On Elliptic and Parabolic Monge-Ampère Equations Giving Rise to a Free Boundary

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1. Introduction

It is well-known that the Monge-Ampère operator has many applications in Geometry and related areas:

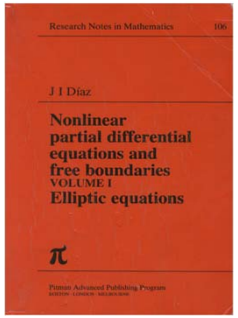
L. Nirenberg: Monge-Ampère equations and some associated problems in Geometry, in *Proceedings of the International Congress of Mathematics*, Vancouver 1974, …

Today we also know that the applications arises in many other areas: optimal transportation, optimal design of antenna arrays, vision, statistical mechanics, front formation in meteorology, financial mathematics,…

C. Budd and V. Galaktionov,: On self-similar blow-up in evolution equations of Monge-Ampère type, *IMA J. Appl. Math.*, (2011), …

In this lecture we shall focus our attention in the **occurrence of a free boundary** (separating the region where the solution u is locally a hyperplane, and so were the Hessian D^2u is vanishing, from the rest of the domain).

G. Díaz and J.I. Díaz, Remarks on the Monge-Ampère equation: some free boundary problems in Geometry, In *Contribuciones matemáticas en homenaje a Juan Tarrés*, M. Castrillón et al. eds., Univ. Complutense de Madrid, Madrid, 2012, 97-126.



Remark 2.25 of the 1985 monograph, …

Cortona 1992 Autumn Course (Talenti, Oliker, Serrin, Kawhol,…)

Personal communications to Nirenberg and Caffarelli (Santander Course 1993)

Now (27 years later !!) we decided to formulate the free boundary parabolic and elliptic problems in connection to the **shape of worn stones** model (W. J. Fiery (1974), R. Hamilton (1993), D. Chopp, L.C. Evans, H. Ishii (1999),…)

$$\frac{\partial \mathbf{P}}{\partial t} = K^{p}\,\mathbf{n}$$

the points **P** of the N-dimensional convex hyper-surface $\Sigma^N(t)$, embedded in R^{N+1} (in the physical case N = 3), under Gauss curvature flow K, with exponent p > 0, moves in the inward direction **n** to the surface with velocity equal to the p power of its Gaussian curvature).

In the special case of $\Sigma^N(t)$=graph of $x_{N+1} = u(x; t)$, x in a convex open set Ω of R^N, the function u satisfies the parabolic Monge-Ampère equation

$$u_t = \frac{\left(\det D^2 u\right)^p}{\left(1+|Du|^2\right)^{\frac{(N+2)p-1}{2}}}.$$

Related problems in 3d-images: V. Caselles and C. Sbert (1996), …

More in general

Evolution Problem: given a bounded open set Ω of R^N, a continuous function φ on $\partial\Omega$, a locally convex function u_0, p > 0, and a continuous function $g \in C([0;+\infty))$ such that

$$g(s) \geq 1 \text{ for any } s \geq 0,$$

find a convex function satisfying,

(PP)

$$\begin{cases} u_t = \dfrac{\left(\det D^2 u\right)^p}{g(|Du|)} & \text{in } \Omega \times \mathbb{R}_+, \\ u(x,t) = \varphi(x), & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x). & x \in \Omega. \end{cases}$$

To simplify the exposition $\varphi(x)$ time-independent.

Formulation of the associate elliptic problem. Difficult (to us) to foresee on purely geometric grounds (Hamilton, …), level set approach (Ishi-Souganidis, …) so that: ***here, the parabolic problem regarded as a Cauchy Problem, semigroup theory,…***

$$\begin{cases} u_t + \mathcal{A}u = 0, & t > 0, \\ u(0) = u_0, \end{cases} \qquad \mathbb{X} = C(\overline{\Omega}) \qquad \mathcal{A}u = -\frac{\left(\det D^2 u\right)^p}{g(|Du|)},$$

$$\frac{u_n - u_{n-1}}{\varepsilon} + Au_n = 0 \quad \text{for } n \in \mathbb{N}, \qquad \det D^2 u_n = \left(g(|Du_n|) \frac{u_n - u_{n-1}}{\varepsilon} \right)^{\frac{1}{p}} \quad \text{in } \Omega.$$

Elliptic problem: under the above assumption on **Ω**, φ, p and g, find a convex function u satisfying,

(EP)
$$\begin{cases} \det D^2 u = g(|Du|) \left[(u-h)^{\frac{1}{p}} \right]_+ & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where h(x) is a given continuous function on $\overline{\Omega}$.

For the evolution problem replace $g(|Du|)$ by $\left(g(|Du|)\right)^{\frac{1}{p}}$.

The operator is degenerate elliptic on the symmetric definite nonnegative matrices: we assume the *compatibility condition*

$$\boxed{h \text{ is locally convex on } \overline{\Omega} \text{ and } h \le \varphi \text{ on } \partial\Omega.} \qquad \text{(C)}$$

Therefore, under (C) the problem becomes

$$\boxed{\begin{cases} \det D^2 u = g(|Du|) (u-h)^{\frac{1}{p}} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}}$$

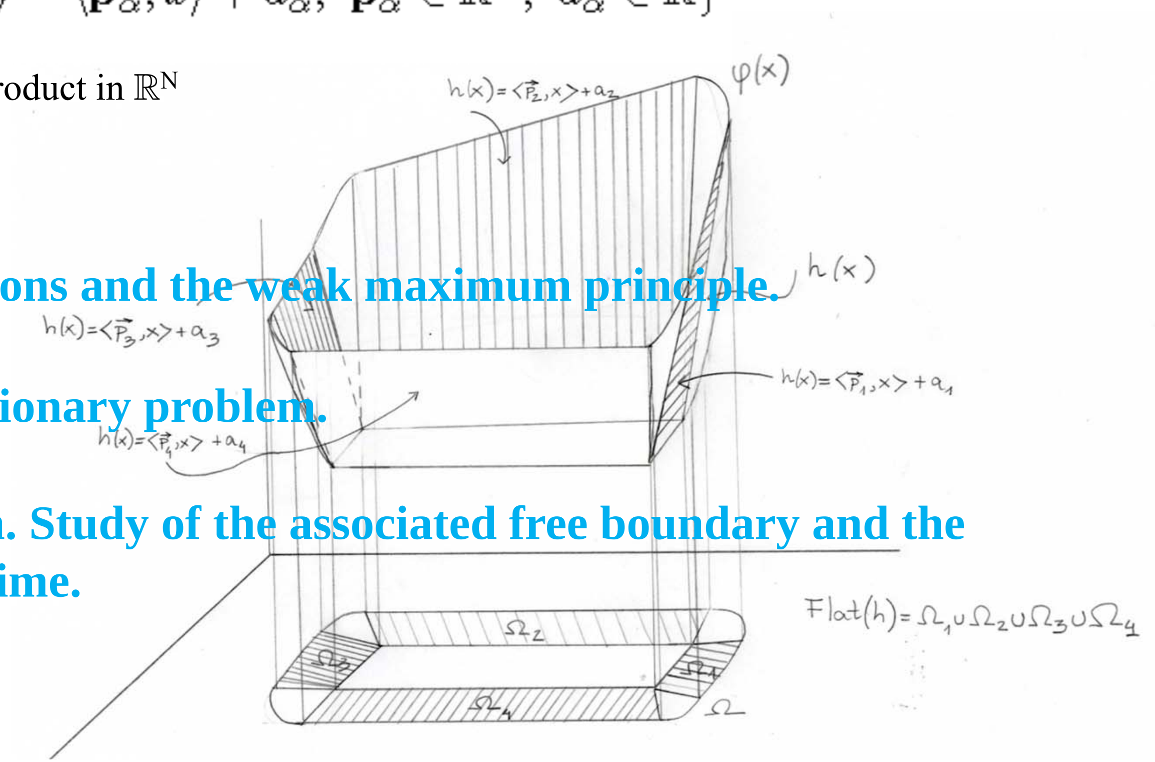
The junction $\mathcal{F}$ between the regions where $[u = h]$ and $[h < u]$ is a **free boundary**: the boundary of the set where $\mathrm{det}\mathrm{D}^2 u > 0$.

Since the interior of the regions $[u = h]$ and $[\mathrm{det}\mathrm{D}^2 u = 0]$ coincide we must have that $\mathrm{D}^2 h = 0$ on the interior of the set $[u = h]$.

Motivated by the applications, as well as by the structure of the equation, the occurrence and localization of a the free boundary will be studied whenever h(x) (respectively u_0(x)) has ***flat regions***

$$\mathrm{Flat}(h) = \bigcup_{\alpha} \{x \in \Omega : h(x) = \langle \mathbf{p}_\alpha, x \rangle + a_\alpha,\ \mathbf{p}_\alpha \in \mathbb{R}^N,\ a_\alpha \in \mathbb{R}\}$$

$\langle\ ,\ \rangle$ denotes the Euclidean inner product in $\mathbb{R}^N$



Plan:

2. On the notion of solutions and the weak maximum principle.

3. Flat regions in the stationary problem.

4. The evolution problem. Study of the associated free boundary and the global flatness in finite time.

2. On the notion of solutions and the weak maximum principle.

Existence results (in the class of C^2 convex functions) for the general elliptic problem

$$\text{(GE)} \qquad \begin{cases} \det D^2 u = H(Du, u, x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

are well known in the literature under suitable assumptions on Ω, $H > 0$ and φ.

Trudinger, N,S., Wang, X.-J.: The Monge-Ampere equation and its geometric applications, in *Handbook of Geometric Analysis*, Vol. I, International Press (2008), 467-524.

A main question arises now both in theory and in applications:

what happens if $H \geq 0$?

Certainly, the elliptic degeneracy occurs and in general the regularity C^2 of solutions cannot be guaranteed. The so called *viscosity solutions* or the *generalized solutions* are suitable notions in order to remove the degeneracy of the operator. In fact, it can be proved that for a convex domain both notions coincide (C.E. Gutierrez (2001)).

Definition 1 (≈A.D. Aleksandrov, (1939)). A convex function u on is a *generalized solution* of (GE) if

$$\mu_u(E) = \int_E H(Du, u, x)\,dx \quad \text{for any Borel set} \quad E \subset \Omega.$$

$$\mu_u(E) \doteq |\partial u(E)| = \text{meas}\{\mathbf{p} \in \mathbb{R}^N : \ \mathbf{p} \in \partial u(x) \text{ for some } x \in E\}.$$

The left hand side have a "classical" sense merely when u is C^1 and convex.
By the structure of the problem, u must be convex on and consequently u is at least locally Lipschitz. While for locally Lipschitz functions the right hand side is well defined, slight but careful modifications are needed to give sense to the left hand side. The progress in this direction is achieved thanks to the notion of *subgradients* of a convex function u:

$$\text{given } \mathbf{p} \in \mathbb{R}^{\mathrm{N}}, \text{ we say } \quad \mathbf{p} \in \partial u(x) \quad \text{iff} \quad u(y) \geq u(x) + \langle \mathbf{p}, y - x \rangle \quad \text{for all } y \in \Omega.$$

Other notion of solutions are available for other type of fully nonlinear equations with non divergence form. The most usual is the so called *viscosity solution*:

Definition 2 (≈ P.L.Lions, M.G .Crandall, L. Caffarelli, I. Hishi, R. Jensen, R. Newcomb, ... (since 1983)). A convex function u is a *viscosity solution* of the inequality

$$\det \mathrm{D}^2 u \geq \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega$$

if for every $\mathcal{C}^2$ convex function Φ on Ω for which

$$(u - \Phi)(x_0) \geq (u - \Phi)(x) \quad \text{locally at } x_0 \in \Omega$$

one has

$$\det \mathrm{D}^2\Phi(x_0) \geq \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), x_0\big).$$

Analogously, one defines the viscosity solution of the reverse inequality

$$\det \mathrm{D}^2 u \leq \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega$$

as a convex function u on Ω such that for every C^2 convex function Φ on Ω for which

$$(u - \Phi)(x_0) \leq (u - \Phi)(x) \quad \text{locally at } x_0 \in \Omega$$

one has

$$\det \mathrm{D}^2 \Phi(x_0) \leq \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), (x_0)\big).$$

Finally, when both properties hold we arrive to the notion of viscosity solution of

$$\det \mathrm{D}^2 u = \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega.$$

Note that the convexity condition on u and are extra assumptions with respect to the usual notion of viscosity solution

Crandall, M.G., Ishii, H., Lions, P.-L.: Users guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc., 27 (1992), 1-67.

This is needed here because the Monge-Ampere operator is only degenerate elliptic on this class of functions. In fact, it can be seen that *the convexity is only required for the correct definition of supersolutions* in viscosity sense.

A very simple (and important fact) was used in our precedent arguments: if $u_1 \in C^2$ and $u_2 - u_1 \in C^2$ are convex functions on a ball B then

$$\det \mathrm{D}^2 u_2 \geq \det \mathrm{D}^2 u_1 \quad \text{in } \mathrm{B}.$$

This simple inequality can be extended to the case u_1 and $u_2 - u_1$ convex function on a ball B, with $u_1 = u_2$ on $\partial \mathrm{B}$, by the "monotonicity formula"

$$\mu_{u_2}(\mathrm{B}) \leq \mu_{u_2}(\mathrm{B}) \tag{17}$$

(see [42]). So that, the Weak Maximum Principle can be extended to the class of generalized solutions

The results of this section apply to the case of a general increasing function $f \in C(\mathrm{R})$ satisfying $f(0) = 0$

$$\det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big) f(u - h) \quad \text{in } \Omega.$$

Theorem 2.6 (Weak Maximum Principle II). *Let $h_1, h_2 \in C(\overline{\Omega})$. Let $u_1, u_2 \in C(\overline{\Omega})$ where u_1 is locally convex in Ω. Suppose*

$$-\det \mathrm{D}^2 u_1 + g\big(|\mathrm{D}u_1|\big) f\big(u_1 - h_1\big) \leq -\det \mathrm{D}^2 u_2 + g\big(|\mathrm{D}u_2|\big) f\big(u_2 - h_2\big) \quad \text{in } \Omega \tag{18}$$

in the generalized solution sense. Then

$$(u_1 - u_2)(x) \leq \sup_{\partial\Omega} [u_1 - u_2]_+ + \sup_{\Omega} [h_1 - h_2]_+, \quad x \in \Omega. \tag{19}$$

In particular,

$$|u_1 - u_2|(x) \leq \sup_{\partial\Omega} |u_1 - u_2| + \sup_{\Omega} |h_1 - h_2|, \quad x \in \Omega, \tag{20}$$

whenever the equality holds in (18).

A first consequence of the general theory and the Weak Maximum Principle is the following *existence and uniqueness result*:

Theorem 2.8. *Let $\varphi \in C(\partial\Omega)$ and assume the compatibility condition (4). Then there exists a unique locally convex function verifying*

Notations of the manuscript: sorry !!

$$\begin{cases} \det \mathrm{D}^2 u = g\left(|\mathrm{D}u|\right) f(u-h) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

in the generalized sense. In fact, one verifies

$$h(x) \leq u(x) \leq \mathrm{U}_\varphi(x), \quad x \in \overline{\Omega}, \tag{21}$$

where U_φ is the harmonic function in Ω with $\mathrm{U}_\varphi = \varphi$ on $\partial\Omega$.

Proof. First we consider the generalized solution of the problem

$$\begin{cases} -\det \mathrm{D}^2 u + g\left(|\mathrm{D}u|\right)\left[f(u-h)\right]_+ = 0 & \text{in } \Omega. \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Since $\mathrm{H}\left(\mathrm{D}u, u, x\right) = g\left(|\mathrm{D}u|\right)\left[f(u-h)\right]_+ \geq 0$ we can apply well known results in the literature. In particular, from [43], it follows the existence and uniqueness of the solution u. The second point is to note that, by construction, the own locally convex function h verifies

$$-\det \mathrm{D}^2 h + g\left(|\mathrm{D}u|\right)\left[f(h-h)\right]_+ \leq 0 \quad \text{in } \Omega.$$

Therefore, by the Weak Maximum Principle and the assumption $h \leq \varphi$ on $\partial\Omega$ we get that

$$h \leq u \quad \text{in } \Omega,$$

whence

$$[f(u-h)]_+ = f(u-h)$$

concludes the existence. The uniqueness also follows from the Weak Maximum Principle. Finally, since u is locally convex, the arithmetic–geometric mean inequality lead to

$$0 \leq \det D^2 u \leq \frac{1}{N}(\Delta u)^N \quad \text{in } \Omega,$$

whence the estimate

$$h(x) \leq u(x) \leq U_\varphi(x), \quad x \in \overline{\Omega}$$

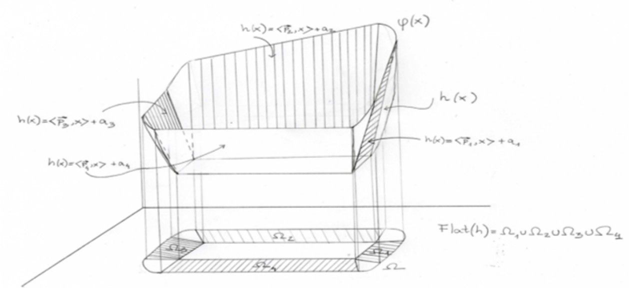
is completed by the weak maximum principe for harmonic functions. □

Remark 2.9. i) As it was pointed out in the Introduction, no sign assumption on h is required in Theorem 2.8. The simple structural assumption (4) implies that $h \leq u$ on $\overline{\Omega}$ and therefore the ellipticity, eventually degenerate, of the equation holds. Thus, the ellipticity holds once h behaves as a lower "obstacle" for the solution u. We note that these compatibility conditions are not required a priori in the Weak Maximum Principles because there we are working with functions whose existence is a priori assumed.

In the next section we shall prove a kind of Strong Maximum Principle which under suitable assumptions will avoid the appearance of the free boundary.

3. Flat regions in the stationary problem.

In this section we focus the attention to a lower "obstacle" function h locally convex on $\overline{\Omega}$ having some region giving rise to the set



$$\mathrm{Flat}(h) = \bigcup_{\alpha} \mathrm{Flat}_{\alpha}(h)$$

where

$$\mathrm{Flat}_{\alpha}(h) = \{x \in \overline{\Omega} : h(x) = \langle \mathbf{p}_{\alpha}, x\rangle + a_{\alpha} \text{ for some } \mathbf{p}_{\alpha} \in \mathbb{R}^{\mathrm{N}} \text{ and } a_{\alpha} \in \mathbb{R}\}. \qquad (23)$$

Since

$$u(y) - \big(\langle \mathbf{p}_{\alpha}, y\rangle + a_{\alpha}\big) \geq u(x) - \big(\langle \mathbf{p}_{\alpha}, x\rangle + a_{\alpha}\big) + \langle \mathbf{p} - \mathbf{p}_{\alpha}, y - x\rangle,$$

thus

$$\mathbf{p} \in \partial u(x) \quad \Leftrightarrow \quad \mathbf{p} - \mathbf{p}_{\alpha} \in \partial\left(u(x) - \big(\langle \mathbf{p}_{\alpha}, x\rangle + a_{\alpha}\big)\right),$$

the equation of the elliptic problem (EP) becomes

$$\det \mathrm{D}^2 u_{\alpha} = \lambda g\big(|\mathrm{D}u|\big) u_{\alpha}^{\frac{1}{\mathrm{p}}}, \quad x \in \mathrm{Flat}_{\alpha}(h), \qquad \boxed{u_{\alpha} = u - \big(\langle \mathbf{p}_{\alpha}, x\rangle + a_{\alpha}\big)}$$

Assumption $g(|\mathbf{p}|) \geq 1$ leads us to study for the auxiliar problem

$$\begin{cases} \det D^2 U = \lambda U^{\frac{1}{p}} & \text{in } B_R(0), \\ U \equiv M > 0 & \text{on } \partial B_R(0), \end{cases} \tag{25}$$

for any $M > 0$. From the uniqueness of solutions, it follows that U is radially symmetric, because by rotating it we would find another solutions. Moreover, by the comparison results U is nonnegative. Therefore, the solution U is governed by a nonnegative radial profile function $U(x) = \widehat{U}(|x|)$ for which some straightforward computations leads to

$$\det D^2 U(x) = \widehat{U}''(r) \left(\frac{\widehat{U}'(r)}{r} \right)^{N-1} = \frac{r^{1-N}}{N} \left[\left(\widehat{U}'(r) \right)^N \right]'. \tag{26}$$

Remark 3.1. For $N = 1$, equation (25) becomes

$$\widehat{U}''(r) = \lambda \widehat{U}^{\frac{1}{p}}$$

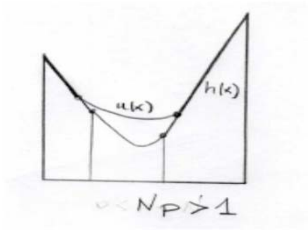
whose annulation set was studied in [26]. Note that for $N > 1$ equation (26) does not coincide with the $(N-1)$–Laplacian considered in [26]. □

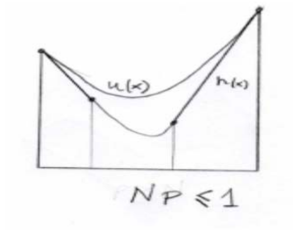
We start by considering the *initial value problem*

(IVP)=(27)
$$\begin{cases} \frac{r^{1-N}}{N} \left[\left(U'(r) \right)^N \right]' = \lambda \left(U(r) \right)^{\frac{1}{p}}, \quad \lambda > 0 \\ U(0) = U'(0) = 0. \end{cases}$$

Obviously, $U(r) \equiv 0$ is always a solution, but we are interested in the existence of nontrivial and non–negative solutions. It will be useful the following result

We shall show that the behaviour of u depends strongly on the exponent p





Lemma 3.2. *Assume* $Np > 1$. *Consider the function*

$$U(r) = \lambda^{\frac{p}{Np-1}} C r^{\frac{2Np}{Np-1}}, \quad r \geq 0, \tag{28}$$

where C *is a positive constant. Let*

$$C_{p,N} = \left(\frac{(2Np)^{N-1}(Np+1)}{(Np-1)^N} \right)^{\frac{p}{Np-1}}. \tag{29}$$

Then,

$$-\frac{r^{1-N}}{N}\left[\left(U'(r)\right)^N\right] + \lambda\left(U(r)\right)^{\frac{1}{p}} = C^{\frac{1}{p}}\left[1 - \left(\frac{C}{\lambda^{\frac{p}{Np-1}} C_{p,N}}\right)^{\frac{Np-1}{p}}\right] r^{\frac{2N}{Np-1}}. \tag{30}$$

Therefore,

(i) *if* $C < \lambda^{\frac{p}{Np-1}} C_{p,N}$ *the function* $U(r)$ *is a supersolution of the equation* (27),

(ii) *if* $C = \lambda^{\frac{p}{Np-1}} C_{p,N}$ *the function* $U(r)$ *is the solution of the equation* (27),

(iii) *if* $C > \lambda^{\frac{p}{Np-1}} C_{p,N}$ *the function* $U(r)$ *is a subsolution of the equation* (27).

Since $Np > 1$, the function

$$U(r) = \lambda^{\frac{p}{Np-1}} C_{p,N} r^{\frac{2Np}{Np-1}}, \quad r \geq 0, \tag{31}$$

enables us to construct functions vanishing in a ball $\mathbf{B}_\tau(0)$

$$v_\tau(x) \doteq U\big([|x| - \tau]_+\big), \quad x \in \mathbb{R}^N, \tag{32}$$

which solves

$$-\det D^2 v_\tau(x) + \lambda\big(v_\tau(x)\big)^{\frac{1}{p}} = 0, \quad x \in \mathbb{R}^N.$$

Moreover, given $M > 0$, it verifies

$$v_\tau(x) = M, \quad |x| = R,$$

once we take

$$\tau = R - U^{-1}(M) = \left(\frac{M}{C_{p,N}}\right)^{\frac{Np-1}{2Np}} \left[\lambda_*^{-\frac{1}{2N}} - \lambda^{-\frac{1}{2N}}\right]$$

with

$$\lambda \geq \lambda_* \doteq \frac{1}{R^{2N}} \left(\frac{M}{C_{p,N}}\right)^{\frac{Np-1}{p}}. \tag{33}$$

Now for the solution of (EP) we may localize a core of the flat region Flat(u) inside the flat subregion Flat_α (h) of the datum h(x).

Theorem 3.3. *Let h be locally convex on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_{\mathrm{R}}(x_0) \subset \mathrm{Flat}_\alpha(h)$ with*

$$0 \le u(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha) \le \mathrm{M} \le \max_{\overline{\Omega}}(u-h), \quad x \in \partial\mathbf{B}_{\mathrm{R}}(x_0), \tag{34}$$

where u is a generalized solution of (7), *for some* $\mathrm{M} > 0$. *Then, if* $\mathrm{Np} > 1$ *and*

$$\lambda \ge \lambda^* \doteq \frac{1}{\mathrm{R}^{2\mathrm{N}}} \left(\frac{\mathrm{M}}{\mathrm{C}_{\mathrm{p,N}}}\right)^{\frac{\mathrm{Np}-1}{\mathrm{p}}},$$

one verifies

$$0 \le u(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha) \le \lambda^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C}_{\mathrm{p,N}} \left(\left[|x - x_0| - \tau\right]_+\right)^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0), \tag{35}$$

where

$$\tau = \left(\frac{\mathrm{M}}{\mathrm{C}_{\mathrm{p,N}}}\right)^{\frac{\mathrm{Np}-1}{2\mathrm{Np}}} \left[\lambda_*^{-\frac{1}{2\mathrm{N}}} - \lambda^{-\frac{1}{2\mathrm{N}}}\right], \tag{36}$$

once we assume that

$$\left(\frac{\mathrm{M}}{\mathrm{C}_{\mathrm{p,N}}}\right)^{\frac{\mathrm{Np}-1}{2\mathrm{Np}}} \lambda^{-\frac{1}{2\mathrm{N}}} < \mathrm{R} \le \mathrm{dist}(x_0, \partial\Omega). \tag{37}$$

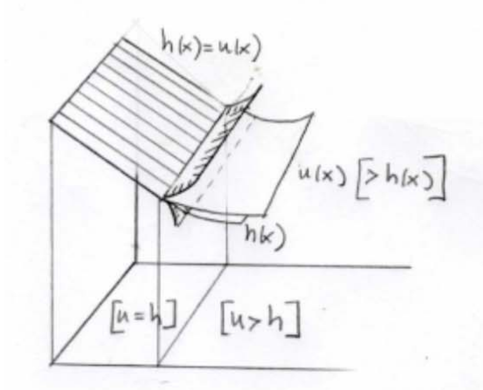
In particular, the function u is locally flat

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \textit{for any } x \in \overline{\mathbf{B}}_\tau(x_0).$$

Remark 3.4. We have proved that under the above assumptions the flat region of u is a non–empty set. Obviously, $\mathrm{Flat}(h) \subset \mathrm{Flat}(u)$ whenever (34) fails, even if $\mathrm{Np} > 1$. We shall examine the optimality of (35) in [24] following different strategies carry out in [26] for other free boundary problems. $\square$

Remark 3.5. We point out that the above result applies to the case in which $\varphi \equiv 1$ and $h \equiv 0$ (the so called "dead core" problem) as well as to cases in which u is flat only near $\partial\Omega$ (take for instance, $h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha$ in Ω and $\varphi \equiv h$ on $\partial\Omega$). $\square$

Theorem 3.3 gives some estimates on the localization of the points inside $\mathrm{Flat}(h)$ where u becomes flat too. The following result shows that if h decays in a suitable way at the boundary points of $\mathrm{Flat}(h)$ then the solution u becomes also flat in those points of the boundary of $\mathrm{Flat}(h)$. In this result the parameter λ is irrelevant, therefore with no loss of generality we shall assume that $\lambda = 1$.



Theorem 3.6. *Let us assume* $\mathrm{Np} > 1$. *Let* $x_0 \in \partial\mathrm{Flat}_\alpha(h)$ *such that*

$$h(x) - \left(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\right) \le \mathrm{K}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_\mathrm{R}(x_0) \cap \left(\mathbb{R}^\mathrm{N} \setminus \mathrm{Flat}(h)\right),$$

and

$$0 \le \max_{|x-x_0|=\mathrm{R}} \left\{u(x) - \left(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\right)\right\} \le \mathrm{C}\mathrm{R}^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}} \tag{39}$$

for some suitable positive constants K *and* R *and* u *is a generalized solution of* (7). *Then*

$$u(x_0) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha. \tag{40}$$

Proof. Define the function

$$V(x) = u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big),$$

which by construction is nonnegative in $\partial \mathbf{B}_R(x_0)$ (see (39). In fact, the Weak Maximum Principle implies that V is non negative on $\overline{\mathbf{B}}_R(x_0)$. Then

$$\begin{aligned}
-\det D^2 V(x) + \big(V(x)\big)^{\frac{1}{p}} &= -\det D^2 u(x) + \big(u(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha)\big)^{\frac{1}{p}}\\
&= -\big(u(x) - h(x)\big)^{\frac{1}{p}} + \big(u(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha)\big)^{\frac{1}{p}}\\
&\le C_p K^{\frac{1}{p}} \big(h(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha)\big)^{\frac{1}{p}}\\
&\le C_p K^{\frac{1}{p}} |x - x_0|^{\frac{2N}{Np-1}}, \quad x \in \mathbf{B}_R(x_0),
\end{aligned}$$

where we have used the classical inequality

$$(a+b)^{\frac{1}{p}} \le C_p \left(a^{\frac{1}{p}} + b^{\frac{1}{p}}\right) \quad \text{for some positive constant } C_p,$$

as well as (38). Moreover, if we take $C < C_{N,p}$ and then K such that

$$C_p K^{\frac{1}{p}} \le C^{\frac{1}{p}} \left[1 - \left(\frac{C}{C_{p,N}}\right)^{\frac{Np-1}{p}}\right]$$

whence

$$-\det D^2 V(x) + \big(V(x)\big)^{\frac{1}{p}} \le -\det D^2 U(|x|) + \big(U(|x|)\big)^{\frac{1}{p}}, \quad x \in \mathbf{B}_R(x_0),$$

for $U(r) = C|x - x_0|^{\frac{2Np}{Np-1}}$ (see (30)). Finally, by choosing R satisfying (39) one has

$$V(x) \le U(|x|), \quad x \in \partial \mathbf{B}_R(x_0),$$

whence the comparison principle concludes by comparison

$$0 \le V(x) \le C|x - x_0|^{\frac{2Np}{Np-1}}, \quad x \in \mathbf{B}_R(x_0),$$

and so $u(x_0) = \big(\langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha\big)$. $\square$

Remark 3.7. The assumption (39) is satisfied if we know that the ball $\mathrm{B_R}(x_0)$ where (38) holds is assumed large enough. The above result is motivated by [26, Theorem 2.5]. By adapting the reasoning used in previous results of the literature (see [1, 3, 27]) it can be shown that the decay of $h(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha)$ near the boundary point x_0 is optimal in the sense that if

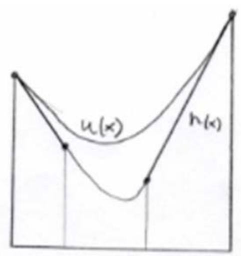
$$h(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha) > \mathrm{C}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}} \quad \text{in a neighbourhood of } x_0$$

then it can be shown that

$$u(x_0) - (\langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha)) > \mathrm{C}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}} \quad \text{for } x \text{ near } x_0.$$

This type of results gives very rich information on the non–degeneracy behavior of the solution near the free boundary. This is very useful to the study of the continuous dependence of the free boundary with respect to the data h and φ (see [27]). $\square$

Now we examine the case in which the solution cannot be flat (i.e: the free boundary cannot appear) independent on the “size” of Ω: obviously it will require the condition $\mathrm{Np} \leq 1$.



$\mathrm{Np} \leq 1$

Lemma 3.8 (Hopf boundary point lemma). *Assume* $\mathrm{Np} \le 1$. *Let u be a nonnegative viscosity solution of*

$$-\det \mathrm{D}^2 u + u^{\frac{1}{\mathrm{p}}} \ge 0 \quad \text{in } \Omega.$$

Let $x_0 \in \partial\Omega$ be such that $u(x_0) \doteq \liminf\limits_{\substack{x \to x_0 \\ x \in \Omega}} u(x)$ *and*

$$\begin{cases} i) & u \text{ achieves a strict minimum on } \Omega \cup \{x_0\}, \\ ii) & \exists\ \mathbf{B}_{\mathrm{R}}(x_0 - \mathrm{R}\mathbf{n}(x_0)) \subset \Omega, \quad (\ \partial\Omega \text{ satisfies an interior sphere condition at } x_0). \end{cases}$$

Then

$$\liminf_{\tau \to 0} \frac{u(x_0 - \tau \mathbf{n})}{\tau} \ge C > 0, \tag{41}$$

where $\mathbf{n}$ stands for the outer normal unit vector of $\partial\Omega$ at x_0 and C is a positive constant depending only on the geometry of $\partial\Omega$ at x_0.

Proof. Let $y = x_0 - \mathrm{R}\mathbf{n}(x_0)$ and $\mathrm{B}_{\mathrm{R}} \doteq \mathrm{B}_{\mathrm{R}}(y)$. As it was pointed out before, equation (7) leads to the study of the differential equation

$$\frac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\left(\Phi'(r) \right)^{\mathrm{N}} \right]' = \left(\Phi(r) \right)^{\frac{1}{\mathrm{p}}}, \quad r > 0,$$

for radially symmetric solutions. We consider now the classical solution of the two point boundary problem

$$\begin{cases} \dfrac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\left(\Phi'(r) \right)^{\mathrm{N}} \right]' = \left(\Phi(r) \right)^{\frac{1}{\mathrm{p}}}, \quad 0 < r < \dfrac{\mathrm{R}}{2}, \\ \Phi(0) = 0, \quad \Phi\left(\dfrac{\mathrm{R}}{2}\right) = \Phi_1 > 0. \end{cases} \tag{42}$$

The existence of solution follows from standard arguments and the uniqueness of solution can be proved as in Theorem 2.6, whence

$$\Phi'(0) \geq 0 \quad \Rightarrow \quad \Phi'(r) > 0 \qquad \Rightarrow \quad \Phi''(r) > 0.$$

Then

$$0 \leq \Phi(r) \leq \Phi_1, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

We note that the singularity at $r = 0$ must be removed by the condition

$$\lim_{r \to 0} \frac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\left(\Phi'(r) \right)^{\mathrm{N}} \right]' = 0. \tag{43}$$

Let r_0 be the largest r for which $\Phi(r) = 0$. We want to prove that $r_0 = 0$ by proving that $r_0 > 0$ leads to a contradiction. In order to do that we multiply (42) by $r^{\mathrm{N}-1}\Phi'(r)$ and get

$$\left[\left(\Phi'(r) \right)^{\mathrm{N}+1} \right]' = (\mathrm{N}+1) \left(\Phi(r) \right)^{\frac{1}{\mathrm{p}}} \Phi'(r) r^{\mathrm{N}-1}, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

Next, since $\Phi'(r_0) = 0 = \Phi(r_0)$, an integration between r_0 and r leads to

$$\begin{aligned} \left(\Phi'(r) \right)^{\mathrm{N}+1} &= \frac{\mathrm{p}(\mathrm{N}+1)}{\mathrm{p}+1} \left(\Phi(r) \right)^{\frac{\mathrm{p}+1}{\mathrm{p}}} r^{\mathrm{N}-1} - \frac{\mathrm{p}(\mathrm{N}+1)(\mathrm{N}-1)}{\mathrm{p}+1} \int_{r_0}^{r} \left(\Phi(s) \right)^{\frac{\mathrm{p}+1}{\mathrm{p}}} r^{\mathrm{N}-2} ds \\ &\leq \frac{\mathrm{p}(\mathrm{N}+1)}{\mathrm{p}+1} \left(\Phi(r) \right)^{\frac{\mathrm{p}+1}{\mathrm{p}}} r^{\mathrm{N}-1}, \quad r_0 < r < \frac{\mathrm{R}}{2}. \end{aligned}$$

Because $\mathrm{Np} \leq 1$, a new integration between r_0 and $\frac{\mathrm{R}}{2}$ yields the conjectured contradiction because

$$\infty = \int_0^{\Phi_1} \frac{ds}{s^{\frac{\mathrm{p}+1}{\mathrm{p}(\mathrm{N}+1)}}} = \int_{r_0}^{\frac{\mathrm{R}}{2}} \frac{\Phi'(r)}{\left(\Phi(r) \right)^{\frac{\mathrm{p}+1}{\mathrm{p}(\mathrm{N}+1)}}} dr \leq (\mathrm{N}+1) \left(\frac{\mathrm{p}(\mathrm{N}+1)}{\mathrm{p}+1} \right)^{\frac{1}{\mathrm{N}+1}} \int_{r_0}^{\frac{\mathrm{R}}{2}} r^{\mathrm{N}-1} dr < \infty.$$

So that, we have proved $\Phi'(0) > 0$ and also

$$0 < \Phi(r) < \Phi_1, \ \Phi'(r) > 0, \quad 0 < r < \frac{\mathrm{R}}{2},$$

as well as $\Phi''(0) = 0$ (see (43)). Hence, straightforward computations on the $\mathcal{C}^2$ convex function $w(x) = \Phi(\mathrm{R} - |x - y|)$, defined in the annulus $\mathcal{O} \doteq \mathrm{B}_{\mathrm{R}} \setminus \overline{\mathrm{B}}_{\frac{\mathrm{R}}{2}}$, prove

$$\begin{cases} \det \mathrm{D}^2 w(x) = f(v(x)), \quad x \in \mathcal{O}, \\ w(x) = \Phi_1, \quad x \in \partial \mathrm{B}_{\frac{\mathrm{R}}{2}}, \\ w(x) = 0, \quad x \in \partial \mathrm{B}_{\mathrm{R}}. \end{cases}$$

Moreover, by construction

$$u(x) > 0, \quad x \in \partial \mathrm{B}_{\frac{\mathrm{R}}{2}} \quad \Rightarrow \quad u(x) \geq w(x), \quad x \in \partial \mathrm{B}_{\mathrm{R}},$$

for Φ_1 small enough. Then the Weak Maximum Principle of Proposition 2.4 implies

$$(u - w)(x) \geq 0, \quad x \in \overline{\mathcal{O}}.$$

that leads to

$$\frac{u(x_0 - \tau \mathbf{n})}{\tau} \geq \frac{\Phi(\mathrm{R} - \mathrm{R}(1 - \tau))}{\tau}, \qquad (\tau \ll 1)$$

whence

$$\liminf_{\tau \to 0} \frac{u(x_0 - \tau \mathbf{n})}{\tau} \geq \Phi'(0) > 0.$$

$\square$

Our main result proving the absence of the free boundary is the following

Theorem 3.10 (Hopf's Strong Maximum Principle). *Assume* $\mathrm{Np} \le 1$. *Let* u *be a nonnegative viscosity solution of*

$$-\det \mathrm{D}^2 u + u^{\frac{1}{\mathrm{p}}} \ge 0 \quad \text{in } \Omega.$$

Then u *cannot vanish at some* $x_0 \in \Omega$ *unless* u *is constant in a neighborhood of* x_0.

Proof. Assume that u is non–constant and achieves the minimum value $u(x_0) = 0$ on some ball $\mathbf{B} \subset \Omega$. Then we consider the semi-concave approximation of u, *i.e.*

$$u^{\varepsilon}(x) \doteq \inf_{y\in\Omega} \left\{ u(y) + \frac{|x-y|^2}{2\varepsilon^2} \right\}, \quad x \in \mathbf{B}_{\varepsilon} \qquad (\varepsilon > 0), \tag{44}$$

where $\mathbf{B}_{\varepsilon} \doteq \{x \in \mathbf{B} : \ \mathrm{dist}(x, \partial\mathbf{B}) > \varepsilon\sqrt{1 + 4\sup_{\mathbf{B}} |u|}\}$. For ε small enough we can assume $x_0 \in \mathbf{B}_{\varepsilon}$. Then u^{ε} achieves the minimum value in $\mathbf{B}_{\varepsilon}$, and $u(x_0) = u^{\varepsilon}(x_0) = 0$. Moreover, u^{ε} satisfies

$$-\det \mathrm{D}^2 u_{\varepsilon} + u_{\varepsilon}^{\frac{1}{\mathrm{p}}} \ge 0 \quad \text{on } \mathbf{B}_{\varepsilon} \tag{45}$$

(see, for instance [43, Proposition 2.3] or [6, 16] for general fully nonlinear equations). By classic arguments, if we denote

$$\mathbf{B}_{\varepsilon}^{+} \doteq \{x \in \mathbf{B}_{\varepsilon} : \ u^{\varepsilon}(x) > 0\},$$

there exists the largest ball $\mathbf{B}_{\mathrm{R}}(y) \subset \mathbf{B}_{\varepsilon}^{+}$ (see [30]). Certainly there exists some $z_0 \in \partial\mathbf{B}_{\mathrm{R}}(y) \cap \mathbf{B}_{\varepsilon}$ for which $u^{\varepsilon}(z_0) = 0$ is a local minimum. Then, Lemma 3.8 implies

$$\mathrm{D}u^{\varepsilon}(z_0) \neq \mathbf{0}$$

contrary to

$$\mathrm{D}u^{\varepsilon}(z_0) = \mathbf{0}, \tag{46}$$

as we shall prove in Lemma 3.13 below. Therefore, u^{ε} is constant on $\mathbf{B} \subset \Omega$, *i.e.*

$$u^{\varepsilon}(y) = u^{\varepsilon}(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

Finally, for every $y \in \mathbf{B}$ we denote by $\widehat{y}$ the point of Ω such that

$$u^{\varepsilon}(y) = u(\widehat{y}) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2$$

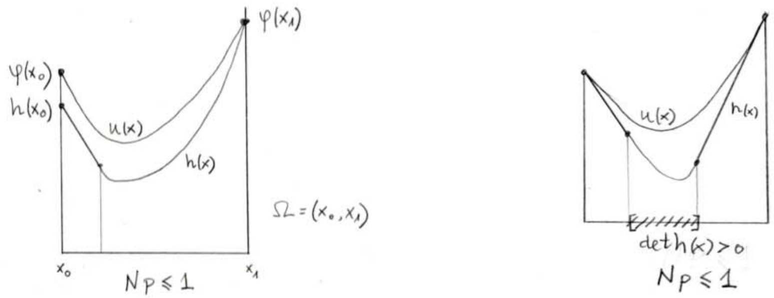
whence

$$u(x_0) = u^{\varepsilon}(x_0) = u^{\varepsilon}(y) = u(y) + \frac{1}{2\varepsilon^2}|y-\widehat{y}|^2 \geq u(x_0) + \frac{1}{2\varepsilon^2}|y-\widehat{y}|^2 \geq u(x_0) \quad \Rightarrow \quad \widehat{y} = y.$$

So that, one concludes

$$u(y) = u^{\varepsilon}(y) = u^{\varepsilon}(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

Corollary 3.11. *Assume* $\mathrm{Np} \leq 1$. *Let* u *be a generalized solution* u *of* (7). *Then if* $u(x_0) > h(x_0)$ *or* $\det \mathrm{D}^2 h(x_0) > 0$ *at some point* x_0 *of a ball* $\overline{\mathbf{B}} \subseteq \overline{\Omega}$ *then* $u > h$ *on* $\overline{\mathbf{B}}$, *consequently the equation* (7) *is elliptic in* $\overline{\mathbf{B}}$. *In particular, if* $\varphi(x_0) > h(x_0)$ *at some* $x_0 \in \partial\Omega$ *or* $\det \mathrm{D}^2 h(x_0) > 0$ *at some point* $x_0 \in \Omega$ *the problem* (5) *is elliptic non degenerate in path-connected open sets* Ω, *provided the compatibility condition* (4) *holds.*



Proof. From Theorem 3.10, both cases imply $u > h$ on $\overline{\mathbf{B}}$. Finally, a continuity argument concludes the proof. $\square$

Remark 3.12. Straightforward computations enable us to extend Lemma 3.8, Theorem 3.10 and Corollary 3.11 to the general case $g(|\mathbf{p}|) \geq 1$, since we know that $u \in \mathrm{W}^{1,\infty}(\Omega)$ (see the comments of Remark 2.9). $\square$

4. The evolution problem. Study of the associated free boundary and the global flatness in finite time.

We start by considering the existence of solution of (2) by means of the accretivity of the operator. The definition of the operator uses odd increasing functions $f \in \mathcal{C}(\mathbb{R})$, such that $f(0) = 0$. Then, we say $u \in \mathrm{D}(\mathcal{A})$ if $u \in \mathcal{C}(\overline{\Omega})$ is a locally convex function on $\overline{\Omega}$ prescribing $\varphi \in \mathcal{C}(\partial\Omega)$ on $\partial\Omega$ and there exists a nonpositive continuous function v in Ω such that u is a generalized solution of

$$\begin{cases} \dfrac{f^{-1}\big(-\det \mathrm{D}^2 u\big)}{g\big(|\mathrm{D}u|\big)} = v & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

or equivalently

$$\begin{cases} \det \mathrm{D}^2 u = f\big(\big(-g\big(|\mathrm{D}u|\big)v\big) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

for a more precise sense. Then we denote by $\mathcal{A}u$ the set of all such $v \in \mathcal{C}(\overline{\Omega})$.

Theorem 4.1. *The operator $\mathcal{A}$ is T-accretive on the Banach space $\mathbb{X} = \mathcal{C}(\overline{\Omega})$ equipped with the supreme norm. In particular,*

$$\begin{aligned} \big\| [u_1 - u_2]_+ \big\| &\leq \sup_{\Omega} \big\| \big[u_1 - u_2 + \varepsilon\big(\mathcal{A}u_1 - \mathcal{A}u_2\big)\big]_+ \big\|, \\ \big\| u_1 - u_2 \big\| &\leq \sup_{\Omega} \big\| u_1 - u_2 + \varepsilon\big(\mathcal{A}u_1 - \mathcal{A}u_2\big) \big\|, \end{aligned} \tag{48}$$

for $\varepsilon > 0$, $u_i \in \mathrm{D}(\mathcal{A})$.

Proof. It is a mere application of Theorem 2.6. □

Certainly, one has

$$\mathrm{D}(\mathcal{A}) \subset \widehat{\mathbb{X}}_\varphi \doteq \{w \in \mathcal{C}(\overline{\Omega}) : \ w \text{ locally convex on } \overline{\Omega} \text{ and } w = \varphi \text{ on } \partial\Omega\}.$$

In fact, we have

Corollary 4.3. *The operator $\mathcal{A}$ satisfies $\overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi$ as well as the range condition*

$$R(\mathrm{I} + \varepsilon\mathcal{A}) \supset \overline{\mathrm{D}(\mathcal{A})}, \quad \varepsilon > 0.$$

Crandall–Liggett generation theorem (see [15]) and Corollary 4.3 enables us to show that $\mathcal{A}$ generates a nonlinear semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $\mathbb{X}$ and

$$S(t)u_0 = \lim_{\substack{n\to\infty \\ \varepsilon n \to t}} (\mathrm{I} + \varepsilon\mathcal{A})^{-1} u_0 \quad \text{for any } u_0 \in \overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi, \tag{49}$$

uniformly for t in bounded subsets of $]0, \infty[$. Furthermore, the mapping $t \mapsto S(t)u_0$ is continuous from $[0, \infty[$ into $\mathbb{X}$. In general the semigroup generated by such accretive operators $\mathcal{A}$ can be regarded as the so called “mild solution” of the Cauchy problem

$$\begin{cases} u_t + \mathcal{A}u = 0, & t > 0, \\ u(0) = u_0, \end{cases} \tag{50}$$

(see [15]). A different characterization is possible.

Proposition 4.4. *Assume $u_0 \in \overline{\mathrm{D}(\mathcal{A})} \subset \mathbb{X}$. Then*

$$u(x,t) = S(t)u_0(x), \quad x \in \Omega, \ 0 < t < \mathrm{T} < \infty, \tag{51}$$

satisfies (PP) in the viscosity sense.

Remark 4.5. Since $u(t) \in \overline{\mathrm{D}(\mathcal{A})}$ the property

$$0 \leq \det \mathrm{D}_x^2 u(t),$$

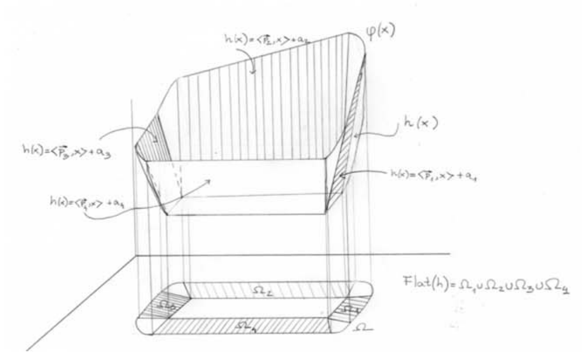
holds in the generalized sense *a.e.* $t > 0$. Note that a priori we merely know that $S(t)\left(\overline{\mathrm{D}(\mathcal{A})}\right) \subset \overline{\mathrm{D}(\mathcal{A})}$ and so the time derivative u_t must be understood in a large sense. Nevertheless, it is possible to apply different regularity results according f see, for instance, [19] and its references. In any case, at least u_t is a nonnegative measure and $u(\cdot, t)$ is a locally convex function. □

Our results on the free boundary begin by studying how a possible region of flatness of the initial datum u_0 shrinks when t increases. We start by considering the interior points of $\mathrm{Flat}(u_0)$. As in Section 3, for $u_0 \in \overline{\Omega}$ we denote

$$\mathrm{Flat}(u_0) = \bigcup_{\alpha} \mathrm{Flat}_{\alpha}(u_0)$$

where

$$\mathrm{Flat}_{\alpha}(u_0) = \{x \in \overline{\Omega} : u_0(x) = \langle \mathbf{p}_{\alpha}, x \rangle + a_{\alpha} \ \text{ for some } \mathbf{p}_{\alpha} \in \mathbb{R}^{\mathrm{N}} \text{ and } a_{\alpha} \in \mathbb{R}\}.$$



Theorem 4.6. *Let* $\mathrm{Np} > 1$ *and*

$$\mathrm{B_R}(x_0) \subset \mathrm{Flat}_\alpha(u_0) \tag{52}$$

for some $\mathrm{R} > 0$. *Then there exists* $t^* = t^*(u_0) > 0$ *such that*

$$u(x_0, t) = \langle \mathbf{p}_\alpha, x_0 \rangle + a_\alpha, \quad 0 \leq t < t^*,$$

where is the solution of (PP).

Proof. We need a suitable local separable supersolution $\overline{\mathrm{U}}(x,t) = \mathrm{U}(|x|)\eta(t)$. The time function $\eta(t)$ is given by

$$\eta'(t) = \delta\big(\eta(t)\big)^{\mathrm{Np}}, \quad t > 0, \quad \text{for some } \delta > 0, \tag{53}$$

whose solution is

$$\eta(t) = \left[\frac{1}{\big(\eta(0)\big)^{\mathrm{Np}-1}} - \frac{\delta}{\mathrm{Np}-1}t\right]^{-\frac{1}{\mathrm{Np}-1}}. \tag{54}$$

Note that $\eta(t)$ blows up at

$$t^*\big(\delta, \eta(0)\big) \doteq \frac{\mathrm{Np}-1}{\delta}\frac{1}{\big(\eta(0)\big)^{\mathrm{Np}-1}}.$$

The spatial dependence is given by the function

$$\mathrm{U}(r) = \delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C_{p,N}}r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \geq 0,$$

(see (31)). In [25] it is proved the regularity

$$\mathrm{D}u \in \mathrm{L}^\infty\big(0, \infty : \mathrm{L}^\infty(\Omega)\big).$$

Then the convexity of the solution enables us to choose $\eta(0)$, δ and R such that

$$\max_{\mathrm{B_R}(x_0)\times\overline{\mathbb{R}}_+} u \leq \eta(0)\delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C_{p,N}}\mathrm{R}^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}. \tag{55}$$

So that, we consider now the function

$$\mathrm{V}(x,t) = u(x,t) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$$

for which

$$\mathrm{V}(x,0)=0, \quad x\in \mathbf{B}_{\mathrm{R}}(x_0),$$

(see (52)) and

$$\mathrm{V}(x,t)\le \overline{\mathrm{U}}(x,t), \quad (x,t)\in \partial\mathbf{B}_{\mathrm{R}}(x_0)\times \big[0,t^*\big(\delta,\eta(0)\big[$$

hold (see (55)). On the other hand, we have that

$$\frac{1}{\delta}\ge g\big(|\mathrm{D}_x\mathrm{V}(x,t)|\big)\ge 1, \quad (x,t)\in \mathbf{B}_{\mathrm{R}}(x_0)\times \big[0,t^*\big(\delta,\eta(0)\big)\big[$$

for a suitable choice of δ. Therefore, one has

$$-\frac{\big(\det \mathrm{D}_x^2\mathrm{V}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x\mathrm{V}|\big)}\le -\delta\big(\det \mathrm{D}_x^2\mathrm{V}\big)^{\mathrm{p}} \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0)\times \big[0,t^*\big(\delta,\eta(0)\big)\big[,$$

and

$$\mathrm{V}_t-\frac{\big(\det \mathrm{D}_x^2\mathrm{V}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x\mathrm{V}(x,t)|\big)}=0\le \overline{\mathrm{U}}_t-\frac{\big(\det \mathrm{D}_x^2\overline{\mathrm{U}}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x\overline{\mathrm{U}}(x,t)|\big)} \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0)\times \big[0,t^*\big(\delta,\eta(0)\big)\big[.$$

Thus, by the comparison principle

$$0\le \mathrm{V}(x,t)\le \overline{\mathrm{U}}(x,t), \quad (x,t)\in \mathbf{B}_{\mathrm{R}}(x_0)\times \big[0,t^*\big(\delta,\eta(0)\big)\big[,$$

Remark 4.7. It is easy to see that the above argument gives a simple estimate on the shrinking of the free boundary

$$\mathcal{F}_\alpha(t)=\partial\{(x,t):\ u(x,t)=\langle \mathbf{p}_\alpha,x\rangle+a_\alpha\} \tag{56}$$

from the rest. Essentially,

$$\limsup_{t\to 0}\operatorname{dist}\big(\mathcal{F}_\alpha(t),\mathcal{F}_\alpha(0)\big)t^{-\frac{1}{\mathrm{Np}-1}}\le \mathrm{C},$$

for some positive constant C. $\square$

The next result shows that if $u_0(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$ arrives to some points of the boundary of its support flat enough, let us say (38), then there exists a "finite waiting time" for those points.

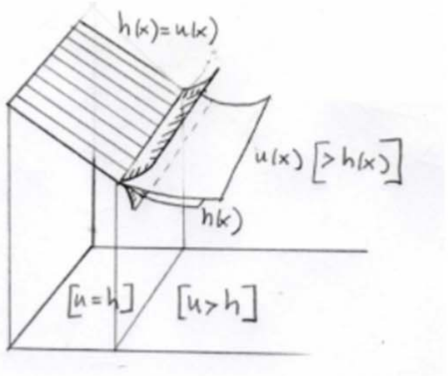
Theorem 4.8. *Let* $\mathrm{Np} > 1$ *and let* $x_0 \in \Omega$ *be such that*

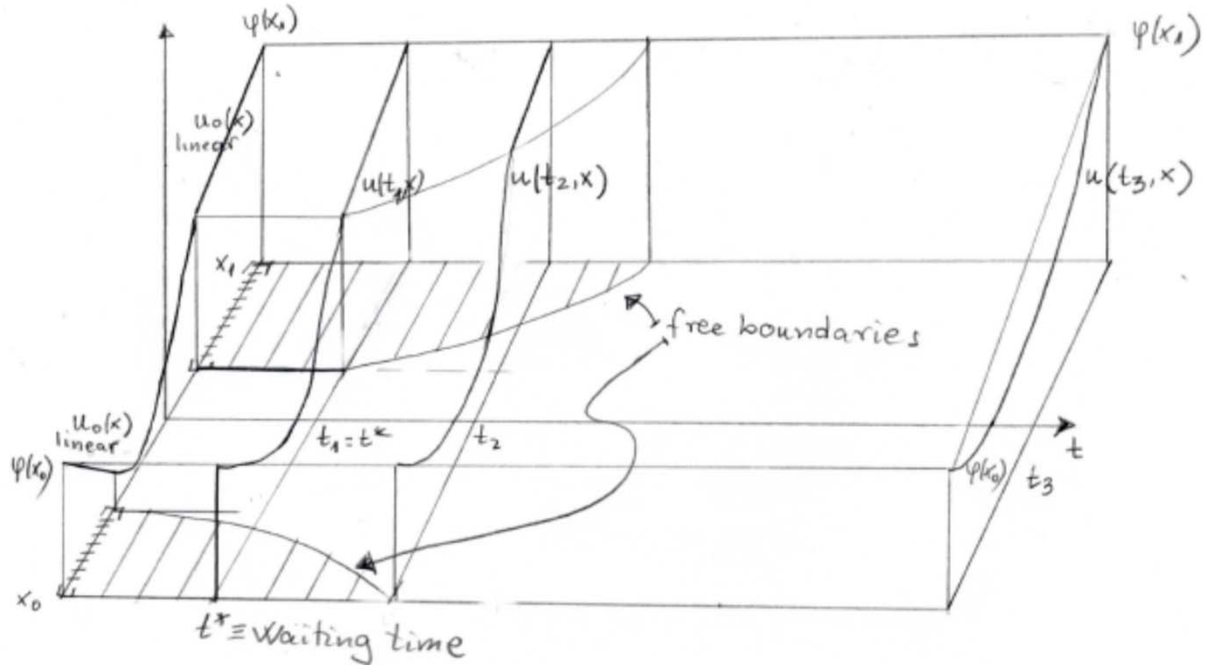
$$u_0(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \le \mathrm{K}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0), \tag{57}$$

for suitable positive constants K *and* R. *Then, there exists* $\widetilde{t} = \widetilde{t}(x_0)$ *such that*

$$u(x_0, t) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha \quad \text{if } 0 \le t < \widetilde{t},$$

where u *is the viscosity solution of* (2).





Proof. As in the above proof we use a local separable supersolution $\overline{\mathrm{U}}(x,t) = \mathrm{U}(|x|)\eta(t)$, where $\eta(t)$ was given in (53) and

$$\mathrm{U}(r) = \mathrm{C}r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \geq 0,$$

for $\mathrm{C} > 0$. Then

$$-\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\left(\mathrm{U}'(r)\right)^{\mathrm{N}}\right] + \delta\left(\mathrm{U}(r)\right)^{\frac{1}{\mathrm{p}}} = \mathrm{C}^{\frac{1}{\mathrm{p}}}\left[1 - \left(\frac{\mathrm{C}}{\delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C}_{\mathrm{p,N}}}\right)^{\frac{\mathrm{Np}-1}{\mathrm{p}}}\right] r^{\frac{2\mathrm{N}}{\mathrm{Np}-1}} > 0$$

for $\mathrm{C} \in \left]0, \delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C}_{\mathrm{p,N}}\right[$ (see (30)). Then the reasonings are similar to those of the proof of Theorem 4.6 because now (57) provides the inequality

$$\mathrm{V}(x,0) \leq \overline{\mathrm{U}}(x,0)$$

before derived from (52). $\square$

Remark 4.9. A similar waiting time result was obtained by Choop, Evans and Ishii (1999) [see also the previous version by Hamilton (1993)] for the special case p = 1 and N = 3 (parametric compact surfaces) under a global formulation on the assumption on the initial datum (two principal curvatures vanish on a subregion). Essentially, they assume

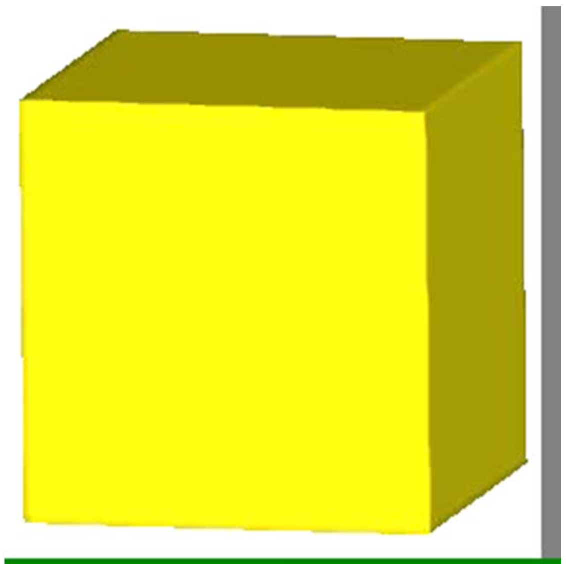
$$u_0 \in C^4([0;+\infty)).$$

Note that for this special case the condition (57) becomes

$$u_0(x) - (\langle \mathbf{p}_\alpha, x\rangle + a_\alpha) \leq \mathrm{K}|x - x_0|^4, \quad x \in \mathbf{B}_\mathrm{R}(x_0). \tag{58}$$

In particular, any C^4 partially flat function satisfies (58) at all points of the boundary of $\mathcal{F}_\alpha(0)$ (see (56)). So, our result can be regarded as a local and generalized version of the result of [14]. □

K. Tso (1985) and then B. Chow (1985) studied problem (PP) (for parametric compact surfaces) and established that the surface Γ_t converge to a point as t ↑ T, for some suitable finite time T.



We continue our study on the evolution of the free boundary by showing that, in the case of a bounded domain Ω, in most of the cases $\mathcal{F}_\alpha(t)$ is shrinking.

Theorem 4.10. *Let* $\mathrm{Np} > 1$ *and assume* x_0 *such that*

$$u_0(x_0) = \langle \mathbf{p}_\alpha, x_0 \rangle + a_\alpha.$$

Then there exists $\widehat{t} > 0$ *such that*

$$u(x_0, t) > \langle \mathbf{p}_\alpha, x_0 \rangle + a_\alpha, \quad t > \widehat{t}, \tag{59}$$

where u *is the viscosity solution of* (2)*.*

As a matter of fact, it is enough to show that

$$u(x_0, \widehat{t}) > \langle \mathbf{p}_\alpha, x_0 \rangle + a_\alpha, \tag{60}$$

because since $u_t \geq 0$ we get (59) for any $t > \widehat{t}$. In order to prove Theorem 4.10 we shall use other suitable supersolution based on the self–similar solution of the Cauchy problem associated to the case $g(s) \equiv \delta$, for suitable $\delta \geq 1$. We start by pointing out that by arguing by adimensionalization we get:

Lemma 4.11. *Let $u(x,t)$ be a viscosity solution of*

$$u_t = \left(\det \mathrm{D}_x^2 u\right)^p \quad \text{in } \mathbb{R}^{\mathrm{N}} \times \mathbb{R}_+. \tag{61}$$

Then the change of scale $x' = \mathrm{L}x$, $t' = \mathrm{T}t$ allows to define

$$u'(x',t') = \mathrm{L}^{\frac{2\mathrm{N}p}{\mathrm{N}p-1}} \mathrm{T}^{-\frac{1}{\mathrm{N}p-1}} u(x,t)$$

which is also a viscosity solution of (61). $\square$

A more deep conclusion on the self–similar solution of (61) is the following

Theorem 4.12. *Assume* $\mathrm{N}p > 1$. *Then, there exists a family of convex compactly supported similarity solutions of* (61) *given by*

The proof of Theorem 4.12 requires the analysis of the correspondent phase–plane system

$$\begin{cases} \dfrac{dq}{d\eta} = -\left[\dfrac{\mathrm{N}^p\beta}{\eta}\left[\dfrac{\sigma}{\beta}\Lambda(\eta) + \eta q^{\frac{1}{\mathrm{N}}}\right]\right]^{\frac{1}{p}}, \\ \dfrac{d\Lambda}{d\eta} = q^{\frac{1}{\mathrm{N}}}, \end{cases}$$

where $q = (\Lambda')^{\mathrm{N}}(\operatorname{sign} \Lambda')$. By simplicity, we do not present here the details but send the reader to [25]. In any case, we can indicate that the proof is a non–difficult variation of some results in the literature (see, for instance, Bernis, Hulshof and Vázquez [7] and Igbida [35]). See also Daskalopoulus and Lee [18] for the case of the "focusing problem" associated to (61). C. Budd and V. Galaktionov (2009) [blow up case].

Proof of Theorem 4.10. As in the proof of Theorem 4.6 we can assume that the solution u has bounded gradient

$$\mathrm{D}u \in \mathrm{L}^{\infty}(\Omega \times]0, \widehat{t}[)$$

for any given $\widehat{t}$. So that $\mathrm{V}(x,t) = u(x,t) - \left(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\right)$ verifies

$$\mathrm{V}_t(x,t) - \delta\left(\det \mathrm{D}_x^2 \mathrm{V}(x,t)\right)^{\mathrm{p}} \geq 0,$$

for a suitable $\delta > 0$, that we suppose here $\delta = 1$ to simplify the notation, otherwise the needed modifications are simple. Let $x_0 \in \Omega$ such that

$$u_0(x_0) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha.$$

Then we consider σ and β for which

$$u(|x - x_0|, t; \sigma, \beta) \leq \mathrm{V}(x,t) \quad \text{for any } t \geq 0 \text{ and } x \in \mathbf{B}_{\mathrm{R}}(x_0)$$

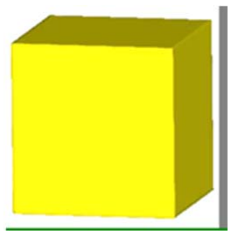
for some $\mathrm{R} > 0$ (see Theorem 4.12). Since

$$u(r_0, t; \sigma, \beta) > 0$$

for any $r_0 > 0$ once that t is large enough, we conclude the result. □

Our last goal is the study of the asymptotic behavior of u as $t \to \infty$ from a peculiar point of view. We start by proving that if $\mathrm{N}p \geq 1$ then the stabilization to a stationary solution requires infinite time. From now on, any locally convex function on Ω such that $\det \mathrm{D}^2 h = 0$, *a.e.* in Ω will be called *flat convex function*. By simplicity we assume $g\left(|\mathbf{p}|\right) \equiv 1$.

To be compared with the results by Tso (1985), Chow (1985) and Chopp, Evans, Ishii (1999) for "parametric compact surfaces".

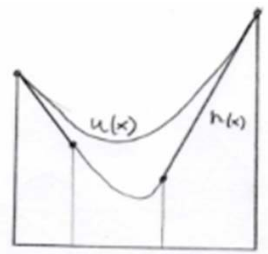


Theorem 4.14. *Assume that* $\mathrm{Np} \geq 1$. *Let* $\overline{h}$ *a flat convex function on* $\overline{\Omega}$ *such that* $\varphi \leq \overline{h}$ *on* $\partial\Omega$. *Then for each* $u_0 \in \overline{\mathrm{D}(\mathcal{A})}$ *such that* $u_0 \leq \overline{h}$ *on* $\overline{\Omega}$ *and* $u_0 < \overline{h}$ *in some set* $\Omega' \subset \Omega$ *with positive measure, there exists a positive constant* $\mathrm{C}_{\Omega'}$ *such that*

$$\begin{cases} \liminf_{t\to\infty} \big(\overline{h}(x) - u(x,t)\big)t^{\frac{1}{\mathrm{Np}-1}} \geq \mathrm{C}_{\Omega'} & \textit{if } \mathrm{Np} > 1, \\ \liminf_{t\to\infty} \big(\overline{h}(x) - u(x,t)\big)e^{t} \geq \mathrm{C}_{\Omega'} & \textit{if } \mathrm{Np} = 1, \end{cases} \quad x \in \Omega', \tag{62}$$

where u *is the viscosity solution of* (2). *Analogously, let* $\underline{h}$ *be a flat convex function on* $\overline{\Omega}$ *such that* $\varphi \geq \underline{h}$ *on* $\partial\Omega$ *verifying* $u_0 \geq \underline{h}$ *on* $\overline{\Omega}$ *and* $u_0 > \overline{h}$ *in some set* $\Omega' \subset \Omega$ *with positive measure, there exists a positive constant* $\mathrm{C}_{\Omega'}$ *such that*

$$\begin{cases} \liminf_{t\to\infty} \big(u(x,t) - \underline{h}(x)\big)t^{\frac{1}{\mathrm{Np}-1}} \geq \mathrm{C}_{\Omega'} & \textit{if } \mathrm{Np} > 1, \\ \liminf_{t\to\infty} \big(u(x,t) - \underline{h}(x)\big)e^{t} \geq \mathrm{C}_{\Omega'} & \textit{if } \mathrm{Np} = 1, \end{cases} \quad x \in \Omega'. \tag{63}$$



$Np \leqslant 1$

Proof. If h be a flat such that $\varphi \leq \overline{h}$ on $\partial\Omega$. Then, one has

$$\overline{h}_t = \big(\det \mathrm{D}_x^2 \overline{h}\big)^{\mathrm{p}}$$

whence $\underline{u}(x,t) = u(x,t) - \overline{h}(x)$ verifies

$$\begin{cases} (\underline{u})_t = \big(\det \mathrm{D}_x^2 \underline{u}\big)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ \underline{u} \leq 0, \ \underline{u} \not\equiv 0 & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \end{cases} \tag{64}$$

in the viscosity sense and

$$\underline{u}(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

Here the key idea is to consider the auxiliar problem

$$\begin{cases} \phi'(t) + \dfrac{2}{\mathrm{m}}\big(k\phi(t)\big)^{\mathrm{Np}} = 0, \quad t \geq 0, \\ \phi(0) = 1, \quad \phi(\infty) = 0, \end{cases} \tag{65}$$

whose solution is

$$\phi(t) = \begin{cases} \left[1 + \dfrac{2}{\mathrm{m}} \dfrac{k^{\mathrm{Np}}}{\mathrm{Np}-1} t\right]^{-\frac{1}{\mathrm{Np}-1}}, & \text{if } \mathrm{Np} > 1, \\ e^{-\frac{2k}{\mathrm{m}} t}, & \text{if } \mathrm{Np} = 1, \end{cases}$$

where k is a positive constant to be choosen and m is a positive constant such that $\overline{h} - u_0 \geq \mathrm{m}$ in some $\mathrm{B}_{2\mathrm{R}} \subset \Omega$. Let $\psi(x_1) \in \mathcal{C}^2$ a non positive function such that

$$\begin{cases} \psi(x_1) = 0, & x \notin \overline{\mathbf{B}}_{2\mathrm{R}}, \\ -\mathrm{m} < \psi(x_1) < -\dfrac{\mathrm{m}}{2} \quad \text{and} \quad \psi''(x_1) \geq 0, & x \in \mathbf{B}_{\mathrm{R}}, \\ -\dfrac{\mathrm{m}}{2} < \psi(x_1) < 0 \quad \text{and} \quad \psi''(x_1) \leq 0, & x \in \mathbf{B}_{2\mathrm{R}} \setminus \overline{\mathbf{B}}_{\mathrm{R}}. \end{cases}$$

Then the function

$$W(x,t) = \phi(t)\psi(x_1), \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+,$$

verifies

$$\begin{cases} W(x,t) < 0, & (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \overline{\mathbb{R}}_+, \\ W(x,t) = 0, & (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \\ W(x,0) = \phi(0)\psi(x_1) \geq -\mathrm{m} > \big(u_0 - h\big)(x), & x \in \mathbf{B}_{2\mathrm{R}}, \\ W(x,0) = \phi(0)\psi(x_1) = 0 \geq \big(u_0 - h\big)(x), & x \in \overline{\Omega} \setminus \overline{\mathbf{B}}_{2\mathrm{R}}. \end{cases}$$

because $\phi(0) = 1$. Moreover, from (65) we get

$$W_t(x,t) + \big(- \det \mathrm{D}_x^2 W(x,t)\big)^{\mathrm{p}} \doteq r(x,t)$$

where

$$r(x,t) \geq \begin{cases} \big(k\phi(t)\big)^{\mathrm{Np}} + \Big(-\big(\phi(t)\big)^{\mathrm{Np}}\Big) \big(\psi''(x_1)\big)^{\mathrm{p}} \geq 0, & x \in \mathbf{B}_{\mathrm{R}}, \\ \phi'(t)\psi(x_1) + \big(\phi(t)\big)^{\mathrm{Np}}\big(- \psi''(x_1)\big)^{\mathrm{p}} \geq 0, & x \in \Omega \setminus \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

for $t > 0$, provided k is large. Then, from (64), comparison results lead to

$$u(x,t) - \overline{h}(x) \leq W(x,t) \leq 0, \quad (x,t) \in \Omega \times \mathbb{R}_+.$$

In particular,

$$\overline{h}(x) - u(x,t) \geq \frac{\mathrm{m}}{2}\phi(t) > 0, \quad (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \mathbb{R}_+. \tag{66}$$

We may repeat the reasoning with a flat function $\underline{h}$ such that $\varphi \geq \underline{h}$ on $\partial\Omega$. So, the function $\overline{u}(x,t) = u(x,t) - \underline{h}(x)$ verifies

$$\begin{cases} (\overline{u})_t = \big(\det \mathrm{D}_x^2 \overline{u}\big)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ \overline{u} \geq 0,\ \overline{u} \not\equiv 0 & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \end{cases}$$

in the viscosity sense and

$$\overline{u}(x,t) \geq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

Now, we consider a non negative function $\psi(x_1) \in \mathcal{C}^2$ such that

$$\begin{cases} \psi(x_1) = 0, & x \notin \overline{\mathbf{B}}_{2\mathrm{R}}, \\ \dfrac{\mathrm{m}}{2} < \psi(x_1) < \mathrm{m} \quad \text{and} \quad \psi''(x_1) \leq 0, & x \in \mathbf{B}_{\mathrm{R}}, \\ 0 < \psi(x_1) < \dfrac{\mathrm{m}}{2} \quad \text{and} \quad \psi''(x_1) \geq 0, & x \in \mathbf{B}_{2\mathrm{R}} \setminus \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

where m is a positive constant such that $u_0 - \underline{h} \geq \mathrm{m}$ in some $\mathbf{B}_{2\mathrm{R}} \subset \Omega$. Arguing as above one proves that the function

$$w(x,t) = \phi(t)\psi(x_1), \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+,$$

verifies

$$\begin{cases} w_t(x,t) + \big(-\det \mathrm{D}_x^2 w(x,t)\big)^{\mathrm{p}} \leq 0 & \text{in } \Omega \times \mathbb{R}_+, \\ w \leq \overline{u} & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \end{cases}$$

provided k is large. Then, we obtain

$$u(x,t) - \underline{h}(x) \geq w(x,t) \geq 0, \quad (x,t) \in \Omega \times \mathbb{R}_+.$$

In particular,

$$u(x,t) - \underline{h}(x) \geq \frac{\mathrm{m}}{2}\phi(t) > 0, \quad (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \mathbb{R}_+. \tag{67}$$

□

Note that, in particular, Theorem 4.14 implies a kind of non flattened global retention property:

$$\begin{cases} u_0(x) < \overline{h}(x), \; x \in \Omega' \subset \Omega & \Rightarrow \quad u(x,t) < \overline{h}(x), \; x \in \Omega' \text{ for all } t \geq 0, \\ \underline{h}(x) < u_0(x), \; x \in \Omega' \subset \Omega & \Rightarrow \quad \underline{h}(x) < u(x,t), \; x \in \Omega' \text{ for all } t \geq 0, \end{cases} \tag{68}$$

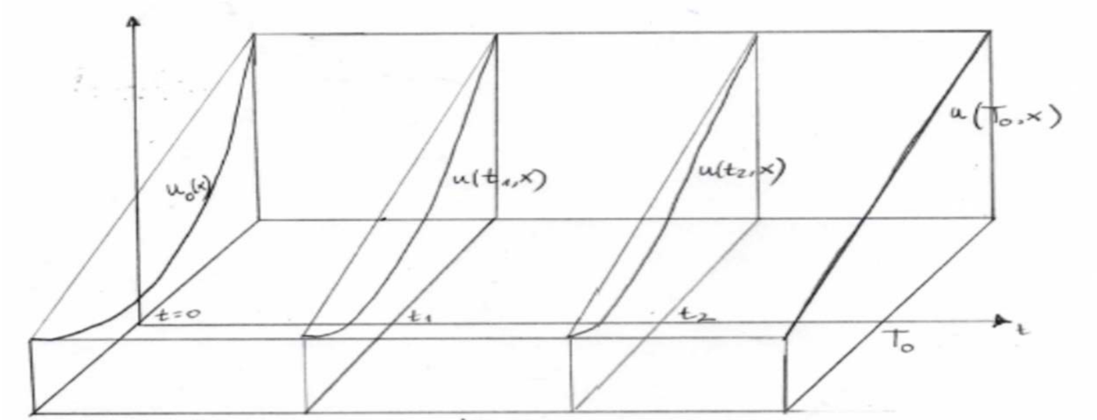
holds. Clearly, the second retention property also follows from $u_t \geq 0$.

Our final result in this paper shows that when Np < 1 the asymptotic behavior is very fast. It is the property of *finite global flattened time*. Again, for simplicity, we assume g=1.

Theorem 4.15. *Let* $h(x) = \langle \mathbf{p}, x \rangle + a$ *on* $\overline{\Omega}$ *and suppose* $\varphi = h$ *in the definition of the operator* $\mathcal{A}$. *Assume* $\mathrm{Np} < 1$. *Then for each* $u_0 \in \overline{\mathrm{D}(\mathcal{A})}$ *such that* $u_0 \leq h$ *on* $\overline{\Omega}$ *there exists a time* T_0, *depending on* $h - u_0$, *such that*

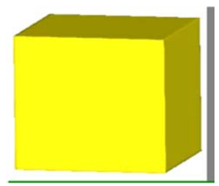
$$u(x,t) = \langle \mathbf{p}, x \rangle + a, \quad x \in \overline{\Omega}, \quad t \geq \mathrm{T}_0.$$

where u *is the viscosity solution of* (2).



Np<1

To be compared with the results by Tso (1985), Chow (1985) and Chopp, Evans, Ishii (1999) for "parametric compact surfaces", but now for "graph surfaces" and Np<1.



Proof. Let us denote $u_h(x,t) = u(x,t) - h(x)$. As in the proof of Theorem 4.14 one verifies (64), thus

$$\begin{cases} (u_h)_t = \left(\det \mathrm{D}_x^2 u_h\right)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ u_h \leq 0, \ u_h \not\equiv 0 & \text{on } \left(\partial\Omega \times \mathbb{R}_+\right) \cup \left(\overline{\Omega} \times \{0\}\right), \end{cases}$$

in the viscosity sense, whence

$$u_h(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

In fact, if $u_0 = h$ one derives the coincidence

$$u_h(x,t) = 0 \quad \text{for any } (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

So that, suppose

$$u_0 \leq h, \ u_0 \not\equiv h.$$

It is clear that the "finite flattened time property" is strongly based on the initial value problem

$$\begin{cases} \mathrm{m}\Theta'(t) = \left(2\Theta(t)\right)^{\mathrm{Np}}, \quad t \geq 0, \\ \Theta(0) = 0 \end{cases}$$

whose solution is

$$\Theta(t) = \left(\frac{2^{\mathrm{Np}}(1-\mathrm{Np})}{\mathrm{m}}\right)^{\frac{1}{1-\mathrm{Np}}} t^{\frac{1}{1-\mathrm{Np}}},$$

provided $\mathrm{Np} < 1$ and m is a positive constant. Then, for each $\mathrm{T}_0 > 0$ the profile function

$$\mathcal{T}(t) = \begin{cases} \Theta\left(\mathrm{T}_0 - t\right) & \text{if } 0 < t \leq \mathrm{T}_0, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies

$$\mathcal{T}'(t)\mathrm{m} + \left(2\mathcal{T}(t)\right)^{\mathrm{Np}} = 0, \quad t > 0. \tag{69}$$

On the other hand, for $\mathrm{R} > 0$ large, we consider the function

$$\zeta(x) = 2^{\mathrm{N}-1}\big(x_1^2 - \mathrm{R}^2\big) \leq 0, \quad x \in \overline{\Omega}$$

which verifies

$$\begin{cases} -m < \zeta(x) < -\mathrm{M} < 0, \quad x \in \overline{\Omega}, \quad -\mathrm{m} \doteq \min\limits_{x\in\overline{\Omega}} \zeta, \quad -\mathrm{M} \doteq \max\limits_{x\in\overline{\Omega}} \zeta \\ \det \mathrm{D}^2\zeta(x) \equiv 2^{\mathrm{N}}, \quad x \in \overline{\Omega}. \end{cases}$$

It enables us to define

$$V(x,t) = \mathcal{T}(t)\zeta(x), \quad (x,t) \in \overline{\Omega} \times \mathbb{R}_+,$$

for which $v(x,t) \leq 0,\ (x,t) \in \partial\Omega \times \mathbb{R}_+$ and $V(x,0) \leq -\Theta\big(\mathrm{T}_0\big)\mathrm{M},\ x \in \overline{\Omega}$, whence

$$v(x,0) \leq \big(u_0 - h\big)(x), \quad x \in \overline{\Omega},$$

provided $\mathrm{T}_0 = \Theta^{-1}\big(\|h - u_0\|_\infty \mathrm{M}^{-1}\big)$. Moreover, for each $(x,t) \in \Omega \times \mathbb{R}_+$ one has

$$v_t(x,t) + \big(-\det \mathrm{D}_x^2 v(x,t)\big)^{\mathrm{p}} \leq -\mathcal{T}'(t)\mathrm{m} + f_{\mathrm{p}}^{-1}\left(-2\big(\mathcal{T}(t)\big)^{\mathrm{N}}\right) = 0$$

(see (69)). Thus

$$v_t(x,t) - \big(\det \mathrm{D}_x^2 v(x,t)\big)^{\mathrm{p}} \leq 0, \quad (x,t) \in \Omega \times [0,\mathrm{T}].$$

This function V can be considered as an eventual test function for the viscosity solution u_h (see (64)), then, arguing as in the proof of Theorem 2.4, we deduce

$$v(x,t) \leq u_h(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times [0,\mathrm{T}[,$$

whence the finite global flattened time property holds. $\square$

Remark 4.16. We end by pointing out that our methods can be applied to the borderline cases for (9) and (11). This will be made in two future papers in which the Monge-Ampère operator is replaced by other nonlinear operators of the Hessian of the unknown such as the kth elementary symmetric functions

$$\mathrm{S}_k[\lambda(\mathrm{D}^2 u)] = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le \mathrm{N}} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \le k \le \mathrm{N},$$

where $\lambda(\mathrm{D}^2 u) = (\lambda_1, \ldots, \lambda_\mathrm{N})$ are the eigenvalues of $\mathrm{D}^2 u$. Note that the case $k = 1$ corresponds to the Laplacian operator while it is a fully nonlinear operator for the other choices of k. The case $k = \mathrm{N}$ corresponds to the Monge–Ampère operator.

Remark 4.17. Some of the above methods (and other different ones as the applications of the symmetric rearrangement) can be applied to other fully nonlinear parabolic equations arising in image processing such as the

$$\frac{\partial u}{\partial t} = \beta(curv(u)) \, \|\nabla u\| \, ,$$

$$curv(u) = div\left(\frac{\nabla u}{\|\nabla u\|}\right) \qquad \beta(s) = \begin{cases} \beta_1 s^p & if \quad s \ge 0 \\ -\beta_{-1}(-s)^p & if \quad s < 0. \end{cases}$$

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Thanks

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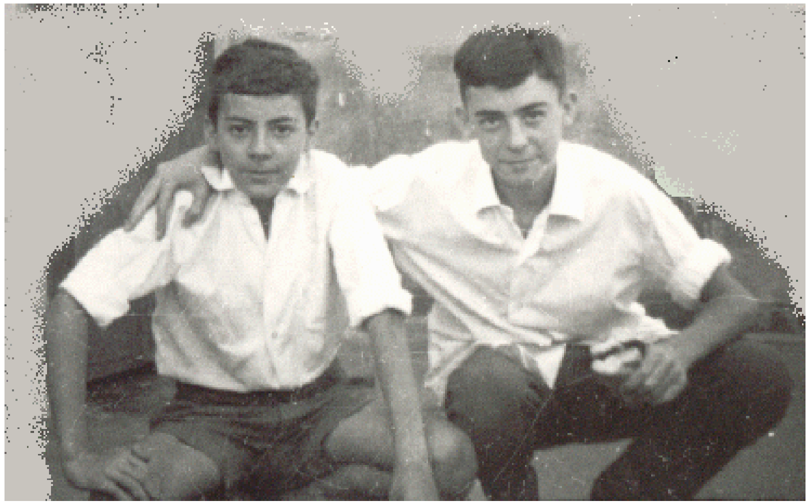

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