Large solutions for a system of elliptic equations arising from Fluid Dynamics

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Montpellier, May 31 , 2005

1. Introduction

Physical problem:

flow of a viscous heat-conducting fluid under the force of gravity

Phenomenon:

temperature fluctuations $\longrightarrow$ density fluctuations

$\longrightarrow$ buoyancy forces

$\longrightarrow$ convective motion

Boussinesq approximation:

fluid is considered "thermally compressible, but mechanically incompressible"

$\longrightarrow$ Navier-Stokes equations plus heat equation

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \operatorname{div}\ (\mu(\theta)D(\mathbf{v})) + \nabla p = \mathbf{F}(\theta), \\ \operatorname{div}\mathbf{v} = 0, \\ \mathcal{C}(\theta)_t + \mathbf{v}\cdot\nabla\mathcal{C}(\theta) - \Delta\varphi(\theta) = |\mu(\theta)D(\mathbf{v})|^2, \end{cases}$$

$\mathbf{v}$ is the velocity field of the fluid,

θ its temperature, p the pressure, $\mu(\theta)$ the viscosity of the fluid,

$D(\mathbf{v}) := \nabla\mathbf{v} + \nabla\mathbf{v}^T$, $\mathbf{F}(\theta)$ the buoyancy force,

$\mathcal{C}(\theta) := \int_{\theta_0}^{\theta} C(s)ds$ $\varphi(\theta) := \int_{\theta_0}^{\theta} \kappa(s)ds$,

$C(\tau)$ and $\kappa(\tau)$ being the specific heat and the thermal conductivity

Neglecting the friction terms

$$C(\theta)_t + \mathbf{u} \cdot \nabla C(\theta) - \Delta\varphi(\theta) = 0,$$

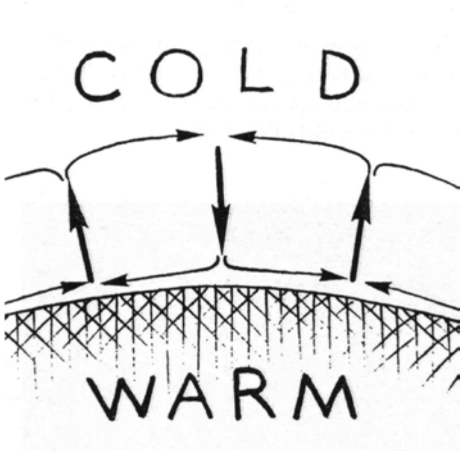
J.I.Díaz, I.I.Vrabie, On a Boussinesq type systems in Fluids Dynamics. *Topological Methods in Nonlinear Analysis*, **4**, 399-416, 1994 (volume dedicated to Jean Leray).

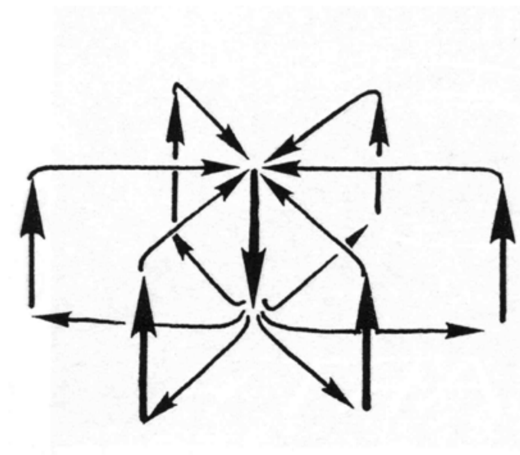
J.I. Díaz, G. Galiano, On the Boussinesq system with nonlinear thermal diffusion*, Nonlinear Analysis*, **30**, 3255-3266, 1997.

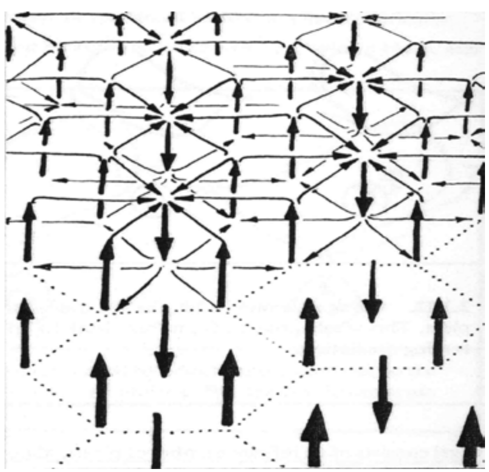
J.I. Díaz, G. Galiano, Existence and uniqueness of solutions of the Boussinesq system with nonlinear thermal diffusion, *Topological Methods in Nonlinear Analysis*, **11**, 58-92, 1998 (volume dedicated to O.A. Ladyzhenskaya),

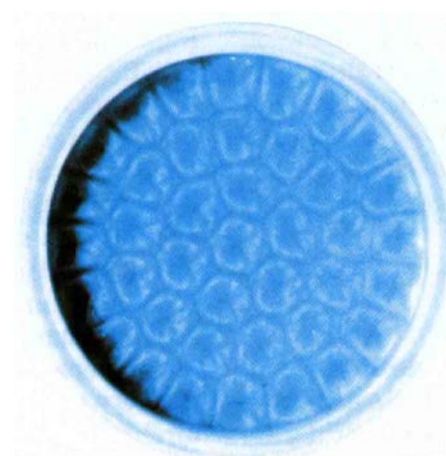
S. Antontsev, J.I. Díaz, S. Shmarev
Energy methods for free boundary problems. Applications to nonlinear PDEs and Fluid Mechanics
Series Progress in Nonlinear Differential Equations and Their Applications, No. 48, Birkäuser, Boston, 2002.

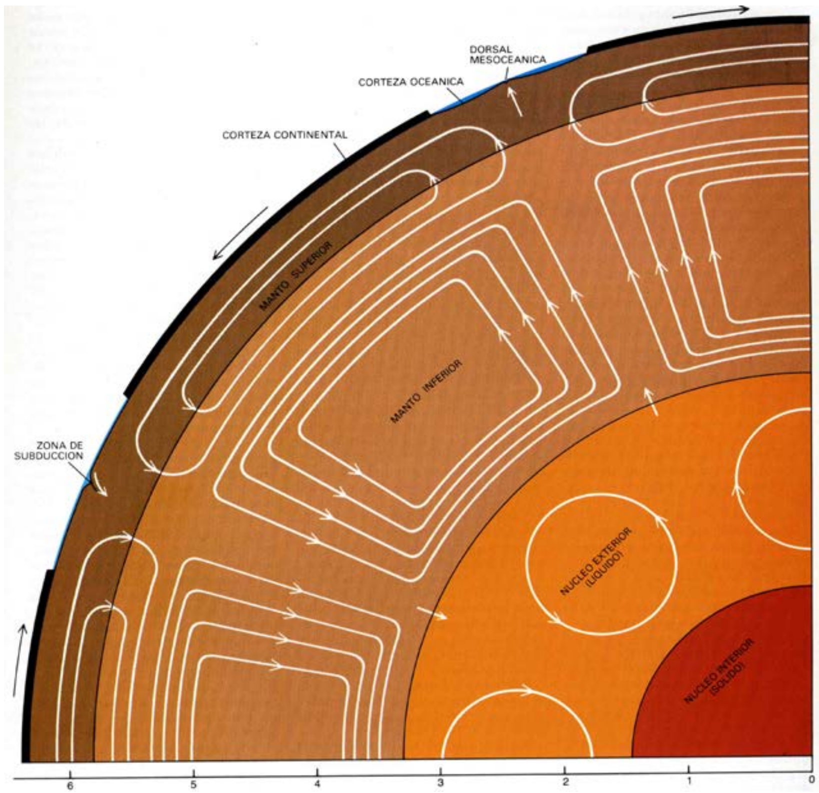
Rayleigh-Benard cells

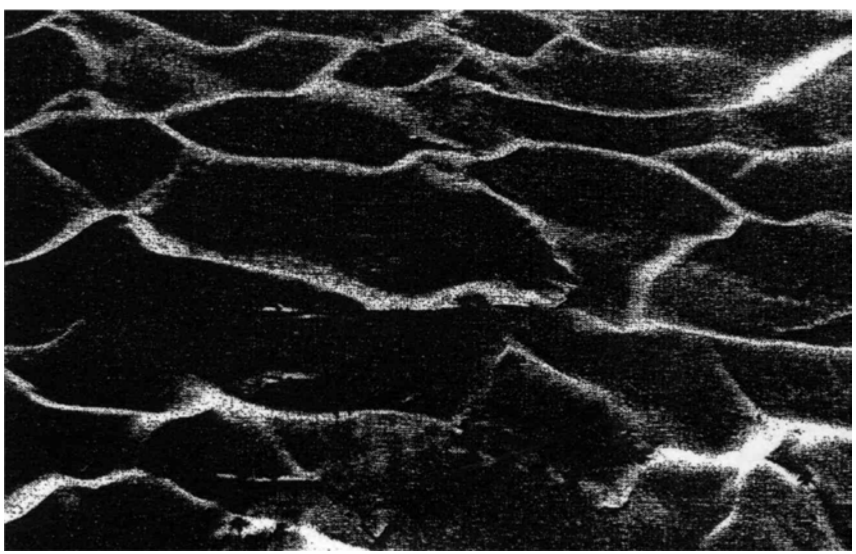


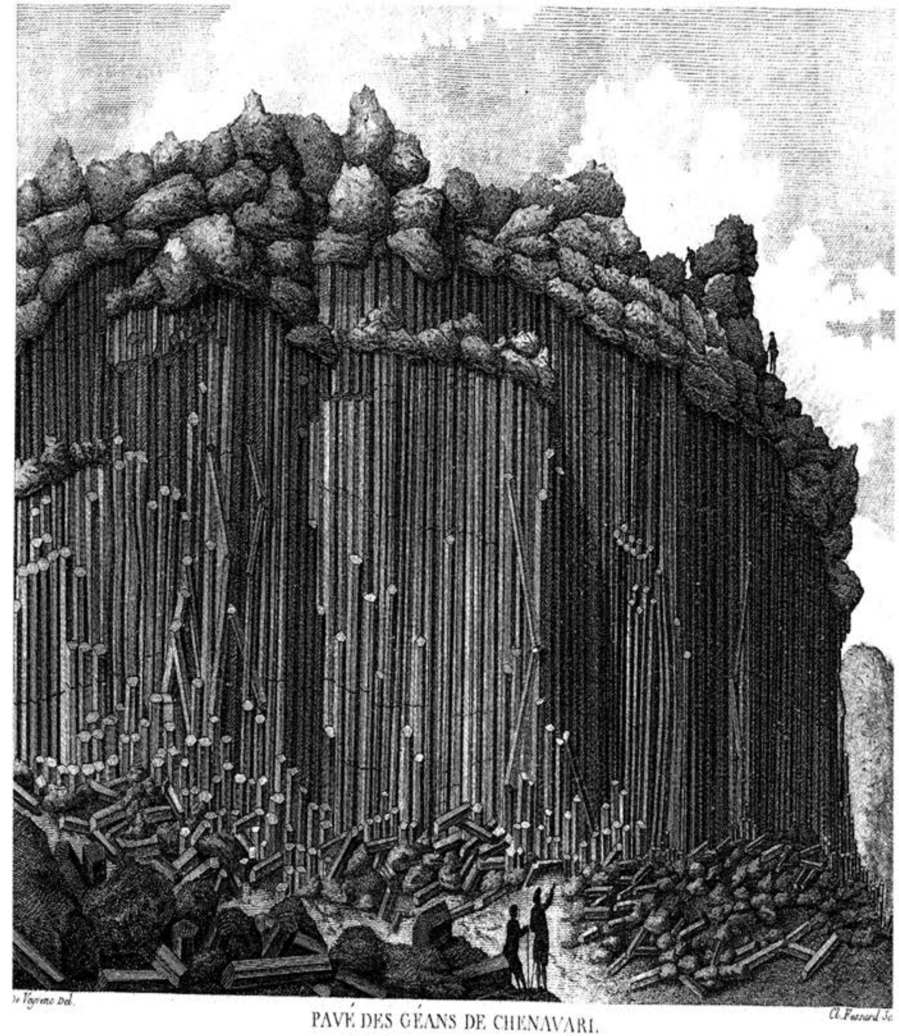












3 | Les cellules hexagonales des orgues de basalte ont de tout temps frappé l'imagination. Leur régularité est quelque peu idéalisée sur cette gravure tirée d'un ouvrage de Faujas de Saint-Fond (1778), un des premiers géologues ayant étudié les volcans d'Auvergne. Les orgues de basalte résultent des craquelures formées (à l'état solide) par refroidissement d'une coulée de lave. Ces craquelures tendent à s'organiser en un réseau hexagonal, à la manière des cellules de convection. Collection P. Thomas.

J.I. Díaz, B. Straughan
Global stability for convection when the viscosity has a maximum,
Continuum Mech. Thermodynamics, **16**, 347-352 2004

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k} 2\nu_0 \Big\{ \big[1 - \gamma (T - T_0)^2\big] D_{ik} \Big\}$$
$$+ 2\nu_1 \frac{\partial}{\partial x_k}\big(|\mathbf{D}|^{2\mu} D_{ik}\big) + g\alpha k_i T,$$

$$\frac{\partial v_i}{\partial x_i} = 0, \qquad \frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \kappa \Delta T.$$

In this lecture: effects of the friction energy source for a simplified model

2. Large solutions.

J. I. Díaz, M. Lazzo, P.G. Schmidt
Large Solutions for a System of Elliptic Equations Arising from Fluid Dynamics, *SIAM Journal on Mathematical Analysis, 2005.*

3. Global existence for

$$C(\theta) = \theta^2 \quad \varphi(\theta) = \theta,$$

J. I. Díaz, J.M. Rakotoson, P.G. Schmidt,

work in progress

2. Large solutions

Model problem:

unidirectional flow in a channel of cross-section Ω

$v = v(t, x)$ velocity

$\theta = \theta(t, x)$ temperature

$t \geq 0$ time, $x \in \Omega$ position

Balance of momentum:

$$v_t - \Delta v = \theta \quad \longleftarrow \quad \text{buoyancy force}$$

Balance of energy:

$$\theta_t - \Delta\theta = |\nabla v|^2 \quad \longleftarrow \quad \text{viscous heating}$$

One motivation for the study of *Large solutions*

Approximate controllability from the boundary with internal observation:

Given $\omega \subset\subset \Omega$ if v and θ are arbitrary on $\partial\Omega$ (the controls)

then $v|_\omega$ and $\theta|_\omega$ form a dense subset of $L^2(\omega)$.

J.L. Lions (1968), J. Blum (1990), D. Tiba (1992),...

Existence of large solutions

$\Rightarrow$Non controllability (*Obstruction phenomenon*)

Díaz (1989, 90, 95, ...: Burger’s equation, semilinear equations,...)

The elliptic system:

$$-\Delta v = \theta, \quad -\Delta\theta = |\nabla v|^2 \quad \text{in } \Omega$$

Large solutions: $|v| = \infty, \; |\theta| = \infty$ on $\partial\Omega$

By maximum principle: $v = \infty, \; \theta = -\infty$ on $\partial\Omega$

Change of variables: $\phi = -\theta$

The problem:

$$\begin{cases} \Delta v = \phi, \quad \Delta\phi = |\nabla v|^2 \quad \text{in } \Omega \\ v = \infty, \quad \phi = \infty \quad \text{on } \partial\Omega \end{cases}$$

Some literature on *Large solutions*

Scalar equations:

Bieberbach (1916), Rademacher (1943), Keller, Osserman (1957), Pohozaev (1957), Loewner and Nirenberg (1974), Bandle and Marcus (1992),..., Veron (1992),..., G. Díaz and Letelier (1993), McKenna, Reichel and Walter (1997), Aftalion and Reichel (1997), Gacía Meilán, J. Sabina, J. López Gómez, A, Lazer, del Pino, Porru, Radulescu,...

Elliptic systems:

Cristea and Radulescu (2002), García-Meilán and Suarez (2003), García-Meilán and Rossi (2004), Ghergu and Radulescu (2004).

The problem in a ball:

$$(1) \qquad \begin{cases} \Delta v = \phi, \quad \Delta \phi = |\nabla v|^2 \quad \text{in } B_R^N(0) \\ v(x),\ \phi(x) \to \infty \quad \text{as } |x| \to R \end{cases}$$

$N \in \mathbb{N},\ R > 0$

Non-uniqueness:

v is determined only up to an additive constant;

w.l.o.g.: $v(0) = 0$

Scaling law:

(v_1, ϕ_1) solution of (1) with $R = R_1$

$\lambda > 0, \quad R_\lambda := R_1/\lambda$

$v_\lambda(x) := \lambda^2\, v_1(\lambda x), \quad \phi_\lambda(x) := \lambda^4\, \phi(\lambda x)$

$\Longrightarrow$ $(v_\lambda, \phi_\lambda)$ solution of (1) with $R = R_\lambda$

Radial solutions:

$v = v(r)\,,\ \phi = \phi(r)\,,\ r = |x|$

$$(2)\qquad \begin{cases} v'' + \frac{N-1}{r}\, v' = \phi \quad \text{in } (0,R) \\ \phi'' + \frac{N-1}{r}\, \phi' = |v'|^2 \quad \text{in } (0,R) \\ v'(0) = \phi'(0) = 0 \\ v(R^-) = \phi(R^-) = \infty \\ v(0) = 0 \quad \text{(w.l.o.g.)} \end{cases}$$

Cauchy problem:

Solving (2), for given $R > 0$, is equivalent to finding initial values $p \in \mathbb{R}$ such that the solution of

$$(3)\qquad \begin{cases} v'' + \frac{N-1}{r}\, v' = \phi, \quad \phi'' + \frac{N-1}{r}\, \phi' = |v'|^2 \\ v'(0) = \phi'(0) = 0, \quad v(0) = 0, \quad \phi(0) = p \end{cases}$$

exists on $[0,R)$ and "blows up" at R:

$$v(r),\ \phi(r) \to \infty \quad \text{as} \quad r \to R^-\,.$$

Scaling law:

- The solution with $\phi(0) = p > 0$ (or $\phi(0) = p < 0$) is a "rescaling" of the solution with $\phi(0) = 1$ (or $\phi(0) = -1$).
- One-one correspondence between initial values $p > 0$ (or $p < 0$) and exit times $R = R_p$: $R_p = R_1/p^{1/4}$ (or $R_p = R_{-1}/|p|^{1/4}$).

Consequence:

The system

$$\Delta v = \phi, \quad \Delta\phi = |\nabla v|^2 \quad \text{in } B_R^N(0)$$

has exactly one large radial solution with $v(0) = 0$ and $\phi(0) > 0$ (or $\phi(0) < 0$) if and only if the solution of the Cauchy problem (3) with $p = 1$ (or $p = -1$) blows up in finite time. In particular, there are at most two large radial solutions (with $v(0) = 0$).

Reduction to a first-order ODE system:

$$\begin{cases} v' = w, & v(0) = 0 \\ w' + \frac{N-1}{r}\, w = \phi, & w(0) = 0 \\ \phi' = \psi, & \phi(0) = p \\ \psi' + \frac{N-1}{r}\, \psi = w^2, & \psi(0) = 0 \end{cases}$$

Drop first equation and replace $N-1$ with a continuous parameter $\mu \geq 0$:

$$(4) \qquad \begin{cases} w' + \frac{\mu}{r}\, w = \phi, & w(0) = 0 \\ \phi' = \psi, & \phi(0) = p \\ \psi' + \frac{\mu}{r}\, \psi = w^2, & \psi(0) = 0 \end{cases}$$

Lemma:

$\mu \geq 0$, $p \in \mathbb{R}$, $p \neq 0$, (w, ϕ, ψ) maximal solution of (4) with interval of existence $[0, R)$

- ϕ is strictly increasing with limit $L \in (p, \infty]$.
- Either $L = 0$ and $R = \infty$ (no blow-up) or $L = \infty$ and $R < \infty$ (blow-up).
- In case of blow-up: $w(r),\ \phi(r),\ \psi(r) \to \infty$

Corollary:

For every $\mu \geq 0$, the solutions of the Cauchy problem (4) with $p > 0$ blow up in finite time.

Theorem 1:

For every $N \in \mathbb{N}$ and $R > 0$, the system

$$\Delta v = \phi, \quad \Delta \phi = |\nabla v|^2 \quad \text{in } B_R^N(0)$$

has a unique large radial solution with $v(0) = 0$ and $\phi(0) > 0$.

Question:

Given $\mu \geq 0$, do the solutions of the Cauchy problem (4) with $p < 0$ blow up in finite time?

Two possibilities:

- global existence and $\phi(r) \to 0$ as $r \to \infty$
- finite exit time R and $\phi(r) \to \infty$ as $r \to R$

Consequence:

finite-time blow-up if and only if ϕ crosses r-axis

Case $\mu = 0$:

$$\phi'' = w^2 \geq 0 \implies \phi \text{ concave up}$$
$$\implies \phi \text{ crosses } r\text{-axis}$$
$$\implies \text{blow-up!}$$

Numerical evidence:

ϕ crosses r-axis (and thus, blows up in finite time) if and only if $\mu < \mu^* \approx 13.755$!

Observation:

Given $\mu \geq 0$, solutions of (4) with $p < 0$ cross the r-axis if and only if one such solution crosses at $r = 1$ (scaling law).

That is: blow-up occurs if and only if the BVP

$$\begin{cases} w' + \frac{\mu}{r} w = \phi, & w(0) = 0 \\ \phi' = \psi, & \phi(1) = 0 \\ \psi' + \frac{\mu}{r} \psi = w^2, & \psi(0) = 0 \end{cases}$$

has a nontrivial solution.

Equivalent fixed-point problem:

$$(5) \qquad \begin{cases} w(r) = \int_0^r \left(\frac{s}{r}\right)^\mu \phi(s)\, ds \\ \phi(r) = -\int_0^r \psi(s)\, ds \\ \psi(r) = \int_0^r \left(\frac{s}{r}\right)^\mu w^2(s)\, ds \end{cases}$$

Proposition:

There exists $\mu^* > 9$ such that (5) has a unique nontrivial solution $u_\mu := (w_\mu, \phi_\mu, \psi_\mu)$ in $X := C([0,1], \mathbb{R}^3)$ for every $\mu \in [0, \mu^*)$. The function $\mu \mapsto u_\mu$ is continuous; its graph is unbounded in $\mathbb{R}_+ \times X$.

Conjecture/Fact:

The nontrivial solution of (5) exists if and only if $\mu < \mu^* \approx 13.755$.

Theorem 2:

For every $N \leq 10$ and $R > 0$, the system

$$\Delta v = \phi, \quad \Delta \phi = |\nabla v|^2 \quad \text{in } B_R^N(0)$$

has a unique large radial solution with $v(0) = 0$ and $\phi(0) < 0$.

Conjecture/Fact:

The solution of Theorem 2 exists if and only if $N \leq 14$.

Remark on a Dirichlet problem:

Solutions of the BVP (5) with $\mu = N - 1$ are in one-one correspondence with radial solutions of the Dirichlet problem

$$(6) \qquad \begin{cases} -\Delta v = \theta, \quad -\Delta\theta = |\nabla v|^2 \quad \text{in } B_1^N(0) \\ v = 0, \quad \theta = 0 \quad \text{on } \partial B_1^N(0) \end{cases}$$

Consequence:

- For every $N \in \mathbb{N}$, (6) has at most one nontrivial radial solution (necessarily positive and radially decreasing).
- Proved: This solution exists if $N \leq 10$.
- Numerical evidence: This solution exists if and only if $N \leq 14$.

Expected blow-up rates:

Fix $\mu \geq 0$ and $p^* \in \mathbb{R}$ such that the maximal solution (w^*, ϕ^*, ψ^*) of the Cauchy problem

$$(7) \qquad \begin{cases} w' + \frac{\mu}{r}\, w = \phi, & w(0) = 0 \\ \phi' = \psi, & \phi(0) = p \\ \psi' + \frac{\mu}{r}\, \psi = w^2, & \psi(0) = 0 \end{cases}$$

with $p = p^*$ blows up in finite time, say (w.l.o.g.) at $r = 1$.

Expected behavior as $r \to 1$:

$$w^*(r) \sim \frac{60}{(1-r)^3}, \quad \phi^*(r) \sim \frac{180}{(1-r)^4}, \quad \psi^*(r) \sim \frac{720}{(1-r)^5}$$

$f(x) \sim g(x)$ $\quad f, g : B_R^N(0) \to \mathbb{R}$ satisfy $f(x)/g(x) \to 1$ as $|x| \to R^-$.

Change of variables:

$$\alpha := \frac{(1-r)^3}{60}\, w, \quad \beta := \frac{(1-r)^4}{180}\, \phi, \quad \gamma := \frac{(1-r)^5}{720}\, \psi$$

$$(8) \quad \begin{cases} (1-r)\left(\alpha' + \frac{\mu}{r}\,\alpha\right) = 3(\beta - \alpha), & \alpha(0) = 0 \\ (1-r)\beta' = 4(\gamma - \beta), & \beta(0) = p/180 \\ (1-r)\left(\gamma' + \frac{\mu}{r}\,\gamma\right) = 5(\alpha^2 - \gamma), & \gamma(0) = 0 \end{cases}$$

$0 < p < p^*$ or $p^* < p < 0$:

Solution (α, β, γ) of (8) exists beyond $r = 1$ and approaches $\bar{0} := (0, 0, 0)$ as $r \to 1$.

$p > p^* > 0$ or $p < p^* < 0$:

Solution (α, β, γ) of (8) ceases to exist before $r = 1$ and approaches $\overline{\infty} := (\infty, \infty, \infty)$.

$p = p^*$:

Solution $(\alpha^*, \beta^*, \gamma^*)$ of (8) exists (and is eventually positive) on $[0, 1)$. To prove expected blow-up rates for (w^*, ϕ^*, ψ^*), we have to show that $(\alpha^*, \beta^*, \gamma^*)$ approaches $\bar{1} := (1, 1, 1)$ as $r \to 1$.

‘Nother change of variables:

$r = 1 - e^{-t}$ (singularity at $r = 1 \longrightarrow t = \infty$)

$a(t) = \alpha(r), \quad b(t) = \beta(r), \quad c(t) = \gamma(r)$

$$(9) \qquad \begin{cases} a' + \frac{\mu}{e^t - 1}\, a = 3(b - a), & a(0) = 0 \\ b' = 4(c - b), & b(0) = p/180 \\ c' + \frac{\mu}{e^t - 1}\, c = 5(a^2 - c), & c(0) = 0 \end{cases}$$

Properties of ODE system:

- autonomous for $\mu = 0$, asymptotically autonomous for $\mu > 0$
- "cooperative" (Kamke condition) for $a \geq 0$; comparison principle
- $\lambda \in [0, 1] \implies (\lambda, \lambda, \lambda)$ supersolution on $(0, \infty)$
- $\lambda \in (1, \infty) \implies (\frac{\lambda+1}{2}, \lambda, \lambda)$ subsolution on (τ, ∞) for some $\tau = \tau(\lambda, \mu) \geq 0$

Lemma 1:

Every unbounded nonnegative solution of the ODE system in (9) blows up in finite time and goes to $\overline{\infty}$.

Proof: Comparison arguments.

Lemma 2:

Every bounded nonnegative solution of the ODE system in (9) converges to either $\bar{0}$ or $\bar{1}$.

Proof for $\mu = 0$:

- System is autonomous, cooperative for $a \geq 0$, with negative divergence.
- Generates monotone, volume-contracting semiflow in $\mathbb{R}^3_+$, with hyperbolic equilibria at $\bar{0}$ (stable node) and $\bar{1}$ (saddle point with 2D stable manifold).
- By Hirsch's results on cooperative systems in $\mathbb{R}^3$ (1989, 1990), every bounded trajectory converges!

Proof for $\mu > 0$:

- System is asymptotically autonomous; limit system is the one with $\mu = 0$.
- General result by Markus on asymptotically autonomous ODE systems (1956): ω-limit sets of bounded solutions are non-empty, compact, connected, attracting, and *invariant under the semiflow of the limit system.*
- By Hirsch's results, the only such subsets of $\mathbb{R}^3_+$ are $\{\bar{0}\}$, $\{\bar{1}\}$, and the closure of the heteroclinic orbit connecting $\bar{1}$ to $\bar{0}$.
- General result by Mischaikow, Smith, and Thieme on asymptotically autonomous systems (1995): ω-limit sets of bounded solutions are also *chain-recurrent under the semiflow of the limit system.*
- This leaves only $\{\bar{0}\}$ and $\{\bar{1}\}$ as possible ω-limit sets of bounded nonnegative solutions!

Lemma 3:

The solution (a^*, b^*, c^*) of (9) with $p = p^*$ is bounded and does not converge to $\bar{0}$.

Proof: Comparison arguments; sub/supersolutions.

Corollary:

The solution (a^*, b^*, c^*) of (9) with $p = p^*$ converges to $\bar{1}$. Thus, the solution (w^*, ϕ^*, ψ^*) of (7) with $p = p^*$ has the expected asymptotic behavior.

Theorem 3:

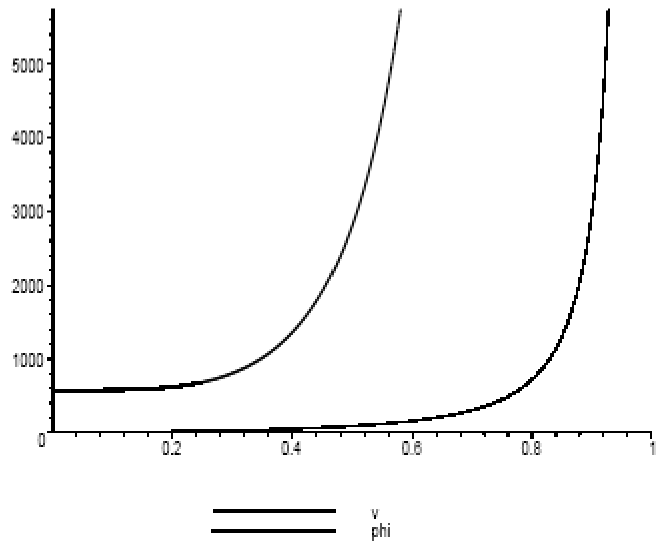
Given $N \in \mathbb{N}$ and $R > 0$, let (v, ϕ) be any large radial solution of the system

$$\Delta v = \phi, \quad \Delta \phi = |\nabla v|^2 \quad \text{in } B_R^N(0)\,.$$

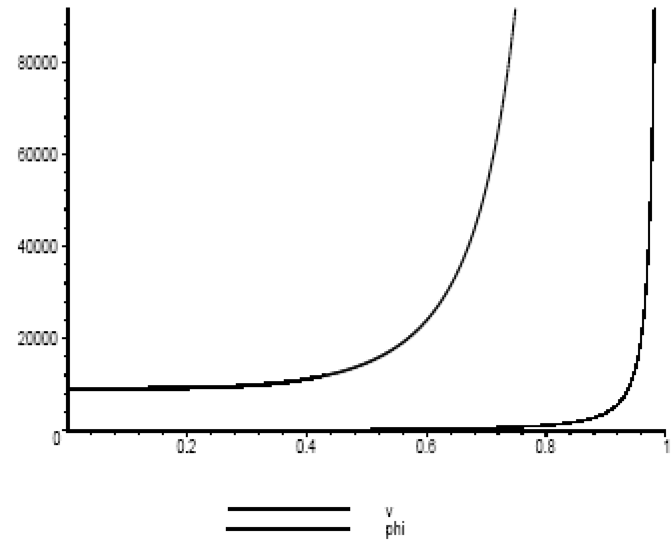
Then, as $|x| \to R$,

$$v(x) \sim \frac{30}{(R - |x|)^2} \quad \text{and} \quad \phi(x) \sim \frac{180}{(R - |x|)^4}\,.$$

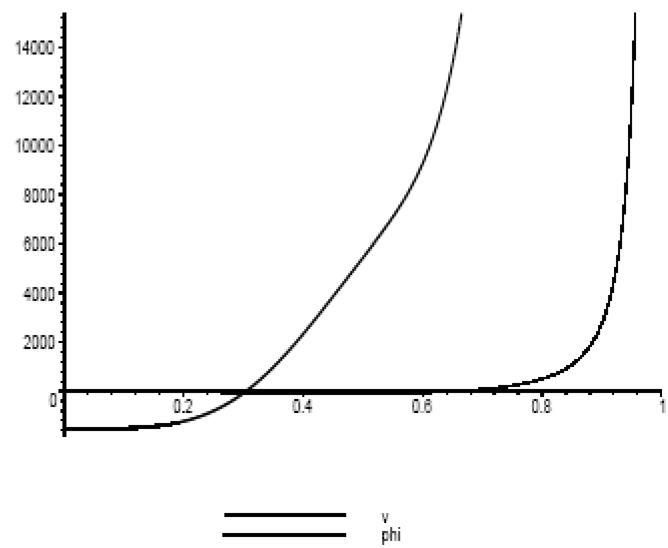
$N = 1$


5000
4000
3000
2000
1000
0
0.2
0.4
0.6
0.8
1
v
phi

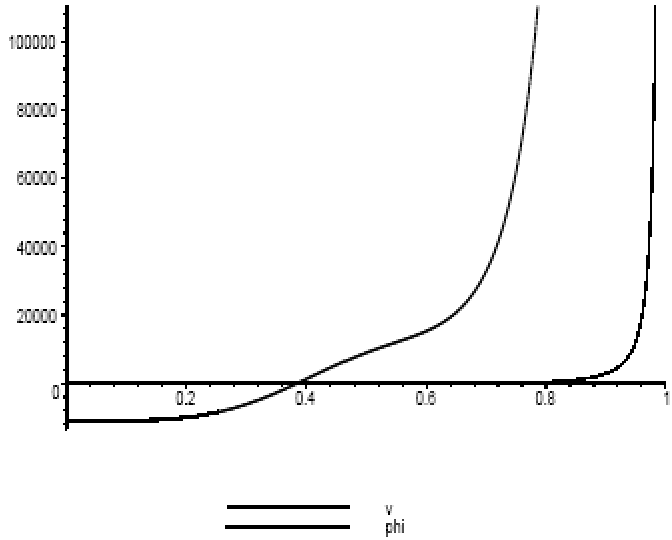
$N = 6$


80000
60000
40000
20000
0
0.2
0.4
0.6
0.8
1
v
phi

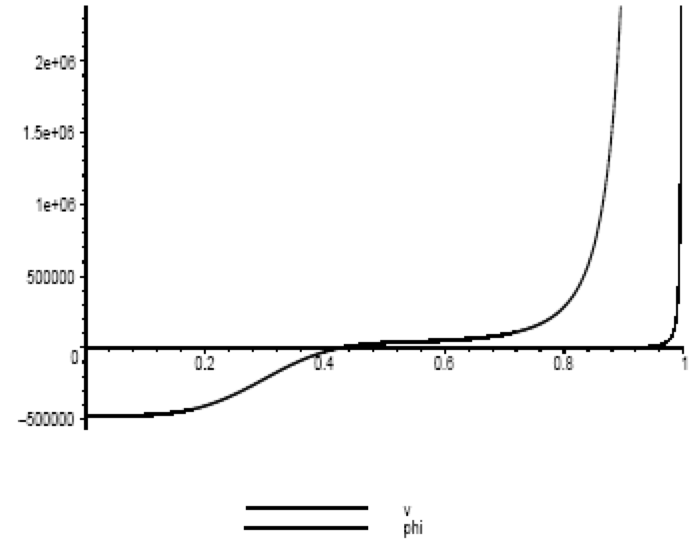
$N = 1$


14000
12000
10000
8000
6000
4000
2000
0
0.2
0.4
0.6
0.8
1
v
phi

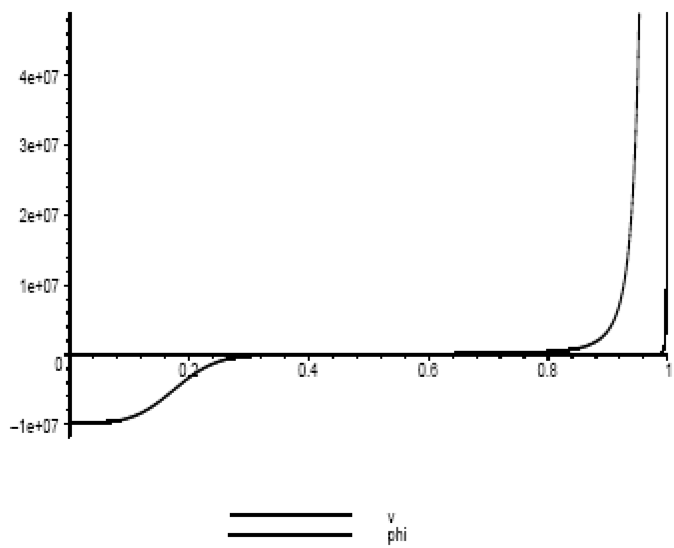
$N = 3$


100000
80000
60000
40000
20000
0
0.2
0.4
0.6
0.8
1
v
phi

$N = 10$


2e+06
1.5e+06
1e+06
500000
0
−500000
0.2
0.4
0.6
0.8
1
v
phi

$N = 14$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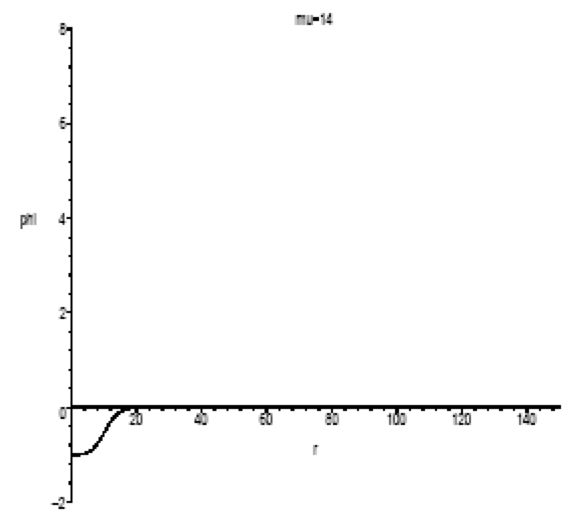
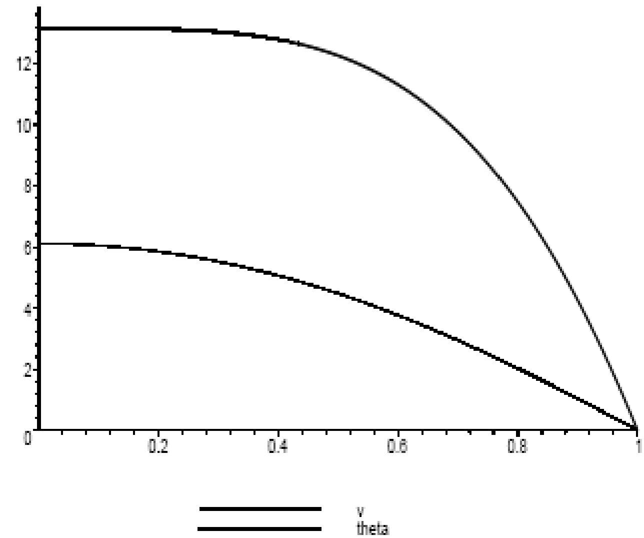
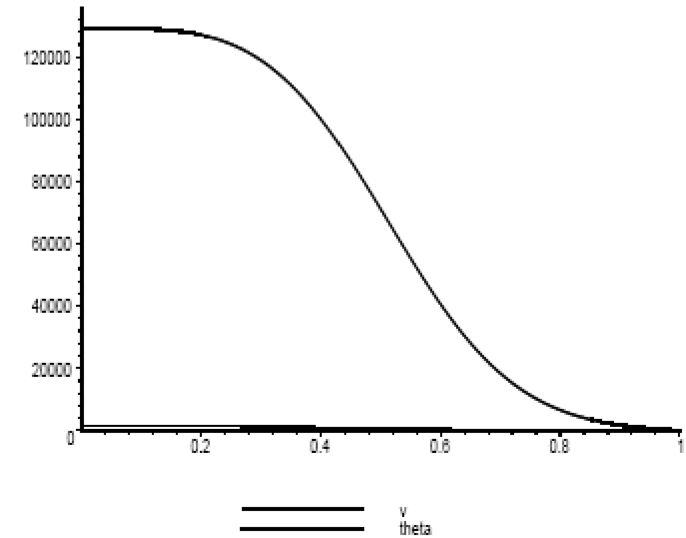
3e+07
2e+07
1e+07
0
−1e+07
0.2
0.4
0.6
0.8
1
v
phi


mu=14
8
6
4
2
0
−2
phi
20
40
60
80
100
120
140
r

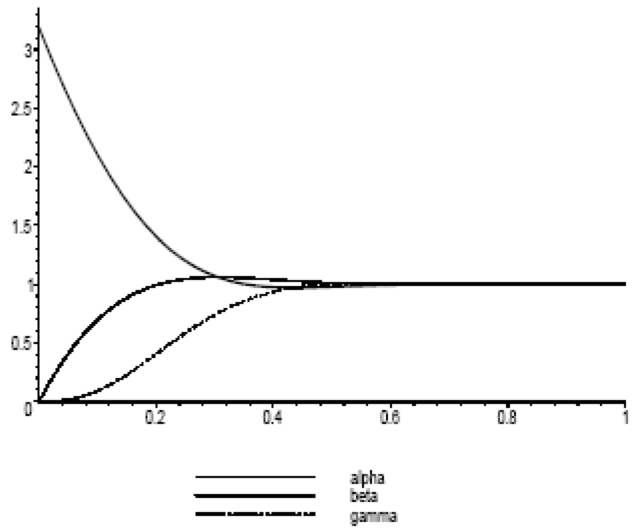
$N = 1$



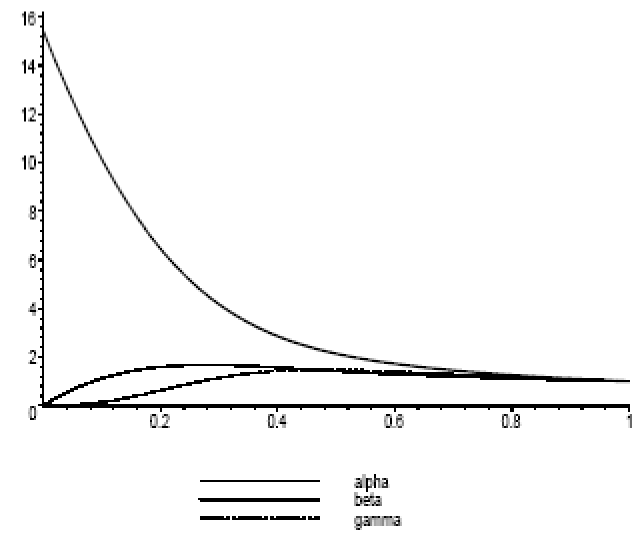
$N = 14$



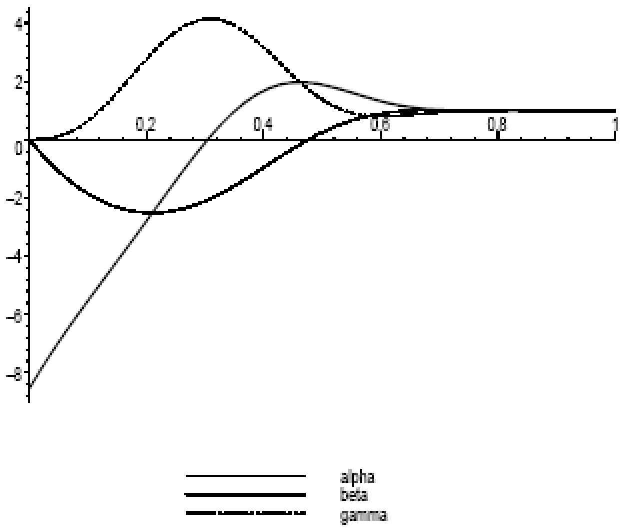
$\mu = 0,\ p/180 = 3.185739467$



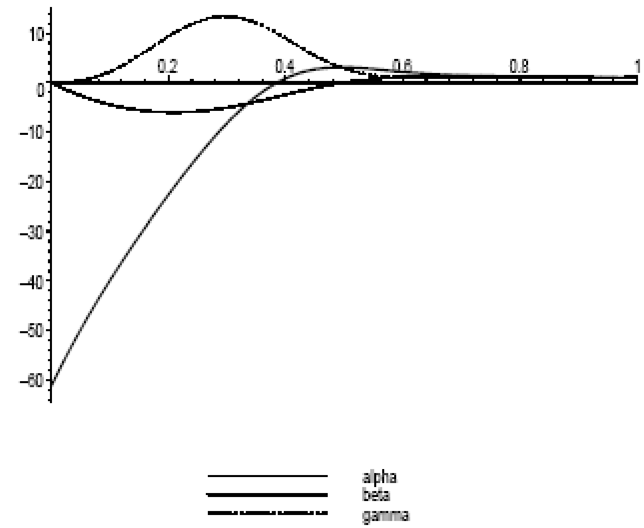
$\mu = 2,\ p/180 = 15.39874038$



$\mu = 0,\ p/180 = -8.525812707$



$\mu = 2,\ p/180 = -61.20559852$



3. Gobal solutions for a nonlinear heat capacity

$$v_t - \Delta v = \rho(\theta) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \qquad v = v_0 \quad \text{at } t = 0,$$

$$\rho(\theta)\,\theta_t - \Delta\theta = |\nabla v|^2 \quad \text{in } \Omega, \qquad \frac{\partial\theta}{\partial\hat{n}} = 0 \quad \text{on } \partial\Omega, \qquad \theta = \theta_0 \quad \text{at } t = 0,$$

Ω is a bounded domain in $\mathbf{R}^N$ $\qquad \rho(\theta) = \theta,$

Multiplying the equation for v by Δv, integrating over Ω, and using Hölder's inequality,

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 \le \int_\Omega \theta^2;$$

integrating the equation for θ,

$$\frac{d}{dt}\int_\Omega \theta^2 = 2\int_\Omega |\nabla v|^2.$$

Adding up,

$$\frac{d}{dt}\int_\Omega (|\nabla v|^2+\theta^2)+\int_\Omega |\Delta v|^2 \le 2\int_\Omega (|\nabla v|^2+\theta^2)$$

and thus,

$$\int_\Omega (|\nabla v|^2+\theta^2) \le e^{2t}\int_\Omega (|\nabla v_0|^2+\theta_0^2).$$

due to Poincare's inequality, it follows that $v \in L^\infty(0,T;H^1(\Omega))$

Integrating (7) with respect to t and using (8),

$$\int_0^t \int_\Omega |\Delta v|^2 \le e^{2t}\int_\Omega (|\nabla v_0|^2+\theta_0^2),$$

hence, $\Delta v \in L^2(0,T;L^2(\Omega))$

It follows that $v \in L^2(0,T;H^2(\Omega))$ and $v_t \in L^2(0,T;L^2(\Omega))$.

Thanks for your attention