On very weak solutions of higher order equations

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UCM, Dedicated to Monique Madaune-Tort in her 60th birthday.

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1. Introduction

- The notion of weak solution of a boundary value problem, on a bounded domain Ω, is associated to functions in some energy space satisfying the equation in a weak form, after multiplying by any test function in such energy space and integrating by parts. Nevertheless, in many relevant cases in the applications the right hand side datum is merely in $L^1_{Loc}(\Omega)$ and a different notion of solution is required. For instance, in the case of second order problems the notion of very weak solution is reduced to functions in $L^1(\Omega)$ satisfying the equation passing the second order derivatives to the test functions.

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- The notion of weak solution of a boundary value problem, on a bounded domain Ω, is associated to functions in some energy space satisfying the equation in a weak form, after multiplying by any test function in such energy space and integrating by parts. Nevertheless, in many relevant cases in the applications the right hand side datum is merely in $L^1_{Loc}(\Omega)$ and a different notion of solution is required. For instance, in the case of second order problems the notion of very weak solution is reduced to functions in $L^1(\Omega)$ satisfying the equation passing the second order derivatives to the test functions.
- Most of the theory on very weak solutions available in the literature deals with second order equations. Recently, sharper results have been obtained, to this case, when the data are merely in $L^1(\Omega, \delta)$, with $\delta = dist\,(x, \partial\Omega))$. That was originally proved by Haim Brezis, at the seventies, in a famous unpublished manuscript *concerning Dirichlet boundary conditions* (see also his 1996 paper with Cazenave, Martel, and Ramiandrisoa).

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- The main goal of this lecture is to present some new results proving that in the case of **higher order** equations **and Dirichlet boundary conditions** the class of $L^1_{Loc}(\Omega)$ data for which the existence and uniqueness of a very weak solution can be obtained is larger than $L^1(\Omega, \delta)$ (the optimal class for the case of second order equations). For instance, for some stationary onedimensional semilinear 4th-order equations we shall prove that the optimal class of data is the space $L^1(\Omega, \delta^2)$. Moreover we shall analyze the optimal solvability also for the case of **other boundary conditions**: something which, as far as we know, was not considered before in the literature.

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- In some sense, the obtained results give an answer to the question about of the greatest weight profile which can support a simple beam such that its two extremes are horizontally supported (for instance to a wall) and do not experience any deflection.

- To fix ideas I will present the results for the relevant model of the Euler-Bernoulli ($\sim$1740) beam model (i. e. a fourth order onedimensional spatial operator) but most of the results remain valid for equations of order $2m$, $m \in \mathbb{N}$. In a first part we shall consider the stationary case:

$$(SP)\begin{cases} \frac{d^4u}{dx^4} = f(x) & x \in \Omega = (0, L), \\ + \text{ boundary conditions } (BC). \end{cases}$$

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- Here we are assuming $I_z E = 1$. We shall consider here only the most classical type of boundary conditions. (BC) corresponds to **two** set of **two** identities (two at $x = 0$ and another two at $x = L$) among the following possibilities

$$\begin{cases} a_0 u(0) = 0, & b_0 u(L) = 0, \\ a_1 u'(0) = 0 & b_1 u'(L) = 0, \\ a_2 u''(0) = 0, & b_2 u''(L) = 0, \\ a_3 u'''(0) = 0, & b_3 u'''(L) = 0. \end{cases}$$

- Here the coefficients are taken such that $a_i, b_i \in \{0,1\}$ and $\sum a_i = 2, \sum b_i = 2$ in order to have a simple way to state general results. Dirichlet conditions (clamped beam) corresponds to

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- Finally, a very often situation corresponds to a cantilever bar ($x = 0$ clamped and $x = L$ free)

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- In a second part I will consider the Euler-Bernouilli transient hyperbolic problem (with a possible damping term)

$$(HP)\begin{cases} \rho\frac{\partial^2 u}{\partial t^2}+\mu\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial x^4}=f(t,x) & t\in(0,T), x\in(0,L), \\ +\text{ boundary conditions,} & t\in(0,T), \\ u(0,x)=u_0(x) & x\in(0,L), \\ u_t(0,x)=v_0(x) & x\in(0,L), \end{cases}$$

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- as well as the so called (Duvaut and Lions 1972) "quasi-static" associated problem (now of a parabolic type) which gives the dynamics decay when λ is large enough

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- We shall see that the optimal weight $w(x)$ in order to solve the above problems is

$$\delta_{\mathbf{ab}}(x) = \max\{\min(a_0, a_1)d(x,0)^2, \min(a_0, a_2)d(x,0), a_3\} \max\{\min(b_0,$$

Notice that, for instance for the Dirichlet problem $[\mathbf{a} = (1,1,0,0), \mathbf{b} = (1,1,0,0)]$, we must take $\delta_{\mathbf{ab}}(x) \sim \delta^2(x)$ with $\delta = dist\,(x, \partial\Omega)$.

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- *A possible* plan for the rest of the lecture:
- 2. Necessary and Sufficient conditions for the existence of solutions for the stationary problem.
- 3. Perturbation results for the stationary operator in $L^1(0, L : \delta_{\mathbf{ab}})$.The semigroup approach for the parabolic problem in $L^1(0, L : \delta_{\mathbf{ab}})$ and remarks on the hyperbolic problem in $L^2(0, L : \delta_{\mathbf{ab}})$.

2. Necessary and Sufficient conditions for the existence of solutions for the stationary problem.

To fix ideas I will consider now the case of Dirichlet boundary conditions.

- **Definition**. Given $f \in L^1_{Loc}(0, L)$ a function $u \in L^1_{Loc}(0, L)$ is a solution of (SP) in $D'(0, L)$ if

$$\left\langle u, \frac{d^4\zeta}{dx^4} \right\rangle_{D'D} = \langle f, \zeta \rangle_{D'D}$$

for any $\zeta \in D(0, L) = C_c^\infty(0, L)$.

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- We introduce now the space associated to the boundary (BC) as

$$V = \overline{\{\zeta \in C^4([0,L]) \colon \zeta \text{ satisfies } (BC)\}}^{W^{4,\infty}(0,L)}.$$

For instance, for the case of Dirichlet boundary conditions $V = W^{4,\infty}(0,L) \cap W_0^{2,\infty}(0,L)$.

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- **Definition**. Given $f \in L^1(0, L : \delta_{\mathbf{ab}})$ a function $u \in L^1(0, L)$ is a "very weak solution" of (SP) and (BC) if for any $\zeta \in V$

$$\int_0^L u(x) \frac{d^4\zeta}{dx^4}(x)dx = \int_0^L f(x)\zeta(x)dx.$$

- **Remark 1**. It is not difficult to show that $\zeta \in V$ implies that $|\zeta(x)| \leq c\delta_{\mathbf{ab}}$ for any $x \in (0, L)$ and so the above identity is well justified.

(Sufficiency) Assume that $a_2 a_3 = 0$ if $b_2 = b_3 = 1$ (respectively, $b_2 b_3 = 0$ if $a_2 = a_3 = 1$). Then, *for any* $f \in L^1(0, L : \delta_{\mathbf{ab}})$ *there exists a unique very weak solution of* (SP) *and* (BC). *Moreover we have the estimate (weak maximum principle)*

$$24L^4 \|u_+\|_{L^1(0,L)} \leq \|f_+\|_{L^1(0,L:\delta_{\mathbf{ab}})}, \tag{1}$$

where, in general, $h_+ = \max(0, h)$. *Moreover* $u \in C^3([0, L])$.
(Strong maximum principle) Let $f \in L^1(0, L : \delta_{\mathbf{ab}})$ *with* $f \geq 0$ *a.e.* $x \in (0, L)$. *Then the very weak solution satisfies)*

$$u(x) \geq C \|f\|_{L^1(0,L:\delta_{\mathbf{ab}})} \delta_{\mathbf{ab}}(x) > 0 \text{ for any } x \in (0, L),$$

for some $C > 0$.

- **Remark 1**. It is not difficult to show that $\zeta \in V$ implies that $|\zeta(x)| \leq c\delta_{\mathbf{ab}}$ for any $x \in (0, L)$ and so the above identity is well justified.
- Main result: **Theorem 1.**

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for some $C > 0$.

(Necessity) Assume that $f \in L^1_{Loc}(0, L)$, *such that* $f \geq 0$ *a.e.* $x \in (0, L)$. *Then if*

$$\int_0^L f(x)\delta_{\mathbf{ab}}(x)dx = +\infty$$

then it can not exists any $u \in C^3([0, L])$ *satisfying* (BC) *being also solution in* $D'(0, L)$ *of* (SP).

Remarks.

1. It is possible to give a physical meaning to the solvability (necessary and sufficient) assumption $f \in L^1(0, L : \delta_{\mathbf{ab}})$. For instance, for the Dirichlet case it means that the momentum function of the shear stress at any interior point x with respect the two extremes must be an integrable function.
2. Theorem 1 extends many previous works in the literature: Aftabizadeh (1986), Gupta (1988), Agarwal (1989), O'Regan (1991), Bernis (1996), Pao (1999), Yao (2008)...
3. The weak maximum principle was first proved in *Chow-Dunninger-Lasota (1973)* but under a non-quantitative version. Estimate (1) is new in the literature.

4 The strong maximum principle extends to fourth order the previous results by Morel and Ostwald (1985) and Brezis-Cabré (1998) It shows the complete blow up (in the whole interval $(0, L)$) when $f \notin L^1(0, L : \delta_{\mathbf{ab}})$ and we truncate it generating $f_n(x) = \min(f(x), n)$. Indeed, now the problem can be solved for f_n since $L^\infty(0, L) \subset L^1(0, L : \delta_{\mathbf{ab}})$ but $u_n(x) \geq C \left\| f_n \right\|_{L^1(0,L:\delta_{\mathbf{ab}})} \delta_{\mathbf{ab}}(x)$ implies that $u_n(x) \nearrow +\infty$ for any $x \in (0, L)$.

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5 It seems possible to extend the above result to the case of several dimensions BUT ON BALLS AND UNDER SYMMETRY CONDITIONS ON f. The maximum principle is false on some ellipsoidal domains (conjecture by Hadamard 1908: proofs by Duffin (1949), Garabedian (1951), ...)

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6 The existence result holds also in the more general class of Radon measures $f \in M(0, L : \delta_{\mathbf{ab}})$: something very useful to justify the engineers study in with the weight on the beam is concentrated in isolated points. Notice that although the usual Radon measure space (without wieight) $M(0, L)$ is a subset of the dual space $H^{-2}(0, L)$ it is not always true (see Brezis-Browder (1982)) that the duality $\langle f, \zeta \rangle_{H^{-2}(0,L),H_0^2(0,L)}$ coincides with the

$\langle f, \zeta \rangle_{M(0,L),C^0([0,L])} = \int_0^L \zeta(x) df$ duality.

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- The existence part can be obtained by using the associated Green function associated to the boundary conditions since formula

$$u(x) = \int_0^L G(x,y) f(y) dy \tag{2}$$

is well justified: it is not difficult to show that $|G(.,y)| \leq C\delta_{\mathbf{ab}}(y)$ (use, for instance, Stakgold (1998)). The regularity and, specially, the L^1-estimate (weak maximum principle) is much more delicate and require several ingredients. The first one is the following "conservation formula":

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- **Lemma 1**. *Let $f \in L^1(0, L : \delta^2)$ and let u be any very weak solution of (SP) and Dirichlet BC. Then*

$$24L^4 \int_0^L u(x) dx = \int_0^L x^2 (L-x)^2 f(x) dx.$$

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- **Lemma 2** *(Chow-Dunninger-Lasota (1973)). Let $f \in L^1_{loc}(0, L)$ such that $f \geq 0$ on $(0, L)$. Let $u \in C^3([0, L])$ solution of (SP) and (BC). Then $u(x) \geq 0$ for any $x \in (0, L)$.*

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- The new key ingredient is an abstract result due to M.G. Crandall and L. Tartar (1980) applied until now only for hyperbolic conservation laws and Hamilton-Jacobi equations.

- **Lemma 3 (**Crandall-Tartar (1980)). *Let* X, Y *two vector lattices and* λ_X, λ_Y *be nonnegative linear functionals on* X *and* Y *respectively. Let* $C \subseteq X$ *and* $f, g \in C$ *imply* $f \vee g \in C$*. Let* $T : C \rightarrow Y$ *satisfy*

$$\lambda_X(f) = \lambda_Y(T(f)) \text{ for } f \in C. \tag{3}$$

Then $(a) \Rightarrow (b) \Rightarrow (c)$ *where* $(a), (b), (c)$ *are the properties:*
(a) $f, g \in C$ *and* $f \leq g$ *imply* $T(f) \leq T(g)$*,*
(b) $\lambda_Y((T(f) - T(g))_+) \leq \lambda_X((f - g)_+)$ *for* $f, g \in C$*,*
(c) $\lambda_Y(|T(f) - T(g)|) \leq \lambda_X(|f - g|)$.
Moreover, if $\lambda_Y(F) > 0$ *for any* $F > 0$*, then* $(a), (b), (c)$ *are equivalent.*

- Application to prove the L^1-estimate:
$C = X = L^1(0, L : \widehat{\delta}^2)$, $Y = L^1(0, L)$,
$\lambda_X(f) = \int_0^L x^2(L-x)^2 f(x)dx$, $\lambda_Y(F) = \int_0^L F(x)dx$.

$$T(f) = 24L^4 u \quad (u \text{ very weak solution of } (SP) \text{ and } (DBC)).$$

Then the identity (3) coincides with Lemma 1 and property (a) is given by Lemma 2. So we get (b) which is the wanted L^1-estimate.

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$\lambda_X(f) = \int_0^L x^2(L-x)^2 f(x)dx$, $\lambda_Y(F) = \int_0^L F(x)dx$.

$$T(f) = 24L^4 u \quad (u \text{ very weak solution of } (SP) \text{ and } (DBC)).$$

Then the identity (3) coincides with Lemma 1 and property (a) is given by Lemma 2. So we get (b) which is the wanted L^1-estimate.
- The proof of the strong maximum principle uses the Green expression (2) and the estimate that $|G(.,y)| \leq C\delta_{\mathbf{ab}}(y)$.

3. Perturbation results for the stationary operator in $L^1(\Omega, \delta_{\mathbf{ab}})$. The semigroup approach for the parabolic problem in $L^1(\Omega, \delta_{\mathbf{ab}})$ and in $L^2(\Omega, \delta_{\mathbf{ab}})$ for the hyperbolic problem.

Many extensions of the above theorem are possible. For instance, the nonlinear problem

$$(NLSP) \begin{cases} \frac{d^4u}{dx^4} + \beta(u) = f(x) & x \in \Omega = (0, L), \\ + \text{ boundary conditions } (BC), \end{cases}$$

arises in many different frameworks: the linear case $\beta(u) = ku$ corresponds to the so called elastic beam (Boggio (1908), Hadamard (1908),.... Monotone non decreasing functions $\beta(u)$ were used in McKenna-Walter (1987), Lazer and McKenna (1990),..., in the modeling of suspension bridges.

Remark 10. A quite curious fact (Dunninger (1981) Swers-Kawhol (2005): the strong maximum principle for the linear equation

$$\frac{d^4u}{dx^4} + ku = f(x)$$

and boundary conditions $a_0 = b_0 = a_2 = b_2 = 1$ is only true for $k \in (-k_0, k_1)$, for some $k_0, k_1 > 0$ depending on L. This also holds for the

Nevertheless, in terms of abstract operators on the Banach space $L^1(\Omega, \delta_{\mathbf{ab}})$ we have

Theorem 2. *Let* $A : D(A) \to L^1(0, L : \delta_{\mathbf{ab}})$ *the operator given by*

$$\begin{cases} D(A) = \{u \in L^1(0, L : \delta_{\mathbf{ab}}) \cap C^3([0, L]) : \frac{d^4 u}{dx^4} \in L^1(0, L : \delta_{\mathbf{ab}}) \text{ and } u \text{ satisf} \\ Au = \frac{d^4 u}{dx^4} \text{ if } u \in D(A). \end{cases}$$

Then,

i) $\exists\ C > 0$ such that

$$C \|u\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq \|Au\|_{L^1(0,L:\delta_{\mathbf{ab}})} \text{ for all } u \in D(A),$$

ii) $\exists\ \omega \in \mathbb{R}$ such that $A + \omega I$ is a m-accretive operator in $L^1(0, L : \delta_{\mathbf{ab}})$, *i.e.* and for any $\lambda > 0$ and $f \in L^1(0, L : \delta_{\mathbf{ab}})$

$$\left\|(I + \lambda(A + \omega I))^{-1} f\right\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq \|f\|_{L^1(0,L:\delta_{\mathbf{ab}})}.$$

In consequence we have

Corollary 1. *For any β maximal monotone graph of R^2 there exists a unique very weak solution of the equation*

$$\frac{d^4u}{dx^4} + \omega u + \beta(u) = f(x)$$

satisfying (BC). *Moreover*

$$\left\| \left[(f - Au + \omega u) - (\widehat{f} - A\widehat{u} + \omega \widehat{u}) \right] \right\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq \left\| \left[f - \widehat{f} \right] \right\|_{L^1(0,L:\delta_{\mathbf{ab}})}.$$

Corollary 2. *Given* $u_0 \in L^1(0, L : \delta_{\mathbf{ab}})$ *and* $f \in L^1(0, T : L^1(0, L : \delta_{\mathbf{ab}}))$ *there exists a unique mild solution of the parabolic problem*

$$(HP) \begin{cases} \mu \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = f(t, x) & t \in (0, T), x \in (0, L), \\ + \text{ boundary conditions,} & t \in (0, T), \\ u(0, x) = u_0(x) & x \in (0, L). \end{cases}$$

Moreover we have the following continuous dependence inequality

$$\|u(t)\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq e^{t\omega}(\|u_0\|_{L^1(0,L:\delta_{\mathbf{ab}})} + \int_0^t \|f(s)\|_{L^1(0,L:\delta_{\mathbf{ab}})}\, ds).$$

- **Remark.** This results improves many results in the literature Damlamian (1978), Friedman-Oswald (1988), Root (1991), Cholewa (1992), Bernis (1994, 1995,...), Cui (1996), Davis (2001), Galaktionov-Pohozaev (2002), Gazzola-Grunau (2009),... The comparison of solutions was shown here by first time in the literature.

Theorem 3 (eventual positivity) *Let* $f \in L^1(0, T : L^1(0, L : \delta_{\mathbf{ab}}))$ *be such that* $f(t, x) \to f_\infty(x)$ *in* $L^1(0, L : \delta_{\mathbf{ab}})$ *as* $t \to +\infty$*, with* $f_\infty(x) > 0$*. Then, for any* $u_0 \in L^1(0, L : \delta_{\mathbf{ab}})$ *there exist a time* $T_0 \geq 0$ *(depending on* u_0*) such that* $u(t, x) > 0$ *for any* $t \geq T_0$ *and for any* $x \in (0, L)$.

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- In general the operator A is not T-accretive (see Remark 10), nevertheless it is possible to show the following result:

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- Finally, concerning the hyperbolic problem, the semigroup approach only works efficiently on Hilbert spaces (counterexample by W. Littman (1963)).
- **Theorem 4**. *Let $u_0 \in H_0^2(0, L : \delta^2)$, $v_0 \in L^2(0, L : \delta^2)$ and $f \in L^2(0, T : H^{-2}(0, L : \delta_{\mathbf{ab}}))$. Then there exists a unique weak solution $u \in C([0, T] : H_0^2(0, L : \delta^2))$, $u_t \in C([0, T] : L^2(0, L : \delta^2))$, $u_{tt} \in L^2(0, T : H^{-2}(0, L : \delta_{\mathbf{ab}}))$ of the problem*

The proof results of the application of the results by Brezis (1971, 1973) and Haraux (1978) to the vectorial operator built through the maximal monotone operator $A + \omega I$ now on the Hilbert space $H = L^2(0, L : \delta^2)$.

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THANKS FOR YOUR ATTENTION