then, we will construct the subsolution in two different pieces

$$U(\theta) = \begin{cases} v_1(\theta) & \text{if } \theta \in (\frac{\pi}{2}, \pi - R_0), \\ v_2(\theta) & \text{if } \theta \in (\pi - R_0, \pi), \end{cases}$$





for some R_0 with $\pi - R_0 \in (\frac{\pi}{2}, \pi)$, and with $v_2(\theta) \in [0,1]$ such that

$$\begin{cases} -v_2''(\theta) + \frac{V_0}{\sqrt{v_2(\theta)}} = \lambda v_2(\theta) & \theta \in (\pi - R_0, \pi), \\ v_2(\pi) = v_2'(\pi) = 0, \end{cases}$$

By the contrary, $v_1(\theta)$ must take into account the singularity of the potential $V(\theta)$ on the interval $(\frac{\pi}{2}, \pi - R_0)$.

In addition, we must guarantee the good matching with the function $u_+(x,y)$ defined on Ω_+, as indicated in the above Proposition. This means that we want to have

$$v_1'(\frac{\pi}{2}) \geq 0$$

since we require $\frac{\partial u_{0,-}}{\partial x}(0,y) \leq 0$ for $y \in (0,1)$, and since $u_{0,-}(0,y) = kr^\alpha U(\theta)|_{\theta=\frac{\pi}{2}}$, we have

$$\frac{\partial u_{0,-}}{\partial x}(0,y) = ... = -Ar^{\alpha-1}U'(\theta)\big|_{\theta=\frac{\pi}{2}} \leq 0.$$

Before to present the details on the construction of $v_1(\theta)$ and $v_2(\theta)$ it is very useful to consider an auxiliary nonlinear ODE of eigenvalue type.

3.3. Bifurcation diagram for a singular eigenvalue ODE problem.

Since the equation (79) can be undertood as a nonlinear eigenvalue problem it is useful to start by considering the following auxiliary related problem:

$$\begin{cases} -U''(s) + \frac{V_0}{\sqrt{U(s)}} = \lambda U(s) & s \in (-R, R), \\ U(\pm R) = 0, \end{cases} \tag{80}$$

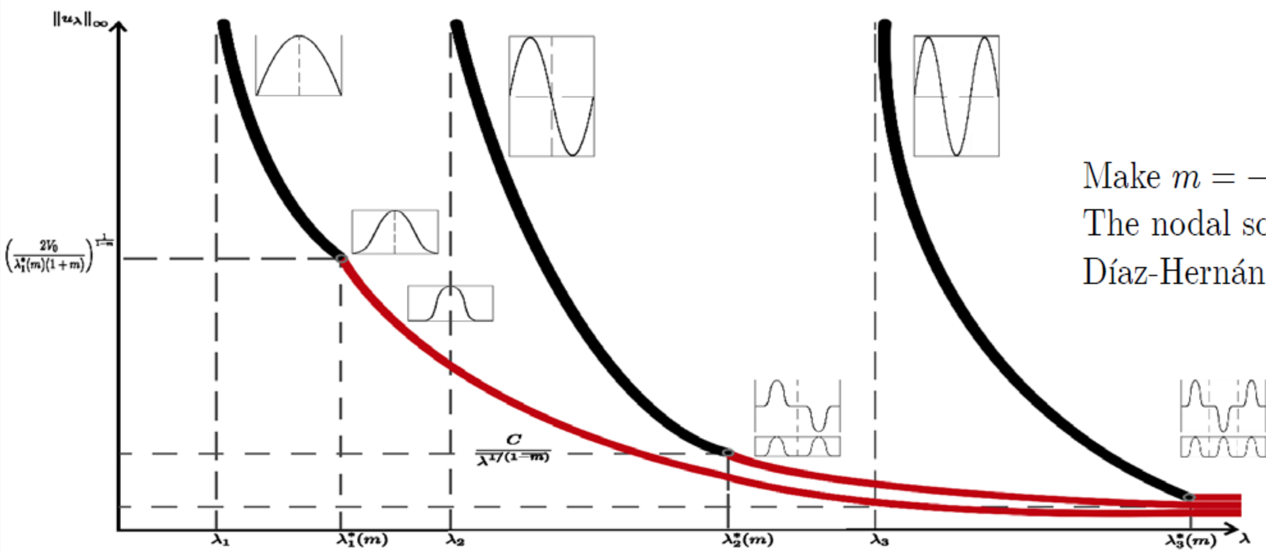
where the positive constants V_0 and R are given. We will show:

Theorem 26 *Given the positive constants V_0 and R then*

i) there is a bifurcation from the infinity for λ near $\lambda_1(R) = (\frac{\pi}{2R})^2$ (the first eigenvalue of the linear problem with $V_0 = 0$),

ii) the bifurcation curve is strictly decreasing (which implies the uniqueness of nonnegative solutions)

iii) the curve is not C^1 for a suitable value $\lambda = \lambda^* > \lambda_1(R)$ corresponding to a "flat solution" (i.e. the solution U is such that $U'(\pm R) = 0$ and $U(s) > 0$).

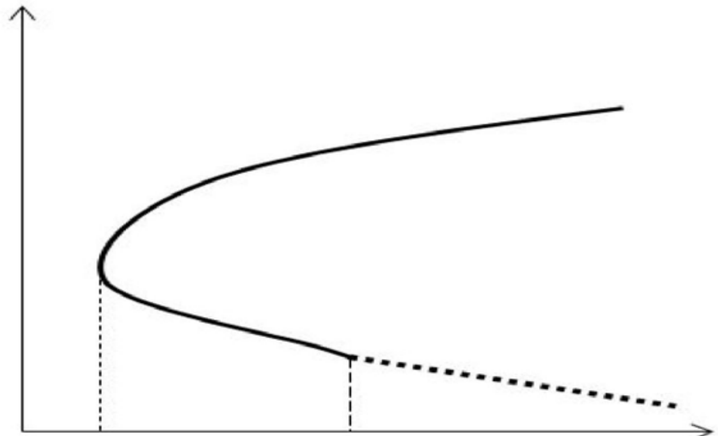


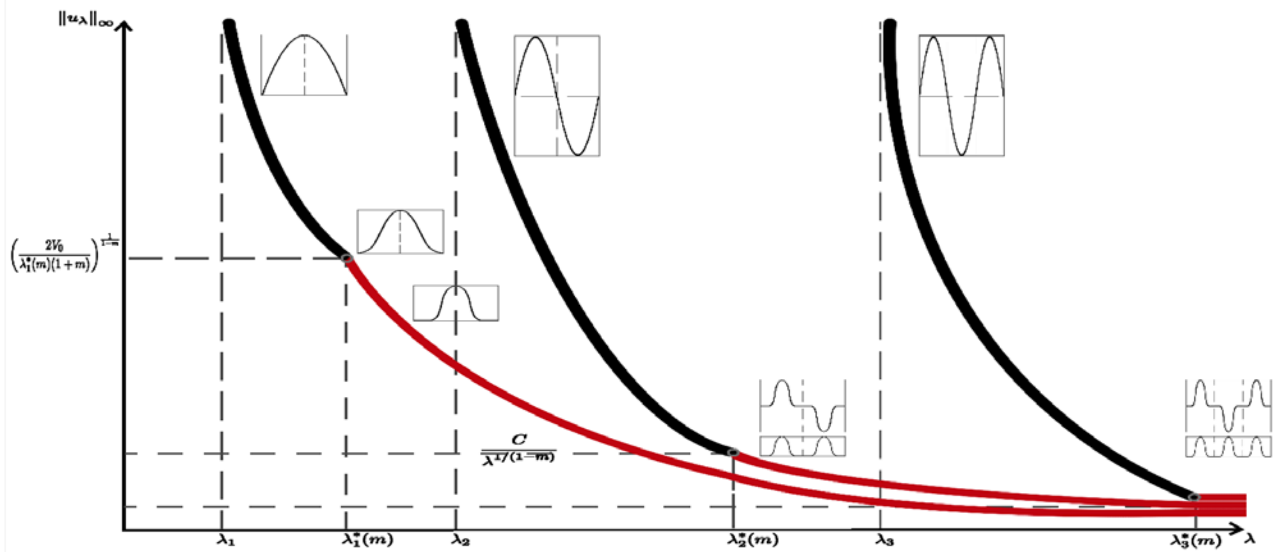
Make $m = -1/2$ in the Figure.
The nodal solutions can be constructed as in Díaz-Hernández-Mancebo (2009) for a related problem.

Then new thing, with respect the above reference, is that instead to consider

$$\begin{cases} -U''(s) + V_0 U(s)^m = \lambda U(s)^q & s \in (-R, R), \\ U(\pm R) = 0, \end{cases} \tag{81}$$

with $-1 < m \leq q < 1$, is that now $m = -1/2$ and $q = 1$ and then the former bifurcation diagram pass to be the one indicated before





By multiplying by u and integrating by parts, we get that there nontrivial solutions may exist only if $\lambda > \lambda_1 = \frac{\pi^2}{4R^2}$, the first eigenvalue to the linear problem

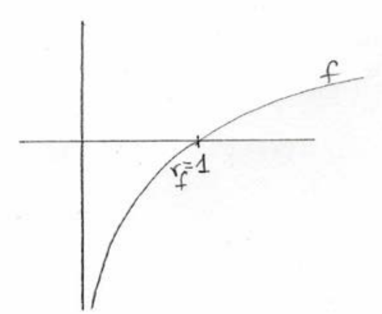
$$\begin{cases} -u'' = \lambda u & \text{in } (-R, R), \\ u(\pm R) = 0. \end{cases} \tag{82}$$

To show the qualitative behaviour of solutions of problem (80) we make the change of variables

$$u_{\lambda,V_0}(x) = \left(\frac{V_0}{\lambda}\right)^{\frac{2}{3}} u(\sqrt{\lambda}x)$$

where u is now the solution of the renormalized problem

$$P(L) \begin{cases} -u'' = f(u) & \text{in } (-L, L), \\ u(\pm L) = 0, \end{cases} \tag{84}$$



where

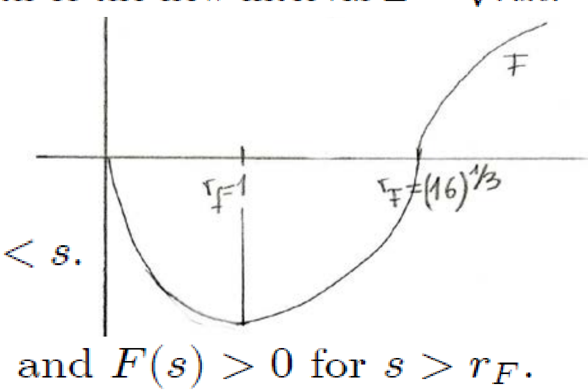
$$L = \sqrt{\lambda}R \qquad f(u) = u - \frac{1}{\sqrt{u}}$$

Notice that now the role of the "eigenvalue" λ is transferred to the length of the new interval $L = \sqrt{\lambda}R$.

We introduce

$$F(r) = \int_0^r f(s)ds = \frac{r^2}{2} - 2\sqrt{r}$$

and note that $f(s) < 0$ if $0 < s < 1 := r_f$ and $f(s) > 0$ if $1 < s$.



On the other hand $F(s) < 0$ if $0 < s < r_F = 4^{2/3}(= 2,51)$ and $F(s) > 0$ for $s > r_F$.

By multiplying by u', integreating by parts and denoting $\mu := \|u\|_{L^\infty}$ for $\mu \in (r_F, \infty)$ we get that a function u is a positive solution of problem $P(L)$ if and only if

$$\frac{1}{\sqrt{2}} \int_{u(x)}^{\mu} \frac{dr}{(F(\mu) - F(r))^{1/2}} = |x|, \text{ for } |x| \leq L,$$

and μ and $L > 0$ are related by the equation

$$\boxed{\gamma(\mu) = L,}$$

where $\gamma : [r_F, +\infty) \to \mathbb{R}$ is given by $\gamma(\mu) := \frac{1}{\sqrt{2}} \int_0^{\mu} \frac{dr}{(F(\mu) - F(r))^{1/2}}$.

Moreover

$$u'(\pm L) = \mp\sqrt{2}\sqrt{F(\mu)}.$$

Thus, $u'(\pm R) = 0$ corresponds to the case in which the maximum of the solution is r_F.

In fact we have:

Theorem 28 *We define with* $F(r) = \frac{r^2}{2} - 2\sqrt{r}$. *Let* $r_F = 4^{2/3}$. *Then the mapping* $\gamma : [r_F, +\infty) \to R$ *has the following properties*

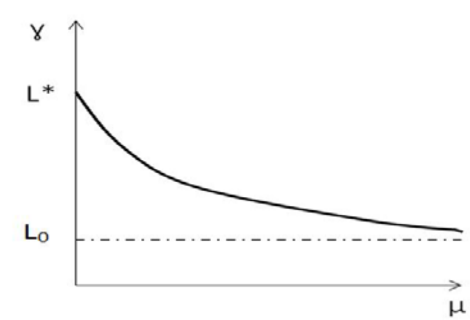
(i) $\gamma \in C[r_F, \infty) \cap C^1(r_F, \infty)$;

(ii) For any $\mu > r_F$ $\gamma'(\mu) < 0$,

(iii) $\gamma'(\mu) \to -\xi$ *as* $\mu \downarrow r_F$, *for some* $\xi > 0$,

(iv) $\lim_{\mu \to +\infty} \gamma(\mu) = \frac{\pi}{2}$.

So, qualitatively, function γ is described by the following figure



Proof of Theorem 28. The proof of property i) is exactly the same than the one presented in D-Hernández-Mancebo (2009). For the proof of (ii) and (iii) we have

$$\gamma'(\mu) = \int_0^\mu \frac{\theta(\mu) - \theta(r)}{(F(\mu) - F(r))^{1/2}} dr$$

where $\theta(t) = 2F(t) - tf(t) = -3\sqrt{t}$, and differentiating we get for, any $t > 0$

$$\theta'(t) = -\frac{3}{2\sqrt{t}} < 0.$$

Hence $\gamma'(\mu) < 0$ for any $\mu > r_F$ (which proves (ii).

For the proof of (iii) it suffices to see that as $\mu \downarrow r_F$, the integrand of $\gamma'(\mu)$ converges pointwise to $(-F(r))^{-3/2}$ near $r = 0$ and in our case $(-F(r))^{-3/2}$ behavies as $r^{-3/4}$ near $r = 0$ and thus $\gamma'(\mu)$ converges to a number $-\xi$ as $\mu \downarrow r_F$, for some $\xi > 0$.

Finally, to prove (iv) we note that

$$\gamma(\mu) \leq \frac{\mu}{2} \int_0^1 \frac{dt}{(\frac{\mu^2}{2}(1 - t^2))^{1/2}} = \int_0^1 \frac{dt}{\sqrt{1 - t^2}} = \frac{\pi}{2}. \tag{88}$$

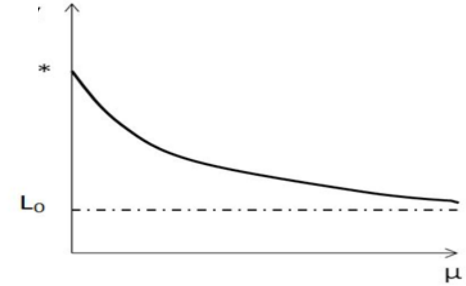
Moreover, we have

$$\gamma(\mu) = \frac{\mu}{\sqrt{2}} \int_0^1 \frac{dt}{\left(\frac{\mu}{\sqrt{2}}((1 - t^2) - \frac{1}{\mu^{3/2}}(1 - \sqrt{t})\right)}$$

and if $\mu \to +\infty$ by using Lebesgue's Theorem we get $\lim_{\mu \to +\infty} \gamma(\mu) = \frac{\pi}{2}$.

Now we define $L_0 = \frac{\pi}{2}$ and L^* given by

$$L^* = \gamma(r_F) = \frac{1}{\sqrt{2}} \int_0^{r_F} \frac{dr}{(-F(r))^{1/2}} = \frac{1}{\sqrt{2}} \int_0^{r_F} \frac{dr}{(-2\sqrt{r} + \frac{r^2}{2})^{1/2}}.$$



$$\boxed{r_F \ = \ 4^{2/3} \ (= \ 2,51).}$$

It can be computed that $\gamma(r_F) = 2,09$ [G.Díaz, 2022: *Gaus-Lobatto rules*, Rehuel Lobatto (1797-1866)]

Corollary 28 *Let*

$$\lambda_1^* = \frac{1}{2R^2} \left(\int_0^{r_F} \frac{dr}{(-F(r))^{1/2}} \right)^2. \tag{93}$$

a) if $\lambda \in (0, (\frac{\pi}{2R})^2)$ there is no positive solution,

b) if $\lambda \in ((\frac{\pi}{2R})^2, \lambda_1^*)$ there is a unique positive solution u_{λ,V_0}. Moreover $\partial u_{\lambda,V_0}/\partial n(\pm R) < 0$ and $\|u_{\lambda,V_0}\|_{L^\infty(-R.R)} = \left(\frac{V_0}{\lambda}\right)^{\frac{2}{3}} \gamma^{-1}(\sqrt{\lambda}R)$,

c) if $\lambda = \lambda_1^*$ there is only one positive solution $u_{\lambda_1^*,V_0}$. Moreover $u'_{\lambda_1^*,V_0}(\pm R) = 0$

$$\|u_{\lambda_1^*,V_0}\|_{L^\infty(-R,R)} = \left(\frac{4V_0}{\lambda_1^*}\right)^{\frac{2}{3}}.$$

d) if $\lambda > \lambda_1^*$, there is a family of nonnegative solutions which are generated by extending by zero the function $u_{\lambda_1^*,V_0}$ outside $(-R,R)$ (and which we label again as $u_{\lambda_1^*,V_0}$). In particular, if $\lambda = \lambda_1^*\omega$ with $\omega > 1$ we have a family $S_1(\lambda)$ of compact support nonnegative solutions with connected support defined by

$$u_{\lambda,V_0}(x) = \frac{1}{\omega^{\frac{2}{3}}} u_{\lambda_1^*,V_0}(\sqrt{\omega}x - z)$$

where the shifting argument z is arbitrary among the points $z \in (-R,R)$ such that support of $u_{\lambda,V_0}(.) \subset (-R,R)$. Moreover, for $\lambda > \lambda_1^*$ large enough we can built, similarly, a subset of $S_j(\lambda)$ of compact support nonnegative solutions with the support formed by j components, with $j \in \{1,2,...,N\}$, for some suitable $N = N(\lambda)$ and then the set of nontrivial and nonnegative solutions of $P(\lambda)$ is formed by $S(\lambda) = \cup_{j=1}^{N} S_j(\lambda)$. In any case those solutions satisfy that

$$\|u_{\lambda,V_0}\|_{L^\infty(-R,R)} = \frac{1}{\omega^{\frac{2}{3}}} \|u_{\lambda_1^*,V_0}\|_{L^\infty(-R,R)} = \frac{1}{\omega^{\frac{2}{3}}} \left(\frac{4V_0}{\lambda_1^*}\right)^{\frac{2}{3}}, \text{ for any } \omega = \lambda/\lambda_1^* > 1$$

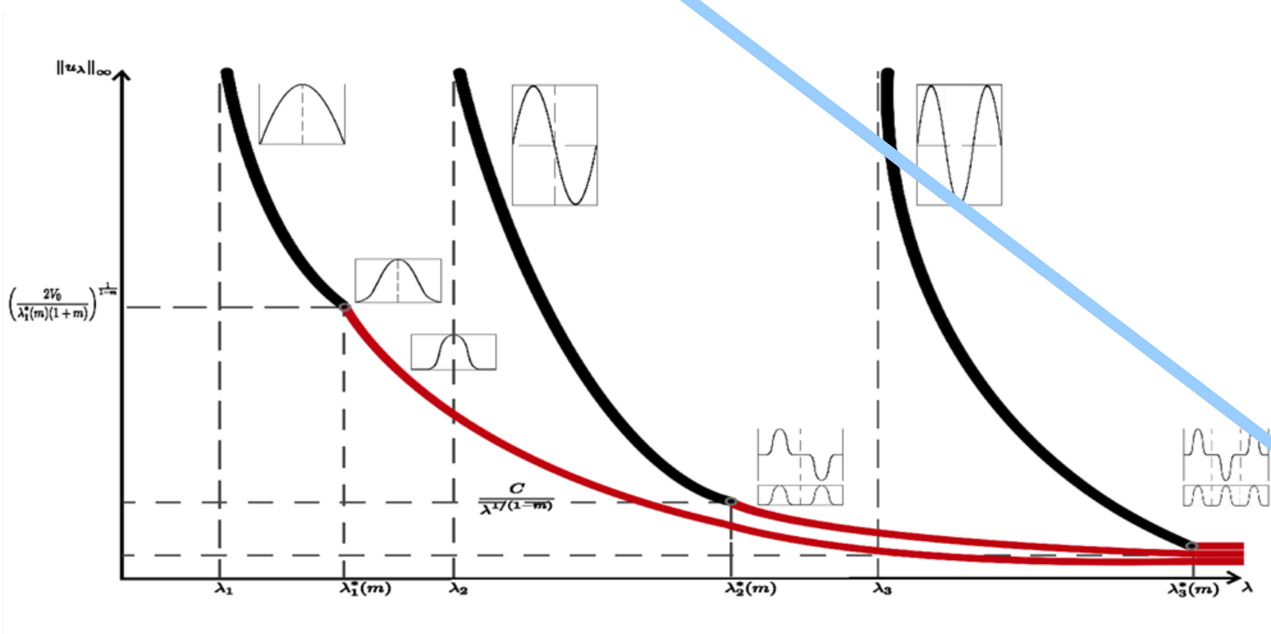
Remark 30 *Once that we know that for $\lambda > \lambda_1^*$ we have that*

$$u_{\lambda,V_0}(x) = \frac{1}{\omega^{\frac{2}{3}}} u_{\lambda_1^*,V_0}(\sqrt{\omega}x - z)$$

then we get that the bifurcating curve Λ is not C^1 at $\lambda = \lambda_1^*$ since $\Lambda'(\lambda_1^*-) = \xi < 0$ and $\Lambda'(\lambda_1^*+) = -\frac{2C}{3(\lambda_1^*)^{5/3}}$, with $C = 4^{2/3}(V_0)^{2/3}$. In addition, for $\lambda > \lambda_1^*$ we can express other norms (different than the L^∞-norm) in terms of λ. For instance we have that

$$\left\| u'_{\lambda,V_0} \right\|_{L^\infty(-R,R)} = C\lambda^{-\frac{1}{6}}$$

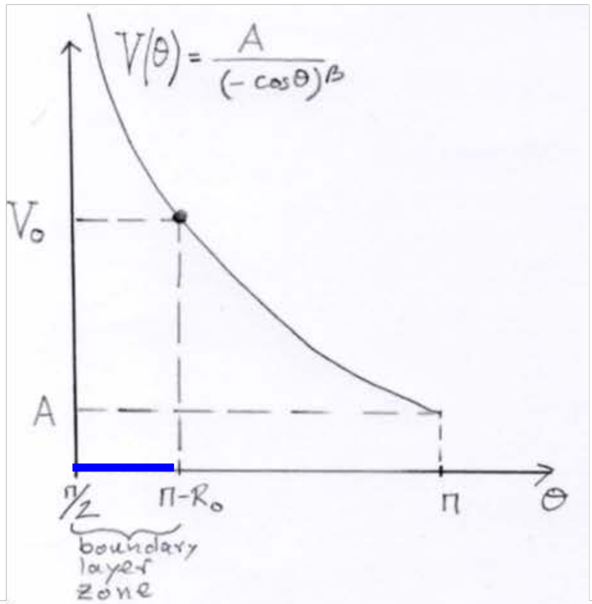
for a suitable constant $C > 0$ independent of λ. This proves that $\left\| u'_{\lambda,V_0} \right\|_{L^\infty(-R,R)} \to 0$ as $\lambda \to +\infty$. Thus, the qualitative description of the bifurcation is as indicated in the indicated. figure

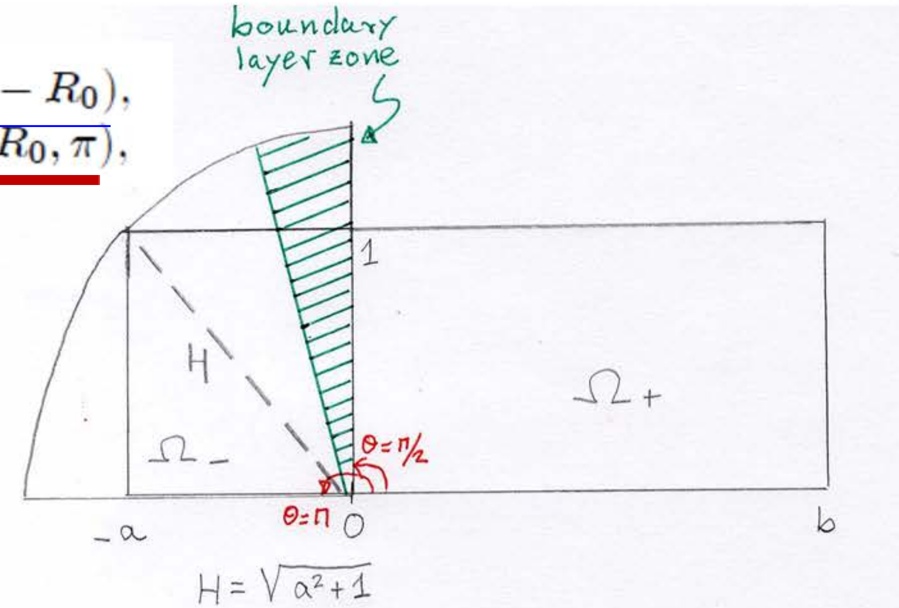


3.4. On the construction of a partially flat subsolution (II).

Notice that the potential $V(\theta)$ is singular only for $\theta = \frac{\pi}{2}$. On the other hand, we will need to match this subsolution with other function which is positive for $\theta = \frac{\pi}{2}$. Then, we will construct the subsolution in two different pieces

$$U(\theta) = \begin{cases} v_1(\theta) & \text{if } \theta \in (\frac{\pi}{2}, \pi - R_0), \\ v_2(\theta) & \text{if } \theta \in (\pi - R_0, \pi), \end{cases}$$





for some R_0 with $\pi - R_0 \in (\frac{\pi}{2}, \pi)$, and with $v_2(\theta) \in [0, 1]$ such that

$$\begin{cases} -v_2''(\theta) + \frac{V_0}{\sqrt{v_2(\theta)}} = \lambda v_2(\theta) & \theta \in (\pi - R_0, \pi), \\ v_2(\pi) = v_2'(\pi) = 0, \end{cases}$$

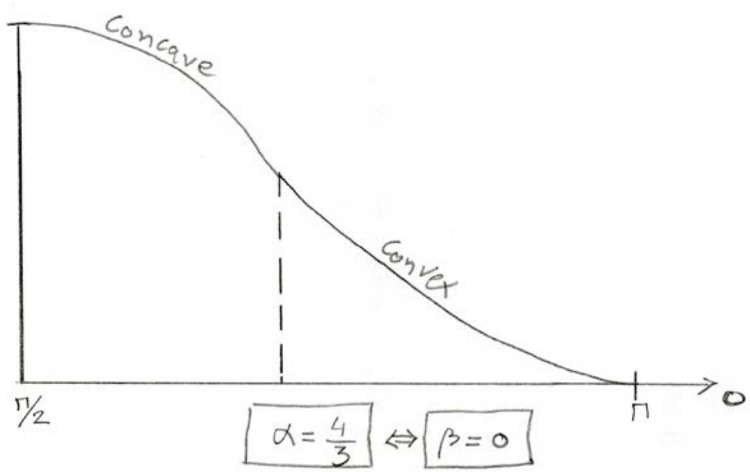
Proof of Theorem A. We recall the change of variable

$$u_{\lambda,V_0}(x) = \left(\frac{V_0}{\lambda}\right)^{\frac{2}{3}} u(\sqrt{\lambda}x),$$

which links the question of the distinguished eigenvalue $\lambda^* = (\alpha^*)^2 = \frac{1}{2R^2}\gamma(r_F)^2$ with a special length L^* for which the solution is flat on the boundary.
The computations $r_F = 4^{2/3}$ $(= 2,51)$ and $\gamma(r_F) = 2,09$ allows to see that the corresponding L^* leads to $\lambda^* = \frac{16}{9}$ and thus $\alpha^* = \frac{4}{3}$ and $R = \frac{\pi}{2}$!!!!!
This corresponds to the case $\beta = 0$ thanks to the reciprocal relation

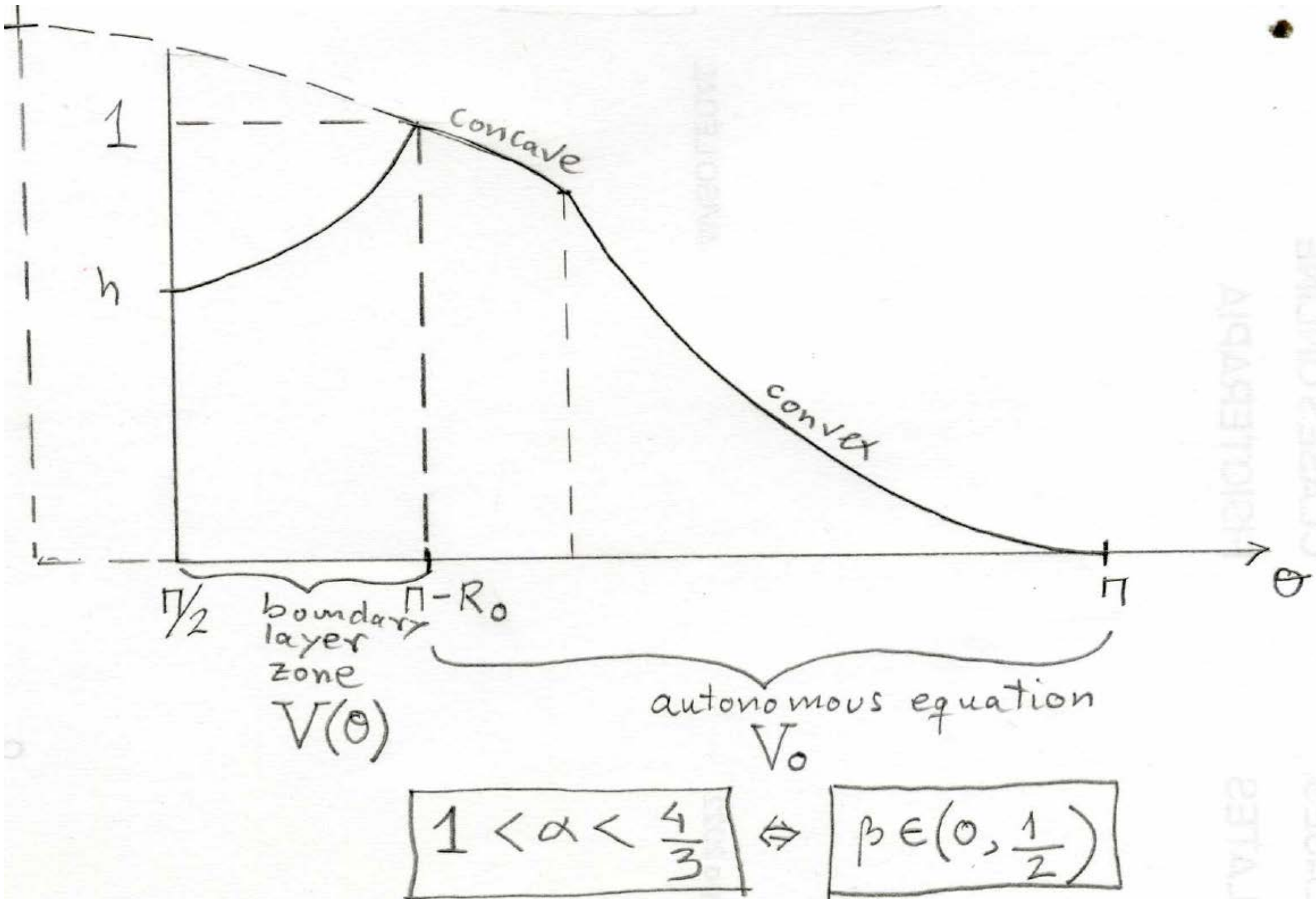
$$\beta = 2 - \frac{3\alpha}{2}.$$

Moreover we must take A small enough (in fact $A = \frac{4}{9}$) and there is a kind of **strange miracle**!!!!!



(Notice that although $\|u_{\lambda,V_0}\|_{L^\infty} > 1$ there is no difficulty with this since we can take $u_{0,-}(x,y) = \phi(r,\theta) = kr^\alpha U(\theta)$, for some $k > 0$ small enough).

If we consider the case of $\beta \in (0, \frac{1}{2})$ (i.e. $\alpha^* \in (1, \frac{4}{3})$) then the problem is not autonomous (it appears $V(\theta) = \frac{A}{(-\cos\theta)^\beta}$), the corresponding $R > 0$, given by the equation $\lambda^* = (\alpha^*)^2 = \frac{1}{2R^2}\gamma(r_F)^2$, is such that $R > \frac{\pi}{2}$ and we must truncate $V(\theta)$ on an interval $(\pi - R_0, \pi)$,for instance by taking $R_0 < R$ such that $u_{\lambda, V_0}(\pi - R_0) = 1$.



We see that the matching between $v_1(\theta)$ and $v_2(\theta)$ must take place at $\pi - R_0$.

The construction of $v_1(\theta)$, in the so called "boundary layer zone", can be carried out as in the one-dimensional problem with a distributed potential $V(\theta)$. Indeed, given $h \in (0,1)$ we take $v_1(\theta)$ as the solution of

$$\begin{cases} -v_1''(\theta) + \frac{V(\theta)}{\sqrt{v_1(\theta)}} \leq 0 & \theta \in (\frac{\pi}{2}, \pi - R_0), \\ v_1(\frac{\pi}{2}) = h, \ v_1(\pi - R_0) = 1. \end{cases}$$

Notice that using that

$$\cos\theta \leq \frac{2}{\pi}(\frac{\pi}{2} - \theta), \text{ if } \theta \in (\frac{\pi}{2}, \pi),$$

then the structural assumption on $V(\theta)$ implies that

$$V(\theta) \leq \frac{A\pi^\beta}{2^\beta(\theta - \frac{\pi}{2})^\beta} := \frac{C}{(\theta - \frac{\pi}{2})^\beta}.$$

and we can take

$$-v_1''(\theta) + \frac{\frac{C}{(\theta - \frac{\pi}{2})^\beta}}{\sqrt{v_1(\theta)}} = 0$$

with the above boundary conditions.

Thus, we can prove that

$$v_1'(\frac{\pi}{2}) \geq 0$$

and that

$$v_1'(\pi - R_0) \geq 0 \geq v_2'(\pi - R_0).$$

("Cubist description" [the coin at $\theta = \pi - R_0$ is far to be real but it is enough for our purposes).

Finally, the boundary inequality

$$u_{0,-}(x,y) = kr^{\alpha}U(\theta) \leq 1 \text{ if } y = 1$$

holds for k small enough such that

$$kH^{\alpha}U(\theta) \leq 1, \text{ where } H = \sqrt{a^2+1}.$$

and the proof is completed.

Notice that the estimate from above on Ω_+ is trivially satisfied since $u_{0,-}(x,y)$ is an harmonic function.

■

3.5. On the construction of a partially flat supersolution.

We make the structural assumption

$$j(x) = \frac{A}{(-x)^{\beta}}, \text{ for } x \in (-a,0) \text{ and } j(x) = 0 \text{ if } x \in (0,b),$$

with

$$0 \leq \beta < 1/2 \text{ and } A > 0 \text{ small enough.}$$

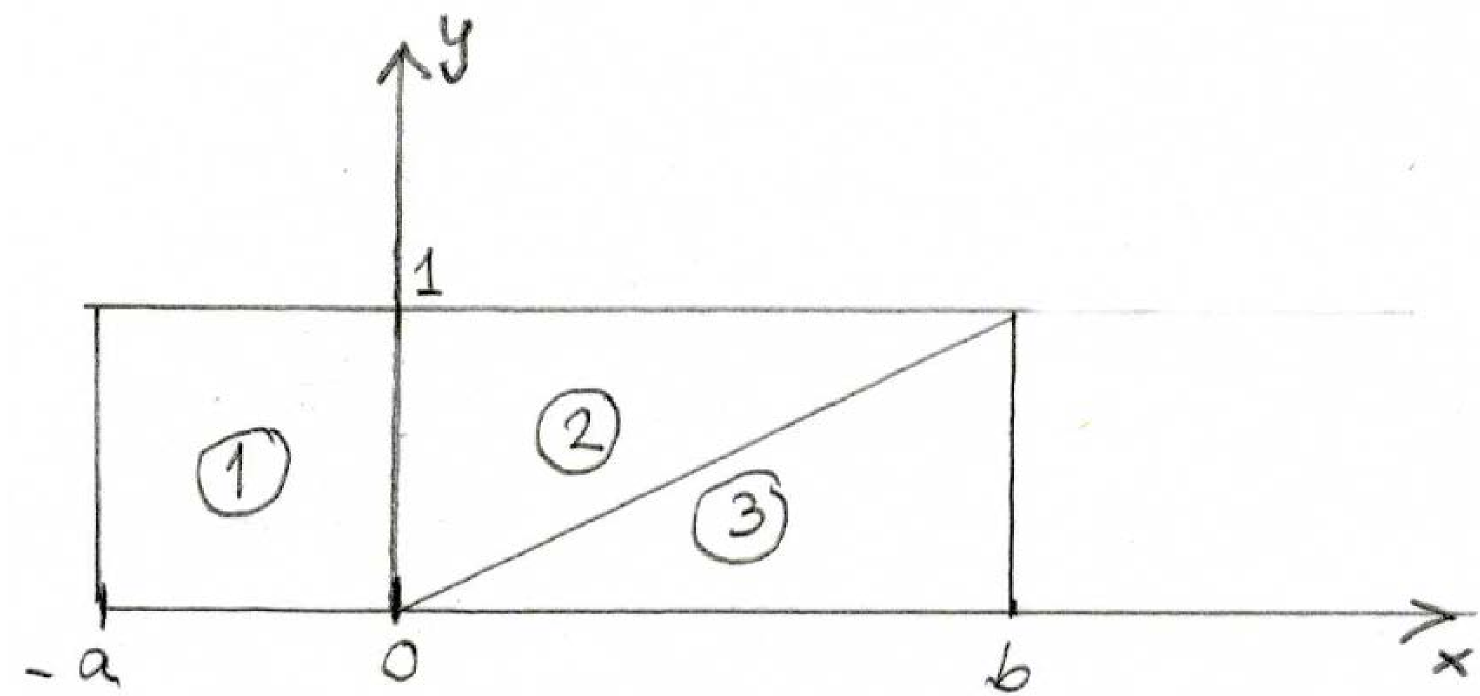
Thus, we already know that the existence of a flat subsolution implies the uniqueness of the solution.

We concentrate our attention now in finding sufficient conditions for the existence of a partially flat supersolution, i.e., being flat only on the cathode region $[-a, a] \times \{0\}$.

Since we only need the inequality $\geq$ in the equation it is enough to use that

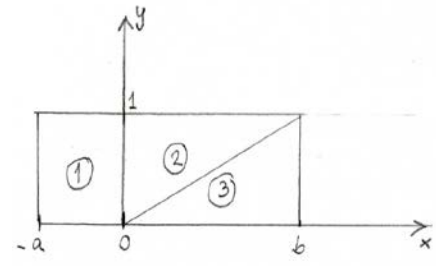
$$j(r,\theta) = \frac{A}{(-r\cos\theta)^{\beta}} \geq V_0 \text{ for } x \in (-a, 0).$$

We will search the supersolution by some matching arguments in three different regions:



In the first region Ω_- will construct the subsolution $u_-^0(x,y)$ in the same form than the subsolution, i.e.,

$$u_-^0(x,y) = \phi(r,\theta) = kr^\alpha U(\theta), \text{ for some } k>0,\ \alpha>1,$$



and thus with $U(\theta)$ solution of

$$-U''(\theta) + \frac{V_0}{\sqrt{U(\theta)}} = \lambda U(\theta) \quad \theta \in (\tfrac{\pi}{2}, \pi).$$

So that, no boundary layer need to be taken into account. This means that, in terms of the proof of Theorem A, $U(\theta) = v_2(\theta)$.

As before, if we consider the case of $\beta \in (0, \frac{1}{2}]$ (i.e. $\alpha^* \in (1, \frac{4}{3}]$) then the corresponding $R>0$, given by the equation $\lambda^* = (\alpha^*)^2 = \frac{1}{2R^2}\gamma(r_F)^2$, is such that $R \geq \frac{\pi}{2}$ ($R = \frac{\pi}{2}$ if $\alpha^* = \frac{4}{3}$) and thus

$$v_2'(\frac{\pi}{2}) \leq 0.$$

The matching with the second region will follow a different argument. In this part the supersolition must be superharmonic and it is searched in the form

$$u_{+,2}^0(x,y) = Kr^\alpha \sin(\alpha\theta),\ \theta \in (\theta_b, \frac{\pi}{2})$$

with $\tan\theta_b = b$. It is not too difficult to check that

$$-\Delta u_{+,2}^0 = K\alpha(\alpha-1)r^{\alpha-2}\sin(\alpha\theta) \geq 0.$$

The matching is now more delicate. For a H^1-matching we must have:

$$\begin{cases} u^0_-(0,y) = u^0_{+,2}(0,y) & y \in (0,1), \\ \nabla u^0_-(0,y) = \nabla u^0_{+,2}(0,y) & y \in (0,1). \end{cases}$$

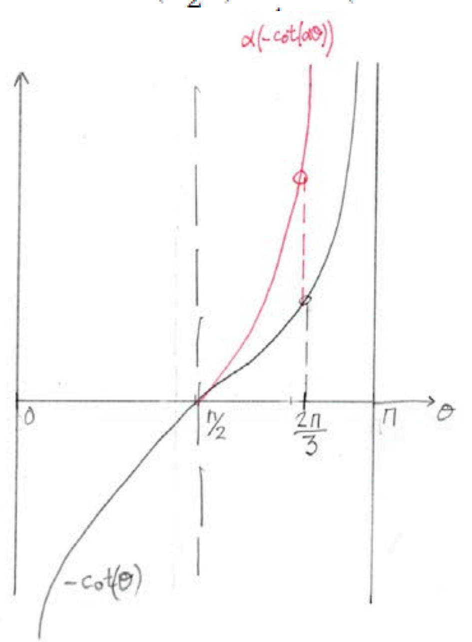
The first condition holds if

$$\frac{k}{K} = \frac{\sin(\frac{\alpha\pi}{2})}{U(\frac{\pi}{2})}.$$

The second condition (once we allow the formation of singularities with a good sign) leads to

$$U'(\frac{\pi}{2}) \geq \frac{K}{k}\alpha\cos(\frac{\alpha\pi}{2}) = \alpha U(\frac{\pi}{2})\frac{\cos(\frac{\alpha\pi}{2})}{\sin(\frac{\alpha\pi}{2})} = \alpha U(\frac{\pi}{2})\cot(\frac{\alpha\pi}{2}).$$

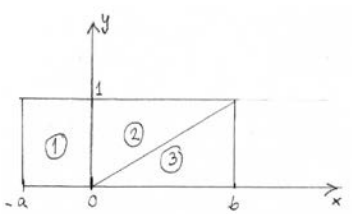
But, $\frac{\alpha\pi}{2} \in (\frac{\pi}{2}, \frac{2\pi}{3}) \subset (0,\pi)$ so that $\cot(\frac{\alpha\pi}{2}) < 0$ (see the figure).

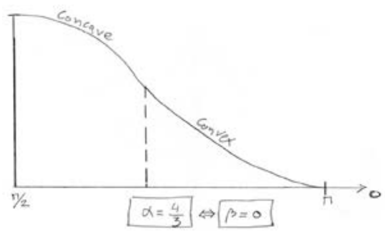


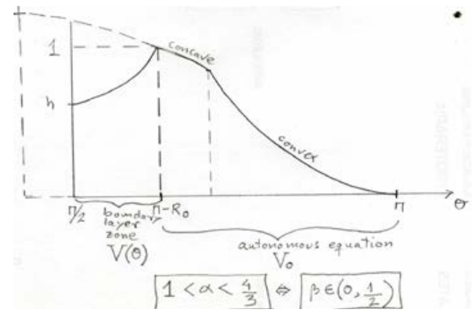
Thus we must investigate for which $\alpha \in (1, \frac{4}{3}]$ we have some kind of Robin type boundary inequality

$$\frac{-U'(\frac{\pi}{2})}{U(\frac{\pi}{2})} \leq \alpha(-\cot(\frac{\alpha\pi}{2})). \qquad \text{(Robin inequality)}$$

This condition trivially holds if $\alpha = \frac{4}{3}$ (since, in that case, $U'(\frac{\pi}{2}) = 0$). Then by continuity in α, there exists $\alpha_0 \in (1, \frac{4}{3})$ such that (Robin inequality) holds for any $\alpha \in (1, \frac{4}{3})$, which justifies our assumption in Theorem C.







No boundary layer in this case

Finally, in the third region, we prolongate the above function $u^0_{+,2}(x, y)$, by a kind of interpolation argument (remember that we must have the boundary condition $u^0_{+,3}(b, y) \geq y$, for any $y \in (0, 1)$). Then we define

$$u^0_{+,3}(x, y) = C_3 r \sin(\alpha\theta) + \widehat{C}_3 r^{\alpha} \sin(\alpha\theta_b) \quad \text{if } (x, y) \in \Omega_{+,3},$$

with $\tan\theta_b = b$. It can be proved that $u^0_{+,3}(x, y)$ is a super-harmonic function $(-\Delta u^0_{+,3} \geq 0)$ and that (if b is large enough) the positive constants C_3 and $\widehat{C}_3$ can be chosen such that $u^0_{+,3}(x, y)$ satisfies the correct inequalities on the boundaries

$$\begin{cases} u^0_{+,3}(b, y) \geq y, & \text{for any } y \in (0, 1), \\ u^0_{+,3}(x, 0) \geq 0 & \text{for any } x \in (0, b), \end{cases}$$

and it correctly matches with $u^0_{+,2}(x, y)$ (generalting, at most, a "good signed" measure) on the matching boundary $\theta = \theta_b$. ■

3.6. Idea of the proof of the uniqueness of partially flat solutions.

We start by considering the associate parabolic problem

$$PP_{a,b,j,u_0} = \begin{cases} u_t - \Delta u + \frac{j(x)}{\sqrt{u}} = 0 & t>0,\ x \in (-a,b),\ y \in (0,1), \\ u(t,x,0) = 0 & t>0,\ x \in (-a,b), \\ u(t,x,1) = 1 & t>0,\ x \in (-a,b), \\ u(t,x,-a) = y^{4/3} & t>0,\ y \in (0,1), \\ u(t,x,b) = y & t>0,\ y \in (0,1), \\ u(0,x,y) = u_0(x,y) & x \in (-a,b),\ y \in (0,1), \end{cases} \tag{105}$$

$$u_0 \in \mathcal{M}(\nu) := \left\{ u \in L^2(\Omega;\delta) \mid \text{such that } u(x,y) \geq C\delta(x,y)^{\nu} \quad \text{in } \Omega, \text{ for some } C>0 \right\},\ \nu \in \left(0, \frac{4}{3}\right].$$

The following result gives the continuous dependence of solutions with respect the initial data (implying, obviously, the uniqueness of solutions) as well as a smoothing effect with respect the initial datum.

We will use strongly some Hilbertian techniques, so we will consider initial data and we will prove that if two solutions are in the class $u, v \in \mathcal{M}(\nu)$. i.e. with $\delta^{-\nu}u, \delta^{-\nu}v \geq C$ then we can estimate the $L^2(\Omega)$-norm of $\delta^{-\gamma}[u(t)-v(t)]_+$ for suitable $\gamma \in (0,1]$ in terms of the $L^2(\Omega;\delta)$-norm of $[u_0 - v_0]_+$.

Notice that this implies, authomatically, an estimate on the $L^2(\Omega)$-norm of $[u(t)-v(t)]_+$.

Theorem 34 *Let $u_0, v_0 \in L^1(\Omega) \cap L^2(\Omega;\delta)$. Let u, v be weak solutions of PP_{a,b,j,u_0} and PP_{a,b,j,v_0}, respectively such that $u(t), v(t) \in \mathcal{M}(\nu)$ for some $\nu \in \left(0, \frac{4}{3}\right]$. Then, for any $t \in (0,\infty)$, we have*

$$\left\|\delta^{-\gamma}[u(t)-v(t)]_+\right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \left\|[u_0-v_0]_+\right\|_{L^2(\Omega;\delta)}, \tag{107}$$

with

$$\gamma := \min\left\{\frac{3\nu}{2}, 1\right\} \tag{108}$$

and for some constant $C>0$ independent of t. In particular, $u_0 \leq v_0$ implies that for any $t \in [0,+\infty)$,

$$u(t,\cdot) \leq v(t,\cdot) \quad a.e. \ in \ \Omega$$

and

$$\left\|\delta^{-\gamma}\left(u(t)-v(t)\right)\right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \left\|u_0-v_0\right\|_{L^2(\Omega;\delta)}. \tag{109}$$

As an application, we will prove the uniqueness of the positive solution for the stationary problem $P_{a,b,j}$ presented in Theorem 25.

Proof of Theorem 25. Let us call u_∞ and v_∞ two solutions of $P_{a,b,j}$ in the class $\mathcal{M}(\nu)$. By taking $u_0 = u_\infty$ and $v_0 = v_\infty$ as initial data in PP_{a,b,j,u_0} and PP_{a,b,j,v_0}, since u_∞ and v_∞ are obviously respective solutions of the mentioned parabolic problems, we get that $u_\infty - v_\infty$ satisfies

$$\left\|\delta^{-1}\left(u_\infty - v_\infty\right)_+\right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \|(u_\infty - v_\infty)_+\|_{L^2(\Omega,\delta)}.$$

Making $t \nearrow +\infty$ and reversing the role of u_∞ and v_∞, we get that $u_\infty = v_\infty$.

Idea of the Proof of Theorem 34. Without lost of generality we can use the notion of mild solution on $L^1(\Omega)$, i.e. $u \in \mathcal{C}([0,T]; L^1(\Omega))$, for any $T > 0$: $j(x)u^{-\beta} \in L^1(\Omega \times (0,T))$ and u fulfills the identity

$$u(\cdot,t) = S(t)u_0(\cdot) - \int_0^t S(t-s)(\chi_{\{u>0\}}u^{-\beta}(\cdot,s) - \lambda u^p)ds, \quad \text{in } L^1(\Omega), \tag{110}$$

where $S(t)$ is the $L^1(\Omega)$-semigroup corresponding to the Laplace operator with the correponding Dirichlet (stationary) boundary conditions.
We shall need need some well-known auxiliary results. The firs one is a singular version of the Gronwall's inequality which is specially useful in the study of non-globally Lipschitz perturbations of the heat equation:

* H. Brézis and T. Cazenave, A nonlinear heat equation with singular initial data, *J. Anal. Math.*, **68**, 277–304, 1996.

Lemma 35 *Let $T > 0$, $A \geq 0$, $0 \leq a, b \leq 1$ and let f be a non-negative function with $f \in \mathrm{L}^p(0,T)$ for some $p > 1$ such that $p'.\max\{a,b\} < 1$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Consider a non-negative function $\varphi \in L^\infty(0,T)$ such that, for almost every $t \in (0,T)$,*

$$\varphi(t) \leq At^{-a} + \int_0^t (t-s)^{-b} f(s)\varphi(s)\, ds. \tag{111}$$

Then, there exists $C > 0$ only depending on T, a, b, p and $\|f\|_{L^p(0,T)}$ such that, for almost every $t \in (0,T)$,

$$\varphi(t) \leq ACt^{-a}. \tag{112}$$

We shall also use some regularizing effects properties satisfied by the semigroup $S(t)$ of the heat equation with zero Dirichlet boundary conditions.

Lemma 36

1. *There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^2(\Omega)$,*

$$\|\nabla S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{1}{2}}\|u_0\|_{L^2(\Omega)}. \tag{113}$$

2. *There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^1(\Omega)$,*

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{N}{4}}\|u_0\|_{L^1(\Omega)}. \tag{114}$$

3. *There exists $C > 0$ such that, for any $t > 0$, any $m \in (0, 1]$ and any $u_0 \in L^2(\Omega; \delta^{2m})$,*

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{m}{2}}\|u_0\|_{L^2(\Omega,\delta^{2m})}. \tag{115}$$

4. *There exists $C > 0$ such that, for any $t > 0$, any $p \in [1, +\infty)$ and any $u_0 \in L^p(\Omega, \delta)$,*

$$\|S(t)u_0\|_{L^p(\Omega)} \leq Ct^{-\frac{1}{2p}}\|u_0\|_{L^p(\Omega,\delta)}. \tag{116}$$

Proof. Properties 1 and 2 are classical (see e.g. L. Véron, Effets régularisants de semi-groupes non-lineaires dans des espaces de Banach, *Ann. Fac. Sci. Toulouse I*, 171-200, 1979).
Property 3 was established in
* J. Dávila and M. Montenegro, Existence and asymptotic behavior for a singular parabolic equation, *Transactions of the AMS*, **357**, 1801–1828, 2004.
They used the function $v_0 := u_0\delta^m$, *i.e.*

$$\|S(t)\delta^{-m}v_0\|_{L^2(\Omega)} \leq Ct^{-\frac{m}{2}}\|v_0\|_{L^2(\Omega)}.$$

Property 4 was proved in
* Ph. Souplet, Optimal regularity conditions for elliptic problems via L^p_δ-spaces, *Duke Math. J.*, **127** (2005), 175–192.

Proof of Theorem 34 (continuation). By the constant variations formula, we know that for any $t \in [0, T]$,

$$u(t) - v(t) = S(t)(u_0 - v_0) + \int_0^t S(t-s)\left(h(u(s)) - h(v(s))\right) ds \quad \text{in } \Omega, \tag{117}$$

where $h(x, u) := j(x)u^{-1/2}$. By the convexity of the function $u \mapsto u^{-1/2}$ and the assumption that $u(t), v(t) \in \mathcal{M}(\nu)$, we deduce that

$$h(x, u) - h(x, v) \leq C j(x) \delta^{-3\nu/2}(u - v)_+ \quad \text{in } \Omega. \tag{118}$$

Thus, if we denote $w := u - v$, we get for any $\tau, t \in [0, T]$ with $\tau \leq t$

$$w_+(t) \leq S(t-\tau)w_+(\tau) + C \int_\tau^t S(t-s)(j(x)\delta^{-3\nu/2}w_+(s))\, ds. \tag{119}$$

We multiply (119) by the weight $\delta^{-\gamma}$, with $\gamma \in [0, 1]$ to be chosen later, and take the L^2-norms. Then,

$$\|\delta^{-\gamma}w_+(t)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma}S(t-\tau)w_+(\tau)\|_{L^2(\Omega)} + C\int_\tau^t \|S(t-s)j(x)\delta^{-[(\beta+1)\nu+\gamma]}w_+(s)\|_{L^2(\Omega)}\, ds.$$

Let us fix $s, t > 0$ and let us call $\psi := S(t-s)j(x)\delta^{-(\beta+1)\nu}w_+(s)$. Then, by Hölder inequality,

$$\|\delta^{-\gamma}\psi\|^2_{L^2(\Omega)} = \int_\Omega \frac{\psi^2}{\delta^{2\gamma}}\, dx \leq \left(\int_\Omega \frac{\psi^2}{\delta^2}\, dx\right)^\gamma \left(\int_\Omega \psi^2\, dx\right)^{1-\gamma}$$

(note that the limit cases $\gamma \equiv 0$ and $\gamma \equiv 1$ are allowed). Then, applying Hardy inequality,

$$\|\delta^{-\gamma}\psi\|_{L^2(\Omega)} \leq C\|\nabla\psi\|^\gamma_{L^2(\Omega)}\|\psi\|^{1-\gamma}_{L^2(\Omega)}.$$

By property 1 of Lemma 36, to $\frac{t-s}{2}$, we get

$$\|\delta^{-[(\beta+1)\nu+\gamma]}S(t-s)w_+(s)\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{\gamma}{2}}\|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)}. \tag{120}$$

Analogously, using property 4 of Lemma 36, and that $j \in L^2(\Omega)$ (remember the assumptions made in Theorem 23)

$$\|\delta^{-\gamma}S(t)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\frac{\gamma}{2}}\|S\left(\frac{t}{2}\right)j(x)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\left(\frac{\gamma}{2}+\frac{1}{4}\right)}\|w_+(0)\|_{L^2(\Omega;\delta)}. \tag{121}$$

In order to apply the singular Gronwall's inequality, we must relate the weights $\delta^{-\gamma}$ and $\delta^{-3\nu/2}$ keeping in mind that $\gamma \in [0,1]$. To do that, we apply property 3 of Lemma 36 for some $m \in [0,1]$. We shall take

$$3\nu/2 = \gamma + m. \tag{122}$$

Indeed, if $(\beta+1)\nu \in (1,2]$, then we take $\gamma = 1$, $m = 3\nu/2 - 1$ and we apply point 3 of Lemma 36 to the initial datum:

$$\|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(m+1)}w_+(s)\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{m}{2}}\|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}. \tag{123}$$

On the other hand, if $3\nu/2 \in [0,1]$, we can take $\gamma = 3\nu/2$ and thus, since $S(t-s)$ is a contraction in $L^2(\Omega)$, we get

$$\|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t-s}{2}\right)j(x)\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}, \tag{124}$$

which corresponds to (122) with $m = 0$. In other words,

$$\gamma = \min\{1, 3\nu/2\}$$

and

$$m = \max\{3\nu/2 - 1, 0\}.$$

Collecting the previous inequalities, we arrive to

$$\|\delta^{-\gamma}w_+(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}}\|w_+(0)\|_{L^2(\Omega;\delta)} + C\int_0^t (t-s)^{-\frac{m}{2}}\|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}.$$

Thus, we can apply Lemma 35 with $a = \frac{2\gamma+1}{4} \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $b = \frac{m}{2}$ and $A = C\|w_+(0)\|_{L^2(\Omega;\delta)}$ to deduce that

$$\|\delta^{-\gamma}w_+(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}}\|w_+(0)\|_{L^2(\Omega;\delta)}.$$

Thanks for your attention