On a Mathematical Model Arising in MHD Perturbed Equilibrium for Stellarator Devices: A numerical Approach

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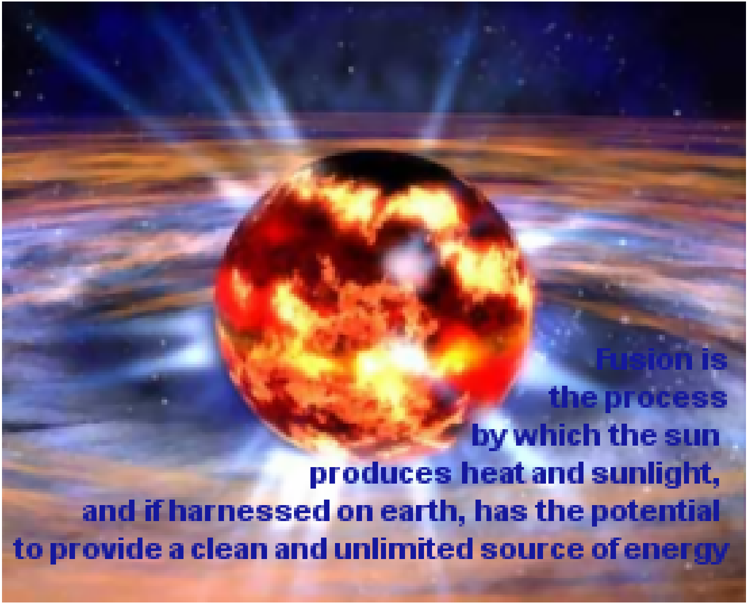
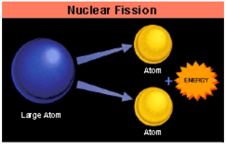
The 2012 International Conference on High Performance Computing and Simulation (HPCS 2012)

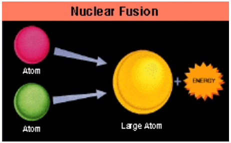
International Workshop on Fusion Distributed Applications (WFDA 2012)

Modeling ... what is the Nuclear fusion?



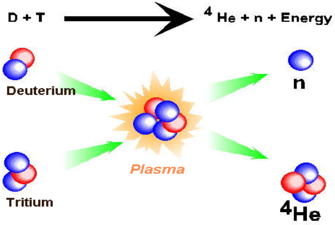
- The nuclear fusion:





- The plasma: A mixture of particles of positive, negative and neuter electrical charge can be consider as an ideal fluid for determining the macroscopic properties.

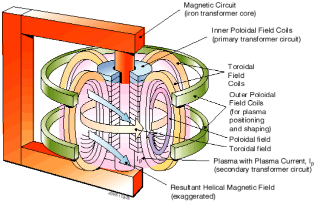
Particles of low mass: Deuterium, Tritium, He,...



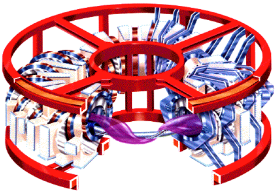
- Magnetic confinement: One need $> 100 * 10^6 C^o$ to obtain an equilibrium state.

Axisymmetric geometry: **Tokamak devices**

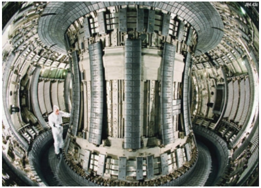
Non axisymmetric geometry: **Stellarator devices**



Sketch of a Tokamak



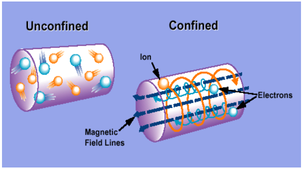
Sketch of TJ-II in the Ciemat-Madrid

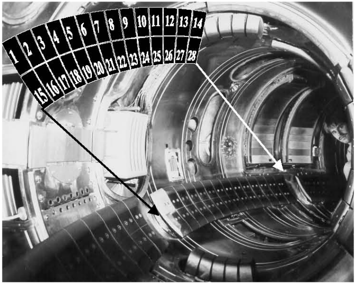


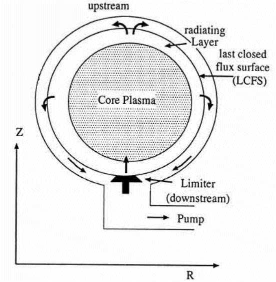
The vacuum vessel

The vacuum vessel

- Difficulties: to determine the conditions on the magnetic field and on the current density in order **to keep the plasma far from the camera walls.**







A way to prevent mechanically this is to introduce a *limiter:* a solid object which determines the boundary of the plasma (limiter plays the role of a *thin obstacle* for the plasma).

Modelling... the 3D stationary model

The plasma as a ideal fluid and use the ideal MHD model.

- Assume that the plasma is a perfect conductor (Ohm's Law).

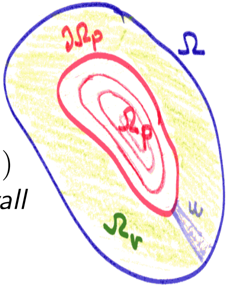
$$
\begin{aligned}
\nabla \cdot \mathbf{B} &= 0, \quad (\textit{Conservation of } \mathbf{B}),\\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J}, \quad (\textit{Ampère's Law}),\\
\nabla P &= \mathbf{J} \times \mathbf{B} \ \text{ in } \Omega_p, \quad (\textit{conservation of momentum})
\end{aligned}
$$

The electromagnetic variables are:	The fluid variables are:
• the magnetic field **B** and • the current density **J**	• the pressure P. • magnetic permeability μ_0.

are satisfied in plasma region.

- Sketch:
- $\mathbb{R}^3 \supset \Omega = \Omega_p \cup \Omega_v$

$$\begin{cases} \Omega_p & := \text{plasma region } (\textit{unknown}) \\ \partial\Omega_p & := \textbf{the free boundary} \\ \Omega_v = \{x : \mathbf{J}(x) = 0\} & := \text{vacuum region} \\ \omega & := \text{the limiter} \end{cases}$$



- **Boundary Conditions**:

$$\mathbf{n}^3 \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega_p = \{x : P(x) = 0\}$$
$$(\Longleftarrow \nabla P || \mathbf{n}^3 \ \text{ and } \ \nabla P(x) \perp \mathbf{B})$$
$$\mathbf{n}^3 \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega. \quad \textit{perfectly conducting wall}$$

- **One Integral Condition**:

  "*the current carrying*" into the plasma.

- **The problem is to find**

$$\boxed{P : \Omega \subset \mathbb{R}^3 \to \mathbb{R}, \ \ \mathbf{B}, \mathbf{J} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3?}$$

As the magnetic field lines are in toroidal nested surfaces, it is useful to *introduce a new coordinates system:*

- Axisymmetric geometry (*Tokamak* devices):

Cylindrical coordinates system (r, φ, z): Let be ψ the magnetic surface, then

$$\underbrace{\begin{array}{c}\mathbf{B}\cdot\nabla\psi = 0\\ \text{(MHD)}\left\{\begin{array}{rl}\nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{B} &= \mu_0\mathbf{J}\\ \nabla P &= \mathbf{J}\times\mathbf{B}\ \ \textbf{in}\ \Omega_p\ \text{(plasma region)}\end{array}\right.\\ \mathbf{B} = \left(B_r, B_\varphi, B_z\right)\ \text{(covariant coordinates)}\end{array}}$$

$$\Downarrow$$

$$-\left[\frac{\partial}{\partial r}\left(\frac{1}{\mu_0 r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu_0}\frac{\partial\psi}{\partial z}\right)\right] = \frac{1}{2\mu_0 r^2}\frac{\partial F^2(\psi)}{\partial\psi} + r\frac{\partial p(\psi)}{\partial\psi}$$

Grad-Shafranov equation

Operator: $-\mathcal{L}\psi := -\mu_0 r \left[\frac{\partial}{\partial r}\left(\frac{1}{\mu_0 r}\frac{\partial \cdot}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu_0}\frac{\partial \cdot}{\partial z}\right)\right] \psi,$

Grad-Shafranov equation

$$\boxed{-\mathcal{L}\psi = \frac{1}{2}\left(F^2(\psi)\right)' + \mu_0 r^2 p'(\psi)} \qquad (\star)$$

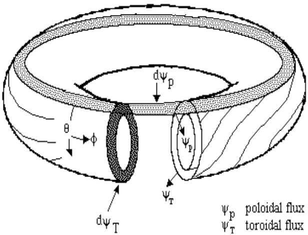
(see Grad–Shafranov equation for Stellarator case [??])

- $\psi := \psi(r,z)$ (is a potential flux and *unknown* function),
- $rB_z := -\frac{\partial \psi}{\partial r}(r,z)$, $rB_r := \frac{\partial \psi}{\partial z}$, $rB_\varphi := F(\psi)$ (F is *unknown*)
- $P := p(\psi)$ the pressure. In the plasma region $p(p) \geq 0$ and in the vacuum region $p(\psi) \leq 0$. (p is a *prescribed* function: $p(\psi) \sim \frac{\lambda}{2}\psi_+^2$).
- **Boundary Conditions**: $\psi = \gamma$ on $\partial\Omega$, and γ is a negative constant.
- **One Integral Conditions**: The known *total current carrying* $I_p$into the plasma:

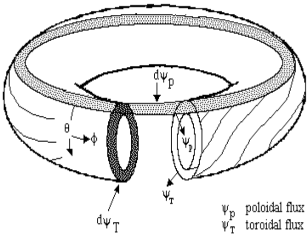
$$\boxed{\int_\Omega \left\{ \frac{1}{2\mu_0 r^2}\left(F^2(\psi)\right)' + p'(\psi)\right\} rdrdz = I_p}$$

- Non axisymmetric geometry (*Stellarator* devices):

Boozer vacuum coordinates system (ρ, θ, ϕ) [Boozer82].
The magnetic field lines becomes "*straights*" in the (θ, ϕ)-plane:



- $\rho = \rho(x, y, z) > 0$ and $\rho = 0$ on the magnetic axis
  ρ is constant on each nested toroid.
- $\theta = \theta(x, y, z)$ is the poloidal angle, is constant on any toroidal circuit but changes by 2π over a poloidal circuit
- $\phi = \phi(x, y, z)$ is the toroidal angle, is constant on any poloidal circuit.



In this *Boozer vacuum coordinates system*, for a vacuum configuration (i.e. without any plasma) the magnetic field $\mathbf{B}_v$ may be written in **contravariant form** as

$$\boxed{\mathbf{B}_v = B_0 \rho \nabla\rho \times \nabla(\theta - t_v(\rho)\phi)}$$

where $t_v(\rho)$ is the so called **vacuum rotational transform** and B_0 is a positive constant.

The **covariant form** of $\mathbf{B}_v$ is

$$\boxed{\mathbf{B}_v = F_v \nabla\phi}$$

where F_v is a *constant* (which customary is taken as positive).

Pass form a 3D to 2D problem: *averaging methods* were used [GreeneJohnson84], [HenderCarreras84].

$$\boxed{f = \langle f \rangle + \underbrace{\tilde{f}}_{\text{rapidly varying part}}}$$

where $\langle f \rangle (\rho, \theta) := \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f d\phi$ is the toroidal averaged .

$$\frac{B^i}{D} = \left\langle \frac{B^i}{D} \right\rangle + \left(\frac{\tilde{B}^i}{D} \right)$$

where B^i are the contravariant components of the **vacuum magnetic field**, $i = \rho, \theta, \phi$, and

$$D = (\nabla\rho \times \rho\nabla\theta) \cdot \nabla\phi \text{ (\textit{Jacobian} of change of coordinates system).}$$

Using a suitable assumption (the Stellarator expansion hypothesis), the conservation of B,

$$\frac{\partial}{\partial\rho}\left(\rho\left\langle\frac{B^{\rho}}{D}\right\rangle\right)+\frac{\partial}{\partial\theta}\left(\left\langle\frac{B^{\theta}}{D}\right\rangle\right)=0,$$

$\Longrightarrow \exists$ averaged poloidal flux function $\psi = \psi(\rho,\theta)$defined by

$$\boxed{\left\langle\frac{B^{\rho}}{D}\right\rangle=\frac{1}{\rho}\frac{\partial\psi}{\partial\theta}\quad\text{and}\quad\left\langle\frac{B^{\theta}}{D}\right\rangle=-\frac{\partial\psi}{\partial\rho}\ .}$$

Also, $\langle B_{\phi}\rangle$ and $\langle p\rangle$ are a functions **only depending** of ψ:

$$\boxed{F(\psi):=\langle B_{\phi}\rangle\quad\text{and}\quad p(\psi):=\langle P\rangle\ \simeq\frac{\lambda}{2}\psi_{+}^{2}\ \ \text{(constitutive law).}}$$

As in [HenderCarreras84] we obtained a Grad–Shafranov type equation for ψ

$$\boxed{-\mathcal{L}\psi = a(\rho,\theta)F(\psi) + F(\psi)F'(\psi) + b(\rho,\theta)p'(\psi)} \tag{1}$$

$$\mathcal{L}\psi := \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}\left(a_{\rho\rho}\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial\rho}\left(a_{\rho\theta}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(a_{\theta\rho}\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(a_{\theta\theta}\frac{\partial\psi}{\partial\theta}\right)\right\}$$

The coefficients:

$$a_{\rho\rho}(\rho,\theta) := \rho \left\langle g^{\rho\rho} \right\rangle (\rho,\theta),\, a_{\theta\theta}(\rho,\theta) := \frac{1}{\rho} \left\langle g^{\theta\theta} \right\rangle (\rho,\theta),$$
$$a_{\theta\rho}(\rho,\theta) := \left\langle g^{\rho\theta} \right\rangle (\rho,\theta) =: a_{\rho\theta}(\rho,\theta)$$

and where $\left\langle g^{i,j} \right\rangle$, $i,j = \rho,\theta$ are the *averaged components of the Riemannian metric associated to the vacuum coordinates system* (all those coefficients are 2π–periodic functions in θ).

$$a(\rho,\theta) := \frac{B_0}{\rho F_v}\left[\frac{\partial}{\partial\rho}(\rho^2 t(\rho)\left\langle g^{\rho\rho}\right\rangle) + \frac{\partial}{\partial\theta}(\rho t(\rho)\left\langle g^{\rho\theta}\right\rangle)\right] \qquad (2)$$

and

$$b(\rho,\theta) := \frac{F_v}{B_0}\left\langle \frac{1}{D} \right\rangle (\rho,\theta). \qquad (3)$$

We remark that $b > 0$ and that usually function a has not any singularity.

- At any place, $\boxed{-\mathcal{L}\psi = \mathbf{J}_T}$:= **toroidal component of current density**, ($\mathcal{L}$ is elliptic [Padial92]).
- **In the vacuum vessel**: $\boxed{-\mathcal{L}\psi = a(\rho,\theta)F_v \quad \text{in } \Omega_v}$
- **In the plasma region,** the following

  Grad–Shafranov equation holds:

$$\boxed{-\mathcal{L}\psi = a(\rho,\theta)F(\psi) + \frac{1}{2}\left(F^2(\psi)\right)' + b(\rho,\theta)p'(\psi) \quad \text{in } \Omega_p} \quad (\bigstar)$$

(see Grad–Shafranov equation for Tokamak case [**??**])

- **In the plasma region,** the following **Grad–Shafranov equation** holds:

and **the problem is to find** ψ and F, such that

$$(P)\begin{cases} -\mathcal{L}\psi = a(\rho,\theta)F(\psi)+\frac{1}{2}\left(F^2(\psi)\right)'+b(\rho,\theta)p'(\psi) \text{ in } \Omega \\ \quad +\text{Boundary Condition} \quad + \quad \text{One Integral Condition} \end{cases}$$

- **Boundary condition**: $\partial\Omega^3$ is assumed to be a *perfectly conducting wall*$\Rightarrow$ $\boxed{\psi = \gamma \equiv \text{constant} < 0 \quad \text{on } \partial\Omega}$
- **One Integral Condition,** "*The current carrying*" into the plasma: for any $s \in [essinf\ \psi, esssup\ \psi]$

$$\int_{\{\psi>s\}} \left[\tfrac{1}{2}\left(F^2(\psi)\right)'+bp'(\psi)\right]\rho d\rho d\theta = j(s_+, \|\psi_+\|_{L^\infty(\Omega)}).$$

"We will replace the $\mathcal{L}$ operator by the *Laplacian* one, Δ."

In this work, we will consider $\boxed{p(\psi) := \frac{\lambda}{2}\psi_+^2}$ (constitutive law).and the ideal Stellarator condition $\Longrightarrow$ $\boxed{j \equiv 0}$.

On the existence and regularity of solution of problem (P)

Given: $\Omega \subset \mathbb{R}^2$ bounded and regular set,
$F_v \in \mathbb{R}$, $F_v > 0$,
$\lambda \in \mathbb{R}$, $\lambda > 0$,
$\gamma \in \mathbb{R}$, $\gamma < 0$,
$a,\ b \in L^\infty(\Omega)$, $b > 0$ a.e. in Ω, $a \not\equiv 0$.

To find: (u, F) $u : \Omega \to \mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ such that
$F(s) = F_v$ for any $s \leq 0$ and satisfying

$$
(\mathcal{P}_I) \begin{cases} -\Delta u = aF(u(x)) + \frac{1}{2}\left(F^2(u(x))\right)' + \lambda b u_+(x) \text{ in } \Omega, \\ u - \gamma \in H_0^1(\Omega), \\ \displaystyle\int_{\{x\in\Omega: u(x)>s\}} \tfrac{1}{2}\left(F^2(u(x))\right)' + \lambda b u_+ dx = 0 \\ \text{for any } s \in \left[ess \inf_\Omega u, ess \sup_\Omega u\right] \end{cases} \tag{4}
$$

Existence of solutions.

Theorem (Diaz, Padial,Rakotoson 1998)

Suppose that $\gamma \leq 0$. *Then there exist* $\Lambda_1, \Lambda_2 > 0$ *such that if*

$$\boxed{\lambda \|b\|_{L^\infty(\Omega)} < \Lambda_1 \quad \text{and} \quad \Lambda_2 < \inf_\Omega |a| F_v},$$

there exist a couple (u, F) *with*

$$u \in V(\Omega) := \left\{ v \in H^1(\Omega) : \Delta v \in L^\infty(\Omega), v_{|\partial\Omega} \leq 0 \right\},$$

$$F \in W^{1,\infty}(]\inf_\Omega u, \sup_\Omega u[), \qquad F(s) = F_v, \ \forall s \leq 0$$

solution of (P). *Moreover,* u *satisfies that*

$$meas\{x \in \Omega : \nabla u(x) = 0\} = 0 \quad \textit{i.e. } u \textbf{ has not flat region}$$

and F *is entirely determined by* u.

Remark

For the numerical simulation, from the hypothesis of the existence result,we will consider a "relative size"on the parameters λ *and* F_v*:*

$$\textit{fixed } b, \textit{ then } \lambda \textit{ small enough } \Longrightarrow \lambda \|b\|_{L^\infty(\Omega)} < \Lambda_1,$$

$$\textit{fixed } a, \textit{ then } F_v \textit{ large enough } \Longrightarrow \Lambda_2 < \inf_\Omega |a| F_v,$$

for a suitable Λ_1 *and* Λ_2

Remark (Díaz-Padial-Rakotoson, 1998)

Derivating the integral condition with respect to s, we can obtain explicitly the unknown function F in terms of the one dimensional decreasing rearrangement of the unknown function u (we will denote by u_*) and the relative rearrangement of the function b with respect of the unknown u (we will denote by b_{*u}), that is

$$\frac{d}{ds} \int_{\{x\in\Omega: u(x)>s\}} \frac{1}{2}\left(F^2\left(u\left(x\right)\right)\right)' + \lambda b u_+ dx = 0$$

$$\Downarrow$$

$$F\left(t\right) \equiv \boxed{F_u(t) = \left[F_v^2 - \lambda \int_0^{t_+} \sigma b_{*u}\left(|u > \sigma|\right) d\sigma\right]_+^{\frac{1}{2}}}$$

and

$$F\left(u\left(x\right)\right) \equiv \boxed{\mathcal{F}_u(x) = \left[F_v^2 - \lambda \int_0^{u_+(x)} \sigma b_{*u}\left(|u > \sigma|\right) d\sigma\right]_+^{\frac{1}{2}}}$$

Remark (...)

Thus, we can rewrite the original problem (recalling $(\mathcal{P}_I)$(4)) with two unknown u, and F as a new **nonlocal problem** $(\mathcal{P}_*)$ *with only one unknownu:*

$$(\mathcal{P}_*)\begin{cases} -\Delta u = a\mathcal{F}_u(x) + \lambda\left[b(x) - b_{*u}(|u > u(x)|)\right] \text{ in } \Omega, \\ u - \gamma \in H_0^1(\Omega), \end{cases} \tag{5}$$

Notation:
$|u > u(x)| = \mathrm{meas}\{y \in \Omega : u(y) > u(x)\} = \int_{\{y \in \Omega : u(y) > u(x)\}} dy.$

One dimensional rearrangement

Definition

Let $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ be a measurable function and let $\Omega_* :=]0, |\Omega|[$. The **Decreasing Rearrangement** of u is the following decreasing real function $u_* : \Omega_* \to \mathbb{R}$:

$$m_u(\sigma) := \text{meas}\{x \in \Omega : u(x) > \sigma\} = |u > \sigma| \text{ (distribution function of } u)$$

$$u_*(s) := \inf\{t \in \mathbb{R} : m_u(\sigma) \leq s\} \text{ (decreasing rearrangement of } u)$$

$$u_*(0) := \operatorname*{esssup}_{\Omega} u := \|u_+\|_{L^\infty(\Omega)} = u_{+*}(0),$$

$$u_*(|\Omega|) := \operatorname*{essinf}_{\Omega} u, \quad \hat{m} := \operatorname*{essinf}_{\Omega} u, \; M := \operatorname*{esssup}_{\Omega} u \; .$$

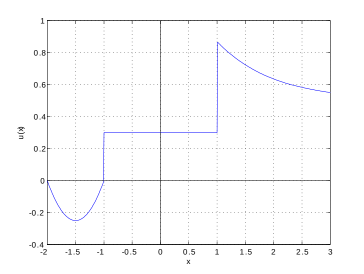
$m_u(\sigma) := \text{meas}\{x \in \Omega : u(x) > \sigma\} = |u > \sigma|$ (*distribution function* of u)

$u_*(s) := \inf\{t \in \mathbb{R} : m_u(\sigma) \leq s\}$ (*decreasing rearrangement* of u)

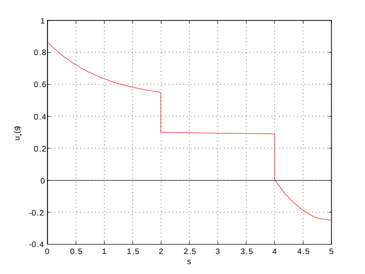
Example

Let be $u : \Omega = (-2, 5) \to R$, such that

$$u(x) = \begin{cases} (x+2)(x+1), & x < -1 \\ 0.3 & -1 \leq x \leq 1 \\ \frac{1}{2} + e^{-x} & 1 < x \end{cases}, \quad u_* : \Omega_* = (0, |\Omega|) \to [\text{ess}\inf_{\Omega} u, \text{ess}\sup_{\Omega} u]$$



The function u



The rearrangement u_*

Figure: *P. Galán*

Relative rearrangement

Definition

Let $b \in L^1(\Omega)$ and a measurable function u in Ω, we set
$w : \bar{\Omega}_* =]0, |\Omega|[\to \mathbb{R}$

$$w(s) = \int_{\{x:u>u_*(s)\}} b(x)\,dx + \int_0^{s-|u>u_*(s)|} \left(b|_{\{u=u_*(s)\}}\right)_*(t)\,dt \ , \quad \text{for } s \in \Omega_*.$$

The **Relative Rearrangement** of b with respect to u is

$$\boxed{b_{*u}(s) := \frac{dw(s)}{ds} = \lim_{\sigma \to 0} \frac{(u+\sigma b)_*(s) - u_*(s)}{\sigma} \qquad \text{in } \Omega_* \ .}$$

Remark: If u has not flat region then $s - |u > u_*(s)| = 0$ and

$$\boxed{b_{*u}(s) := \frac{d}{ds}\int_{\{x:u>u_*(s)\}} b(x)\,dx}.$$

Numerical Approximation

- We compute the numerical solution of $(\mathcal{P}_*)$ [5] using the *finite element method* combined with *a fixed point algorithm*.
- Let D_h be a partition of Ω such that $D_h = \{Q_i\}_{i=1}^{N_e} \subset \bar{\Omega}$, where Q_i are rectangles and N_e the number of finite elements in the partition. Then the finite element subspaces V_h is defined as

$$V_h = \left\{ v_h \in C\left(\bar{\Omega}\right) : v_{h|Q_i} \in P_1\left(Q_i\right) \ \forall i = 1, 2, \ldots, N_e \right\}$$

and $V_{h0} = V_h \cap H_0^1(\Omega)$, where $P_1(Q_i)$ is the set of polynomials $\sum_j c_j p_j(x)\, q_j(y)$ where p_j and q_j are polynomials of degree ≤ 1.

- Let $u_h^{(0)} \in V_h$ such that $u_h^{(0)} - \gamma \in V_{h0}$, $\Longrightarrow$
the discretized problem consists in solving, for any $k = 0, 1, 2, \ldots;$

$$\text{find } u_h^{(k+1)} \in V_h \text{ such that } u_h^{(k+1)} - \gamma \in V_{h0}$$

and

$$\left(\nabla u_h^{(k+1)}, \nabla v_h\right) = \left(g_h^{(k)}, v_h\right) \text{ for all } v_h \in V_{h0}, \tag{6}$$

where $g_h^{(k)} \in V_h$ is an approximation of the function

$$aF\left(u_h^{(k)}\right) + \frac{1}{2}\left(F^2\right)'\left(u_h^{(k)}\right) + \lambda b\left(u_h^{(k)}\right)_+ .$$

Note that when solving problem (6), we first need to compute the function $g_h^{(k)}$ in the right hand side

Computing the function g

The used algorithm is the following:

1. Start with a given function $u_h^{(0)} \in V_h$ (without flat region).
2. Step k. Given $u_h^{(k)}$ by (6)

   Then, $u_h^{(k)}$ has not flat region.

   a) *Obtain an approximation of the distribution function* $m_{u_h^{(k)}}$. Let $T = \{t_0 = \max_\Omega u_h^{(k)} > t_1 > ... > t_{hz} = 0\}$ a mesh of interval $[0, \max_\Omega(u)]$.
   We sort the array of mapping of $u_h^{(k)}$ on x in the mesh of $\bar{\Omega}$
   $m_{u_h^{(k)}}(t_i) = |u_h^{(k)}(x) > t_i| \approx \sum_{\{x : u_h^{(k)}(x) > t_i\}} weight(x)$
   (where the weighted function is taking accordingly either $x \in \partial\bar{\Omega}$ or $x \in \mathring{\Omega}$).

b) *Obtain the decreasing rearrangement* $\left(u_h^{(k)}\right)_*$. Since $u_h^{(k)}$ has not flat region, $\left(u_h^{(k)}\right)_*(\cdot) = m^{-1}_{u_h^{(k)}}(\cdot)$.

3. *Obtain the relative rearrangement* $b_{\left(u_h^{(k)}\right)_*}$.

Since $u_h^{(k)}$ has not flat region, $\Longrightarrow$compute $b_{\left(u_h^{(k)}\right)_*}$ by discrete integration and differentiation.

1^ointegrating b over $\{x \in \Omega : u_h^{(k)}(x) > \left(u_h^{(k)}\right)_*(s)\}$ for all $s \in \Omega_*$

2^odifferentiating with respect to s, with $\left(u_h^{(k)}\right)_*(s) \in T$.

Compute $F := F_{u_h^{(k)}}$. By trapezoidal integration role for any $t \in T$.

Obtain $g_h^{(k)}$. We derivate $\frac{1}{2}F^2_{u_h^{(k)}}$ in T, $\Longrightarrow$ by lineal interpolation of $\left(\frac{1}{2}F^2_{u_h^{(k)}}\right)'$ and $F_{u_h^{(k)}} \Longrightarrow u_h^{(k)}(x)$ in the mesh of $\bar{\Omega} \Longrightarrow$compute $g_h^{(k)}$.*Solve discretized problem* (6) by Conjugate Gradient.

4) Stopping criterion.

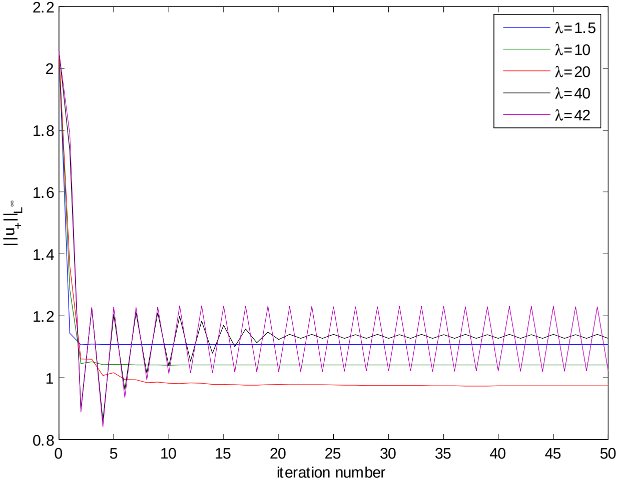
- The authors have implemented *their own code* using the *C language* to solve the PDE by using the **finite element method**, as well as for the determination of distribution function of averaged poloidal flux.
- The partition D_h used by the finite element method consists in a regular rectangular mesh with $h = \frac{1}{32}$, i.e., 4096 elements and 4225 nodes.
- The associated linear system is solved using Conjugate Gradient. The CPU time consumed to compute the full algorithm is *less than one second* on an Intel Core i7 at 2.67 Ghz processor.
- The test problems: $\Omega = [-1,1] \times [-1,1]$, $\gamma = -1.5$, $F_v = 10$, $B_0 = 1$, $\lambda = 1.5, \lambda = 10, \lambda = 20, \lambda = 40, \lambda = 42$,

$$a(x,y) = \frac{B_0}{F_v} \frac{5(\sin \pi x \cdot \sin \pi y + 2)}{\sqrt{x^2 + y^2 + 1}}$$

$$b(x,y) = \frac{F_v}{B_0} \frac{1}{(x^2 + y^2 + 1)(\cos(\arctan(y) + 2))}.$$

$\lambda = 1.5 \Longrightarrow$ converging (very fast)
$\lambda = 10 \Longrightarrow$ converging (fast)
$\lambda = 20$

$\lambda = 40 \Longrightarrow$ divergent (oscillating)
$\lambda = 42 \Longrightarrow$ divergent (oscillating)



$\lambda = 1.5 \Longrightarrow$ converging (very fast)

$\lambda = 10 \Longrightarrow$ converging (fast)

$\lambda = 20 \Longrightarrow$ converging (slowly)

$\lambda = 40 \Longrightarrow$ divergent (oscillating)

Future work

The methodology concerning the numerical results of this work can be applied in many different contexts:

- Action of a *limiter.*
- Current carrying Stellarators models
- Evolution problem, where even time implicit schemes could be considered due to the fast convergence of the algorithm of the stationary model.
- Nonlocal formulations arising in Tokamaks

That enhancements on the code that could require a *more powerful computing platform and the possibilities of distributing the execution of the problem in parallel or distributed tasks* could be designed in the light of similar works in the literature on parallel computing for nonlinear elliptic partial differential equations.

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