

C.2.

**On the von Neumann problem and the
approximate controllability of
Stackelberg-Nash strategies for some
environmental problems**

J.I. Díaz

Lisbone, July, 14, 2004

1. Introduction

$$\frac{\partial \mathbf{y}}{\partial t} + A(\mathbf{y}) = \text{sources} + \text{sinks} + \text{actions}$$
$$\mathbf{y}(\cdot, 0) = \mathbf{y}_0(\cdot),$$

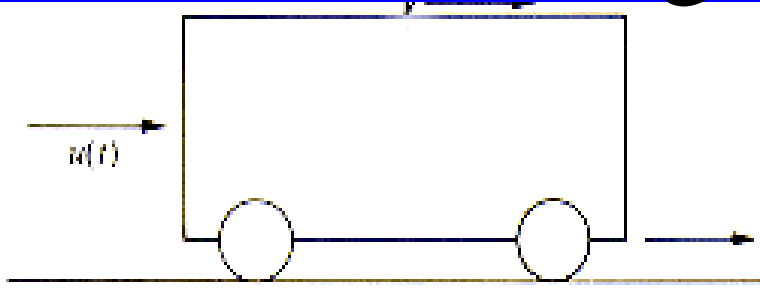
(“actions” = active controls=**unknowns**)

Suppose we are not satisfied with the initial state and that it would be “better” to be near (ideal target state) \mathbf{y}^T : **Given** $T > 0$, can we "drive the system" (by choosing the actions) in such a manner that $\mathbf{y}(\cdot, T)$ let as close as possible to \mathbf{y}^T ?

* a concrete \mathbf{y}^T : **inverse problem**

* \mathbf{y}^T arbitrary in a functional space X : **approximate controllability in X**

bang-bang controls



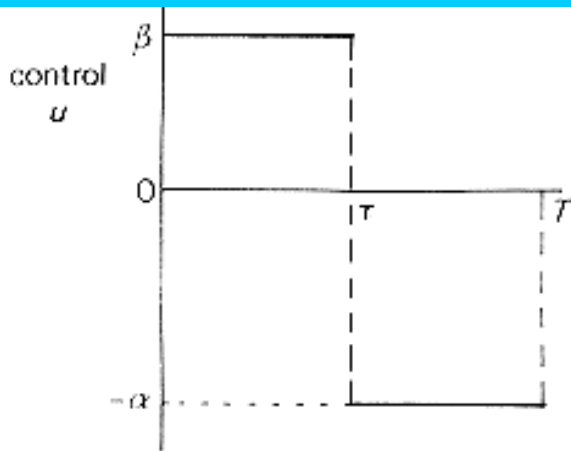
$$V(0)=0$$

$$V(T)=0$$

$$\text{Acceleration } u(t), \quad -\alpha \leq u(t) \leq \beta$$

$$v'(t) = u(t)$$

Problem: find $u(t)$ such that T minimum



ODE with discontinuous coefficients

C. Caratheodory (1873-1950),.....

Some examples of antropogenerated actions

Local actions: cloud seeding



tradicional sigue siendo objeto de controversia.

LOS EXPERIMENTOS DE SIEMBRA

de nubes con hielo seco fueron realizadas por Vincent Schaefer (izquierda, delante). Actualmente, los generadores acoplados en aviones ligeros (abajo) arrojan cristales de yoduro de plata.

EL PAÍS, jueves 13 de mayo de 1999

DEPORTES

El Parma se corona sin rival

El **Marsella** se rinde al primer contratiempo y acaba goleado en la final de la **Copa de la UEFA**

| | |
|----------|---|
| MARSELLA | 0 |
| PARMA | 3 |

Olimpico de Marsella: Porato; Blondeu, Issa, Blanc, Domoraud, Da Silva (Camara, m. 46); Brando, Bravo, Gourvenec, Pires; y Maurice.

Parma: Buffon; Thuram, Sensi, Cannavaro; Fuser, Dino Baggio, Boghossian, Vanoli; Verón (Fiore, m. 76); Chiesa (Balbo, m. 72) y Crespo.

Goles: 0-1. M. 26. Blanc cabecea pillado hacia su portero, Hernán Crespo adivina la cesión, se anticipa y bate a Porato por arriba.

0-2. M. 36. Vanoli ajusta un cabezazo al palo izquierdo tras un centro preciso de Fuser.

0-3. M. 55. Verón centra desde la derecha, Crespo deja pasar el balón y Chiesa fusila a la escuadra.

Árbitro: Dallas (Escocia). Mostró tarjeta amarilla a Blondeau.

65.000 espectadores en el estadio Luzhnikí de Moscú. Final de la Copa de la UEFA. Campeón, el Parma.

JOSÉ MIGUÉLEZ
No es el Parma un equipo voraz, de esos que siempre quieren más y más. Por eso la final de Moscú concluyó en 3-0. Simplemente en 3-0. La superioridad italiana fue mucho más grande que el resultado. Pero se sintió tan seguro, tan dueño de



Los jugadores del Parma celebran el triunfo con la Copa de la UEFA. / REUTERS

El alcalde de Moscú ordenó quitar las nubes

1972: Cambridge, EE.UU.



10:37 LST-16,100'



11:12 LST-14,250'
26 min. después de la inseminación



11:20 LST-16,100'
34 min. después de la inseminación

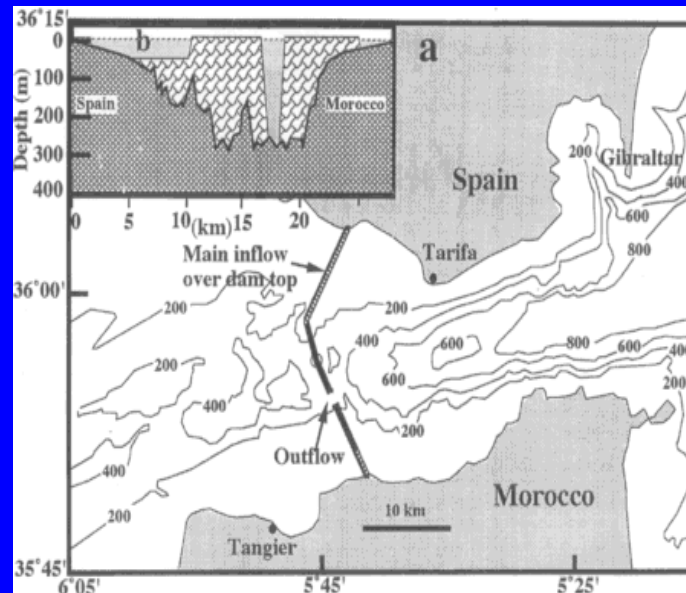
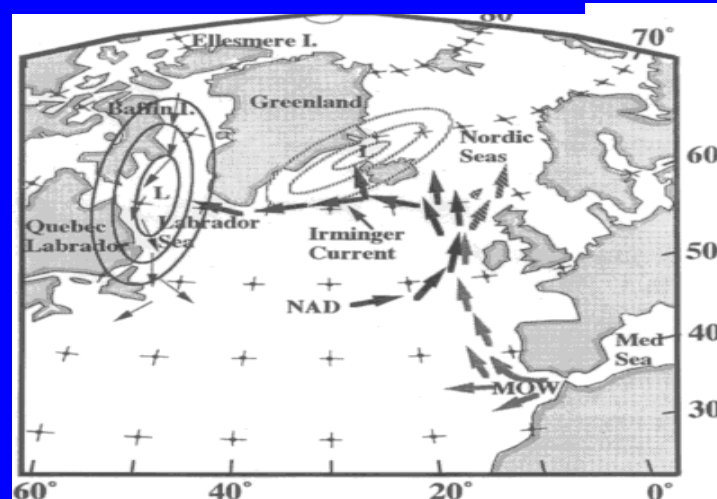


11:31 LST-16,200'
45 min. después de la inseminación

Global actions: a dam near Gibraltar

R.G. Thomson

American
Geophysical
Union, 1997



Milankovitch insolation
at 25°N decreases

African monsoons
weaken, Nile discharge
diminishes

Mediterranean hydrologic
deficit increases

Mediterranean salinity
rises, Outflow at
Gibraltar increases

Outflow water upwelling
off Scotland increases

Warm surface water
diverted to Labrador
Sea, cold upwelling
water enters Nordic Seas

Warmer Labrador Sea,
greater advection of
moisture into Canada

Canadian ice-sheet growth
begins as Nordic Seas
and Europe cool

The von Neumann problem



John von Neumann (1955): *Probably intervention in atmospheric and climate matters will come in a few decades, and will unfold on a scale difficult to imagine at present*

To act on the atmospheric climate by acting on the albedo : **the von Neumann problem**

A single “player” and a single control v (terminology of Control Theory and Games Theory).

In a first part of the lecture, I will report some recent results (Díaz, 2001) on a mathematical formulation (raised to me by J.L. Lions: fax of 15 may 1999).

Energy Balance Models, M.I. Budyko and W. D. Sellers, (1969):

$$y_t - \Delta y + By + C = QS(x)(\beta(y) + v(x)H(y)) \text{ in } (\mathcal{M} - \mathcal{I}) \times (0, T)$$

$$y = u_i \text{ on } \Gamma \times (0, T)$$

$$y(0, \cdot) = y_0(\cdot) \text{ on } (\mathcal{M} - \mathcal{I}),$$

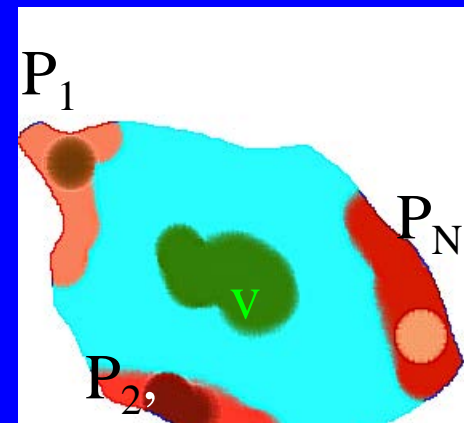
find $v(x)$ such that $y(T; v) = y^T$.

The second part of the lecture: a different problem, many players (nonnecessarily cooperating among them).

An example (an academic scenario) : to maintain “clean ” a resort lake), Ω . Now, the state of the system is $y = (y_1, \dots, y_n)$, $y_i(x, t)$ concentrations of various chemical products in the lake or of living organisms (solutions of a set of diffusion-convection equations).

Several local agents or local plants P_1, P_2, \dots, P_N .

Each plant decide its policy w_i . But, there is a different action v made by a representative authority or general manager (**leader**) in contrast to the rest of the players **followers**).



Formulation introduced in Economy by
H von Stackelberg (1934)



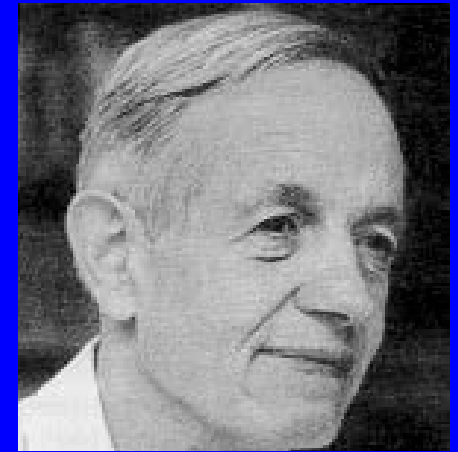
The general goal of the manager (control \mathbf{v}): to “drive the system” at time T , $\mathbf{y}(T:\mathbf{v})$, as close as possible to an ideal state \mathbf{y}^T . Each plant has (essentially) the same goal but it will be particularly careful to the state \mathbf{y} near its location.

Let ρ_i be a smooth function given in Ω .

$$\rho_i(x) \geq 0, \rho_i = 1 \text{ near } P_i.$$

Then P_i try to choose such that $\mathbf{y}(T:\mathbf{v})$, will be “close” to $\rho_i \mathbf{y}^T$ and to achieve this at minimum cost.

$$J_i(v; w_1, \dots, w_N) = \frac{1}{2} \|w_i\|^2 + \frac{\alpha_i}{2} \|\rho_i(\mathbf{y}(\cdot, T) - \mathbf{y}^T)\|^2,$$



The “local” controls w_1, w_2, \dots, w_N assume that the leader has made a choice v and try to find an equilibrium (J.F.Nash,1951) of their cost J_i , i.e., one looks for w_1, w_2, \dots, w_N (as functions of v) such that

$$\left. \begin{aligned} J_i(v; w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_N) &\leq J_i(v; w_1, \dots, w_{i-1}, \widehat{w}_i, w_{i+1}, \dots, w_N) \\ , \forall \widehat{w}_i, \text{ for } i &= 1, \dots, N. \end{aligned} \right\}$$

The leader v wants now that the global state (i.e. the state $y(.,T)$ in the whole Ω) let as close as possible from y^T

But,..., not always there exists a Nash equilibrium. We shall prove later some sufficient conditions for the existence and uniqueness of it.

We shall show is that if there is existence and uniqueness of a Nash equilibrium for the followers, then the leader can control the system (in the sense of approximate controllability).

Remark.

Stackelberg's type strategy is not the only one possible! One could also replace the Nash equilibrium by a Pareto equilibrium for the followers (see, for instance, J.L. Lions (1986)). Here all the controls w_i agree to work in a strategy where v is the leader, and they agree to work in the context of a Nash equilibrium. Their personal (selfish) interest are expressed in the cost functions J_i as we shall see later.

Remark. Rio (1992), Agenda 21, International Scientific and Technological Community, IPGC,...

Remark. A general reference:

J.L. Lions, *El planeta Tierra*, Espasa, Madrid, 1992

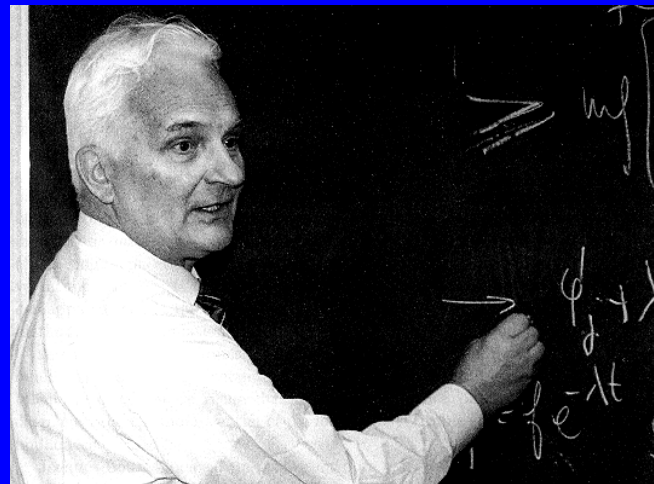
The plan of the rest of this part of the course is the following:

2. The von Neumann problem

J.I. Díaz: Modelos matemáticos en Climatología; la conjetura de von Neumann, In *Les Mathematiques y les seus aplicacions*, Editorial de la UPV, Valencia, 2001, pp. 67-98

3. The Stackelberg-Nash strategies: the optimal leader action.

J. I. Díaz and J.L. Lions, On the Approximate Controllability of Stackelberg-Nash Strategies. In *Ocean Circulation and Pollution Control: A Mathematical and Numerical Investigation (Proceedings of the Diderot Video-conference Amsterdam-Madrid-Venice, December, 1999)* Springer Verlag, 2004 (J.I. Díaz, ed.) pp.17-28.



2. The von Neumann problem

Energy Balance Models have a diagnostic character and intended to understand the evolution of the global climate on along time scale.

Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters

Used in the study of the Milankovitch theory of the ice-ages

(R.North, 1984)

The distribution of temperature $y(x,t)$ is expressed pointwise after a standard average process, where the spatial variable x is in the Earth's surface

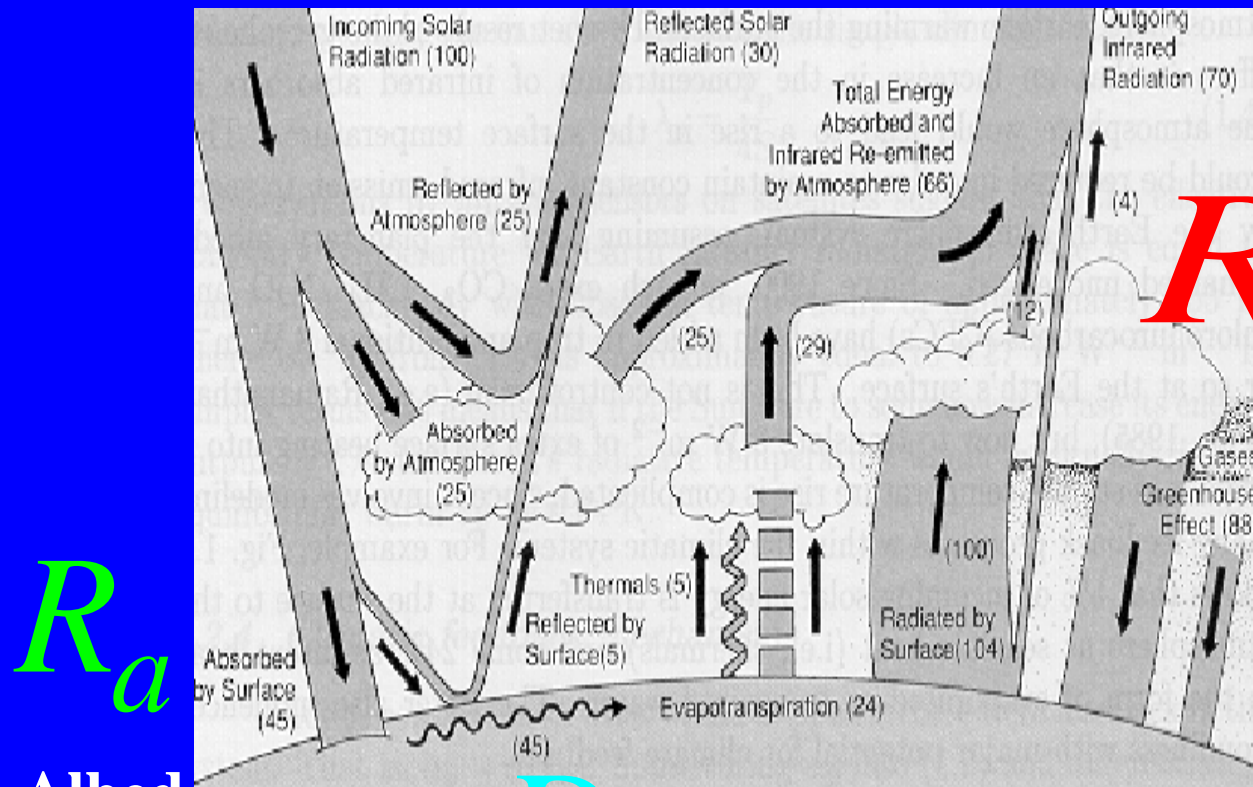
$$y(x,t) = \frac{1}{2\tau |B(x)|} \int_{t-\tau}^{t+\tau} \int_{B(x)} T(y,s) dy ds$$

S. Arrhenius (1896),

W.D. Sellers(1969), M.I. Budyko (1969),....

The Energy Balance on the Earth's surface:

$$c \frac{\partial y}{\partial t} = R_a - R_e + D$$



R_a
Albedo

D

R_e

Greenhouse effect

Constitutive laws

$$R_a = QS(x)\beta(u)$$

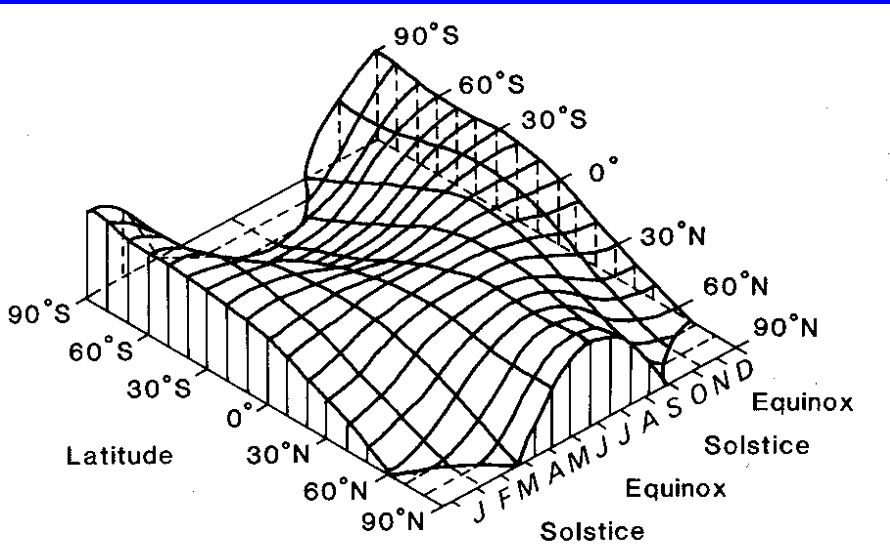


Fig. 2.8. The variation of insolation (at the top of the atmosphere) as a function of

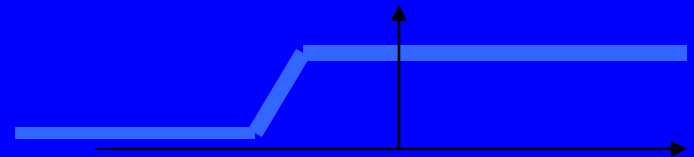
$$\beta(u) = (1 - a(u)) \text{ coalbedo}$$

$$\beta(u) = \begin{cases} 0.38 & \text{if } u \ll -10 \\ 0.71 & \text{if } u \gg -10 \end{cases}$$

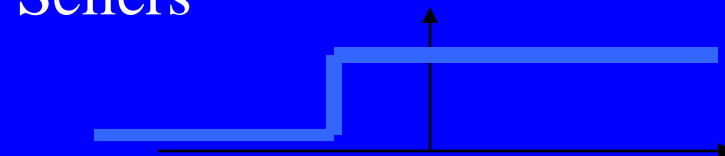


Eye in the sky. The Earth Radiation Budget Satellite, a space age effort to measure the radiation balance between the earth and the sun.

Earth Radiation Budget Satellite



Sellers



Budyko $u = -10^\circ$

On R_e :

Sellers

$$R_e = \sigma u^4 \text{ Stefan-Boltzman}$$

Budyko

$$R_e = A + Bu \text{ Newton}$$

Empirical relation, Depends of the greenhaus gases, anthropogénicos changes,... (internal variables)

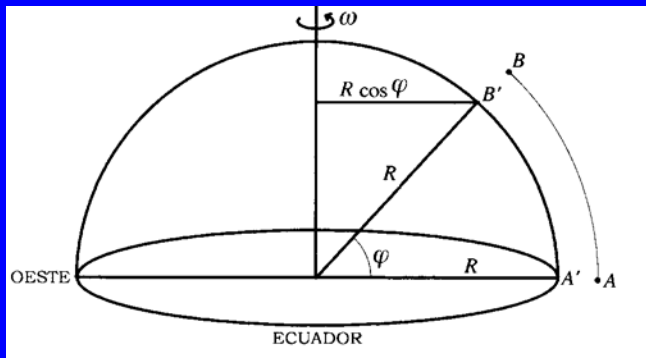
On the diffusion operator, D : a jerarchy.

0-dimensional model $D=0$

$$c \frac{du}{dt} = Q\beta(u) - R_e(u)$$

1-dimensional model

$$D = \frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} \left(k \cos \varphi \frac{\partial u}{\partial \varphi} \right) = \frac{\partial}{\partial x} \left(k(1-x^2) \frac{\partial u}{\partial x} \right)$$

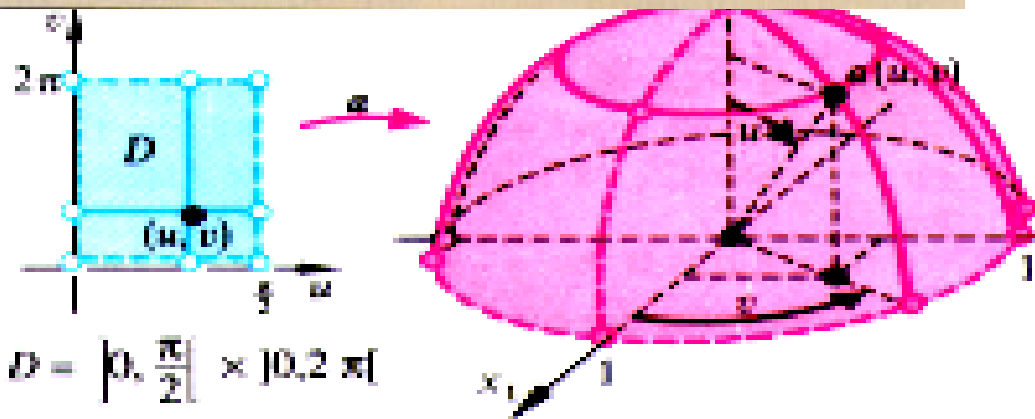
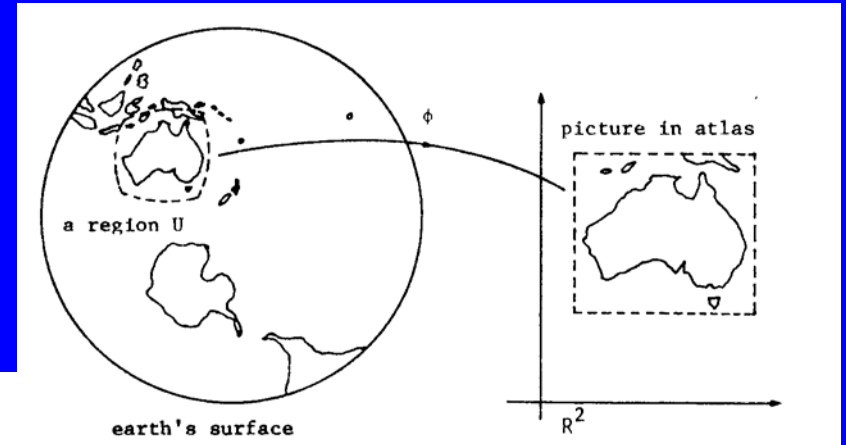
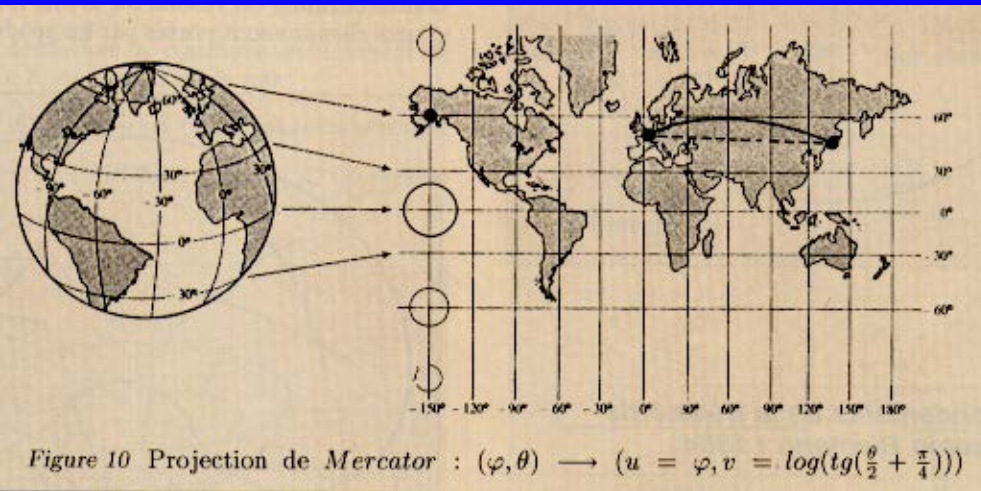


$$x = \cos \varphi$$

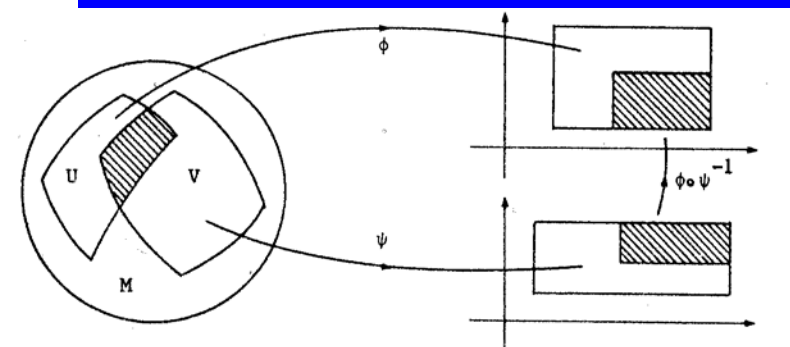
In planetary scales $O(10^4 \text{ Km})$ the velocity is eliminated by using the eddy diffusive approximation

Bidimensional model

$$D = \operatorname{div}(k(x)\nabla u)$$



$$\alpha(u, v) = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ 1 \end{pmatrix}$$



$$c \frac{\partial y}{\partial t} = Q\beta(y) + R_e(x, y) + \operatorname{div}(k(x)\nabla y)$$

$$y(x, t_0) = y_0(x)$$

Mathematical treatment (North, Hetzer, Drazin, D, Tello, Badii, Arcoya, Hernandez, ...)

J.I. Díaz and J. L. Lions (eds.), *Mathematics, Climate and Environment*, Masson, Paris, 1993.

J.I. Díaz and J. L. Lions (eds.), *Environment, Economics and Their Mathematical Models*, Masson, Paris, 1994.

J.I. Díaz (ed), *The Mathematics of Models for Climatology and Environment*, NATO ASI Series, Springer Verlag, 1997.

R. Dautray and J.I. Díaz, (2004)

The von Neumann conjecture

Artificial modification of the coalbedo near the ice cups.

To simplify the exposition

$$R_e(x,r) = Br + C, \quad B > 0.$$

x-dependent Sellers type coalbedo function (Henderson-Sellers).

$$\beta(x, u) = \begin{cases} \beta_i(x) & u < u_i, \\ \beta_i(x) + \left(\frac{u - u_i}{u_w - u_i}\right)(\beta_w(x) - \beta_i(x)) & u_i \leq u \leq u_w, \\ \beta_w(x) & u > u_w, \end{cases}$$

The modification proposed by von Neumann corresponds to

$$\beta(x, u: v) = \beta(u) + v(x)\chi_\omega(x)H(u)$$

$$H(u) = \begin{cases} 0 & u < u_i, \\ \left(\frac{u - u_i}{u_w - u_i}\right) & u_i \leq u \leq u_w, \\ 1 & u > u_w. \end{cases}$$

We assume that the modification is out side the ice cups

$$\omega = \mathcal{M} - \mathcal{I}.$$

$$\begin{aligned} y_t - \Delta y + By + C &= QS(x)(\beta(y) + v(x)H(y)) && \text{in } (\mathcal{M} - \mathcal{I}) \times (0, T), \\ y &= u_i && \Gamma \times (0, T), \\ y(0, \cdot) &= y_0(\cdot) && \text{on } (\mathcal{M} - \mathcal{I}), \end{aligned}$$

Inverse Problem: find $v(x)$ such that $y(T;v)=y^T$

Theorem

Let $y_0 \in C^{2,\alpha}(\overline{\mathcal{M} - \mathcal{I}})$, $y_0 = u_i$ on Γ and

$$\Delta y_0 - B y_0 - C + Q S(x) \beta(y_0) \leq 0 \quad \mathcal{M} - \mathcal{I}.$$

Let $y^T \in C^{2,\alpha}(\overline{\mathcal{M} - \mathcal{I}})$, $y^T = u_i$ on Γ and

$$\Delta y(T : 0) \leq \Delta y^T \quad \mathcal{M} - \mathcal{I}.$$

$$\Delta y^T \leq \Delta y(T : 0) + Q S(x) (\beta_w - \beta_i) (H(y(T : 0)) + \delta) \quad \mathcal{M} - \mathcal{I}.$$

Then, there exists

$v \in C^{0,\alpha}(\overline{\mathcal{M} - \mathcal{I}})$ with $v(x) \in [-\mu, 0]$, $\forall x \in (\mathcal{M} - \mathcal{I})$

such that $y(T:v)=y^T$ on $\mathcal{M}-\mathcal{I}$

Remark. To decrease the temperature

$$v(x) \in [-\mu, 0], \quad x \in \mathcal{M} - \mathcal{I}, \quad \mu = \beta_w - \beta_i.$$

and so, $y(T : -\mu) \leq y^T \leq y(T : 0)$ on $(\mathcal{M} - \mathcal{I})$.

Idea of the proof.

The existence of $\mathbf{v}(\mathbf{x})$ is reduced to find a fixed point for

$$(\mathcal{T}v)(x) = \frac{y_t(T : v) + By(T : v) + C - \beta(y(T : v)) - \Delta y_d + QS(x)(v(\gamma y(T : v) + \delta))}{QS(x)(H(y(T : v)) + \gamma y(T : v) + \delta)}$$

$$\gamma > \frac{\beta_w - \beta_i}{u_w - u_i}, \quad \delta > \gamma(-u_i)$$

T is a decreasing operator on a suitable “interval”: Amann fixed point theorem on Hölder spaces.

3. The Stackelber-Nash strategies for the approximate controllability

Let A be a second order elliptic operator

$$A\varphi = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} (a_{i,j}(x) \frac{\partial \varphi}{\partial x_j}) + \sum_{i=1}^N a_i(x) \frac{\partial \varphi}{\partial x_i} + a_0(x) \varphi$$

with smooth coefficients and

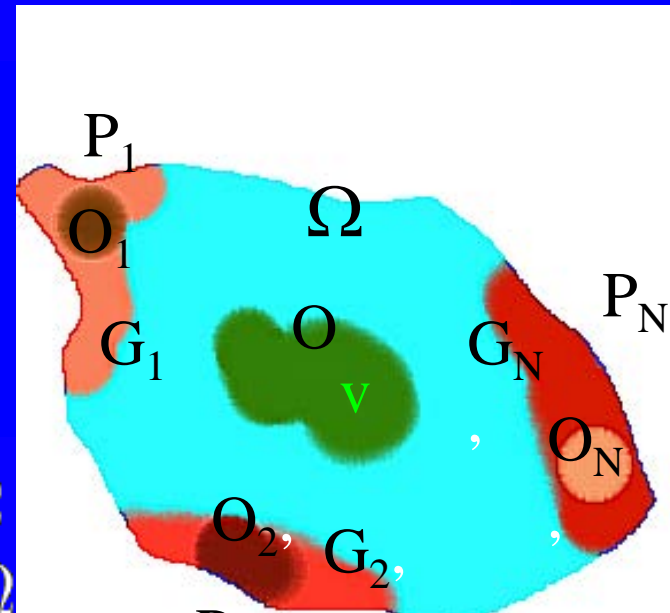
$$\sum_{i,j=1}^N a_{i,j}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^N \xi_i^2, \quad \alpha > 0, \quad x \in \overline{\Omega}.$$

We assume the state equation

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^N w_i \chi_i$$

χ is the characteristic function of $\mathcal{O} \subset \Omega$

χ_i is the characteristic function of $\mathcal{O}_i \subset \Omega$



Remark. The control of the leader is distributed in O and the control of the follower i is distributed in O_i .

Remark. Since the system is linear there is no restriction in assuming the initial state, sources and sinks to be zero.

We assume that the boundary conditions $y = 0$ on $\partial\Omega \times (0, T)$.

We introduce

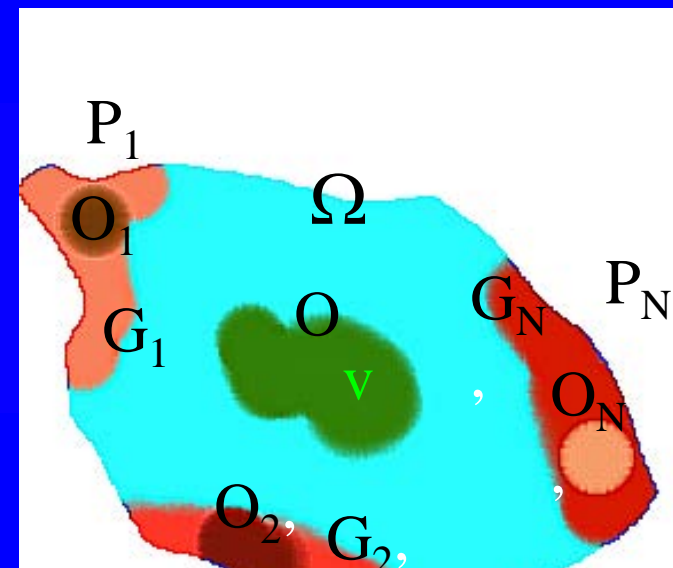
$$\left. \begin{array}{l} \rho_i \in L^\infty(\Omega), \rho_i \geq 0, \\ \rho_i = 1 \text{ in a domain } G_i \subset \Omega. \end{array} \right\}$$

G_i represents the region of the lake the plant P_i is particularly interested (the place near P_i for instance!). If P_i is selfish, then $\rho_i = 0$ outside.

We define the cost function J_i

$$J_i(v; w_1, \dots, w_N) = \frac{1}{2} \int_0^T \int_{O_i} w_i^2 dx dt + \frac{\alpha_i}{2} \left\| \rho_i y(T; v, \mathbf{w}) - \rho_i y^T \right\|^2,$$

$\|\cdot\|$ is the norm in $L^2(\Omega)$



We assume

$$v \in L^2(\mathcal{O} \times (0, T)), w_i \in L^2(\mathcal{O}_i \times (0, T))$$

Given $v \in L^2(\mathcal{O} \times (0, T))$, we now define

$$\mathbf{w} = \{w_1, \dots, w_N\}, \text{ a Nash equilibrium for the cost functions } J_1, \dots, J_N .$$

This Nash equilibrium can be defined as a function of \mathbf{v} :

$$\mathbf{w} = \mathbf{w}(v) \text{ or } w_i = w_i(v), \quad i = 1, \dots, N.$$

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^N w_i(v)\chi_i$$

The initial problem for this system admits a unique solution $y(\mathbf{x}, t; \mathbf{v}, \mathbf{w}(\mathbf{v}))$

Theorem. Assume that there exists a unique Nash equilibrium. Then, when \mathbf{v} spans $L^2(\mathcal{O}) \times (0, T)$, the functions $\mathbf{y}(\mathbf{x}, t; \mathbf{v}, \mathbf{w}(\mathbf{v}))$ describe a dense subset of $L^2(\Omega)$. In other words, there is approximate controllability when a Stackelberg-Nash strategy is followed.

A non constructive proof of the theorem.

The Nash equilibrium is characterized by

$$\int_0^T \int_{\mathcal{O}_i} w_i \widehat{w}_i dx dt + \alpha_i \int_{\Omega} \rho_i^2 (y(T; v, \mathbf{w}) - y^T) \widehat{y}_i(T) dx = 0 \quad \forall \widehat{w}_i,$$

where

$$\left. \begin{aligned} \frac{\partial \widehat{y}_i}{\partial t} + A \widehat{y}_i &= \widehat{w}_i \chi_i, \\ \widehat{y}_i(0) &= 0, \text{ in } \Omega, \quad \widehat{y}_i = 0 \text{ in } \partial\Omega \times (0, T). \end{aligned} \right\}$$

We introduce the adjoint state \mathbf{p}_i
$$\left. \begin{aligned} -\frac{\partial p_i}{\partial t} + A^* p_i &= 0 \text{ in } \Omega \times (0, T), \\ p_i(x, T) &= \rho_i^2(x) (y(x, T; v, \mathbf{w}) - y^T(x)) \text{ in } \Omega, \\ p_i &= 0 \text{ in } \partial\Omega \times (0, T), \end{aligned} \right\} \quad A^* \text{ adjoint of } A.$$

Multiplying and integrating by parts

$$\int_{\Omega} \rho_i^2 (y(T; v, \mathbf{w}) - y^T) \widehat{y}_i(T) dx = \int_0^T \int_{\Omega} p_i \widehat{w}_i \chi_i dx dt,$$

so that

$$\int_0^T \int_{O_i} (w_i + \alpha_i p_i) \widehat{w}_i dx dt = 0, \quad \forall \widehat{w}_i,$$

$$\text{i.e., } w_i + \alpha_i p_i \chi_i = 0.$$

We want to show that the set described by $\mathbf{y}(\mathbf{x}, t; \mathbf{v}, \mathbf{w}(\mathbf{v}))$ is dense.

By the Hahn-Banach theorem, it suffices to show that if \mathbf{f} is given in $L^2(\Omega)$, and we assume that

$$(y(\cdot, T; v), f) = 0, \quad \forall v \in L^2(\Omega) \text{ then, } \mathbf{f} = \mathbf{0}.$$

Let us define the auxiliary adjoint problem

$$\left. \begin{aligned} -\frac{\partial \varphi}{\partial t} + A^* \varphi &= 0, \\ \frac{\partial \psi_i}{\partial t} + A \psi_i &= -\alpha_i \varphi \chi_i, \\ \varphi(T) &= f + \sum_i \psi_i(T) \rho_i^2, \\ \psi_i(0) &= 0, \\ \varphi = 0, \psi_i = 0 &\text{ in } \partial\Omega \times (0, T). \end{aligned} \right\}$$

Multiplying the first (resp. the second) equation by \mathbf{y} (resp. \mathbf{p}_i) we get

$$\begin{aligned} & -(f + \sum_i \psi_i(T) \rho_i^2, \mathbf{y}(T)) + \\ & \int_0^T \int_{\Omega} \varphi \left(\frac{\partial \mathbf{y}}{\partial t} + A \mathbf{y} \right) dx dt + \\ & \sum_i (\psi_i(T), \mathbf{p}_i(T)) + \\ & + \sum_i \int_0^T \int_{\Omega} \psi_i \left(-\frac{\partial \mathbf{p}_i}{\partial t} + A^* \mathbf{p}_i \right) dx dt \\ & = - \sum_i \alpha_i \int_0^T \int_{\Omega} \varphi \mathbf{p}_i \chi_i dx dt. \end{aligned}$$

Then

$$-(f, y(T)) + \int_0^T \int_{\Omega} \varphi v \chi dx dt = 0.$$

Therefore $\varphi = 0$ on $\mathcal{O} \times (0, T)$.

Using Mizohata's Unique Continuation theorem

$$\varphi = 0 \text{ on } \Omega \times (0, T).$$

and $\psi_i = 0$ in $\partial\Omega \times (0, T)$ imply that

$$\psi_i = 0 \text{ in } \Omega \times (0, T), \forall i = 1, \dots, N,$$

so that $f = 0$ and the proof ends.

A criterion for the existence and uniqueness of Nash equilibria:

Let

$$\left. \begin{aligned} \mathcal{H}_i &= L^2(\mathcal{O}_i \times (0, T)), \\ \mathcal{H} &= \prod_{i=1}^N \mathcal{H}_i, \\ L_i \widehat{w}_i &= \widehat{y}_i(T) \end{aligned} \right\} \text{which defines } L_i \in L(\mathcal{H}_i; L^2(\Omega)).$$

Since \mathbf{v} is fixed, one can write

$$y(T; v, \mathbf{w}) = \sum_{i=1}^N L_i w_i + z^T, \quad z^T \text{ fixed.}$$

With these notations we can rewrite

$$J_i(v; \mathbf{w}) = \frac{1}{2} \|w_i\|_{\mathcal{H}_i}^2 dxdt + \frac{\alpha_i}{2} \left\| \rho_i \left(\sum_j L_j w_j - \eta^T \right) \right\|^2$$

$\eta^T = y^T - z^T$. Then $\mathbf{w} \in \mathcal{H}$ is a *Nash equilibrium* iff

$$\begin{aligned} (w_i, \widehat{w}_i)_{\mathcal{H}_i} + \alpha_i (\rho_i (\sum_j L_j w_j - \eta^T), \rho_i L_i \widehat{w}_i) &= 0. \\ \forall i = 1, \dots, N, \quad \forall \widehat{w}_i \end{aligned}$$

Or, equivalently

$$w_i + \alpha_i L_i^* \left(\rho_i^2 \sum_{j=1}^N L_j w_j \right) = \alpha_i L_i^* (\rho_i^2 \eta^T), \forall i = 1, \dots, N,$$

$L_i^* \in \mathcal{L}(L^2(\Omega); \mathcal{H}_i)$ is the adjoint of L_i

Or, vectorially

$$\left. \begin{aligned} \mathbf{L} \mathbf{w} &= \text{given in } \mathcal{H} \\ \mathbf{L} &\in \mathcal{L}(\mathcal{H}; \mathcal{H}) \\ (\mathbf{L} \mathbf{w})_i &= w_i + \alpha_i L_i^* \left(\rho_i^2 \sum_{j=1}^N L_j w_j \right). \end{aligned} \right\}$$

Proposition. Assume that $\alpha = \alpha_i \quad \forall i$ and

$\alpha \|\rho_i - \rho_j\|_{L^\infty(\Omega)}^2$ is small enough, for any $i, j = 1, \dots, N$.

Then \mathbf{L} is invertible. In particular there is a unique Nash equilibrium

Proof. We can write

$$\begin{aligned} (\mathbf{L}\mathbf{w}, \mathbf{w}) &= \sum_{i=1}^N \|w_i\|_{\mathcal{H}_i}^2 + \alpha \left\| \sum_{i=1}^N \rho_i L_i w_i \right\|^2 \\ &+ \alpha \sum_{i,j=1}^N (\rho_i - \rho_j)^2 (L_j w_j, L_i w_i) \end{aligned}$$

Then, by applying Young's inequality

$$(\mathbf{L}\mathbf{w}, \mathbf{w}) \geq \gamma \|\mathbf{w}\|_{\mathcal{H}}^2, \text{ for some } \gamma > 0.$$

The conclusion is now consequence of the Lax-Milgram theorem.

Remark. It is possible to show (see Díaz-J.L. Lions (1999)) that the assumptions are optimal in some suitable sense.

Remark: There are infinite controls \mathbf{v} leading to the approximate controllability.

The optimal leader action: a constructive proof . Given $\delta > 0$

we want to find the best leader control \mathbf{v} in the sense that

$$\inf_{v \in L^2(\omega \times (0, T))} \left\{ \frac{1}{2} \int_{O \times (0, T)} |v|^2 dx dt, \quad y(T, v) \in y^T + \delta B_{L^2(\Omega)} \right\}, \quad \text{B unit ball.}$$

Theorem. i) The minimum \mathbf{v} is given by $v = \varphi \chi$

from the unique solution $\{y, p_1, \varphi, \psi_1\}$ of the Optimality System

$$\frac{\partial y}{\partial t} + Ay + \sum_{i=1}^N \alpha_i p_i \chi_i = \varphi \chi,$$

$$-\frac{\partial p_i}{\partial t} + A^* p_i = 0$$

$$-\frac{\partial \varphi}{\partial t} + A^* \varphi = 0,$$

$$\frac{\partial \psi_i}{\partial t} + A \psi_i = -\alpha_i \varphi \chi_i$$

$$\varphi(T) = f + \sum \psi_i(T) \rho_i^2, \quad p_i(T) = \rho_i^2 (y(T) - y^T),$$

$$y(0) = 0, \quad \psi_i(0) = 0$$

$$y = p_i = \varphi = \psi_i = 0 \text{ on } \Sigma$$

with \mathbf{f} given by the minimization problem

$$\inf \{I(f) : f \in L^2(\Omega)\},$$

$$I(f) = \frac{1}{2} \int_{O \times (0,T)} |\varphi|^2 dxdt + \delta \left\| \widehat{f} \right\|_{L^2(\Omega)} - \int_{\Omega} f y^T dx$$

ii) The minimization dual problem has a unique solution

Idea of the Proof. i) Let

$$F(v) = \frac{1}{2} \int_{O \times (0,T)} |v|^2 dxdt$$

$$G(f) = \left\{ \begin{array}{ll} 0 & \text{if } f \in y^T + \delta B_{L^2(\Omega)} \\ +\infty & \text{otherwise on } L^2(\Omega) \end{array} \right\}$$

Then, an equivalent formulation is

$$\inf_{v \in L^2(O \times (0,T))} (F(v) + G(Lv)), \quad \text{where } Lv = y(T:v).$$

By Fenchel and Rockafellar's (1967) duality

$$\inf_{v \in L^2(O \times (0, T))} (F(v) + G(Lv)) = - \inf_{f \in L^2(\Omega)} (F^*(L^* f) + G^*(-f))$$

where L^* is the adjoint operator and F^* the conjugate function

$$F^*(\varphi) = \sup_{\hat{\varphi}} ((\varphi, \hat{\varphi}) - F(\hat{\varphi})).$$

But, $F^*(\varphi) = \varphi$, $G^*(f) = \delta \left\| \hat{f} \right\|_{L^2(\Omega)} + \int_{\Omega} f y^T dx$, and $L^* f = \varphi \chi$

which gives conclusion i). The proof of ii) comes from the fact that $I(f)$ is strictly convex, continuous and coercive (by the Unique Continuation theorem).

Remark. f is characterized as the unique solution of the Variational Inequality

$$(y(T; f) - y^T, \hat{f} - f) + \delta \left\| \hat{f} \right\| - \delta \|f\| \geq 0, \\ \forall \hat{f} \in L^2(\Omega).$$

Remark: Non linear state equations 1. Sublinear case

A) Controllability via linearisation and fixed point arguments for state equations with A given by .

$$Ay = -\Delta y + f(y), \text{ or } Ay = -\Delta y + \operatorname{div} \mathbf{f}(y)$$

f, \mathbf{f} sublinear at the infinity,

$$|f(s)| \leq C_1 + C_2|s| \quad \forall s \in \mathbb{R}, |s| > \overline{M}$$

Henry (1976), Fabre, Puel, Zuazua (1992),...

B) Controllability via a penalized optimal control problem.

$$J_k(v) = \frac{1}{2} \|v\|_{L^2(O \times (0, T))}^2 + \frac{k}{2} \|y(T; v) - y_T\|_{L^2(\Omega)}^2$$

passing to the limit, as k increases to infinity.

(Idea of J.L. Lions, Málaga, 1991)

Non linear state equations 2. Superlinear case.

Obstruction

Semilinear equation: Henry (Bamberger), 1976,
energy method

Universal super and subsolutions over the exterior to the
control subdomains

Burger equation

J.I.Díaz, *Comptes Rendus de l'Academie des Sciences de Paris*, t. 1
312, Série I, 519-522, 1991, and J.I.Díaz and A.M.Ramos, *Revista de la
Real Academia de Ciencias Exactas, Físicas y Nat. de Madrid*, Tomo
LXXXIX, 11-30, 1995

H. Brezis and E.H. Lieb, Long Range Atomic
Potentials in Thomas-Fermi Theory, *Commun. Math.
Phys.*,. 65,(1979), 234-246.

Remark. Some related numerical experiences: control for EBM

J.I. Díaz and A. M. Ramos, In CD-Rom Proceedings European Congress Computational Methods in Applied Sciences and Engineering (ECCOMAS 2000).

$$y_t - y_{xx} + y^3 = \arctgy + u(t)\delta_{1/2} \text{ in } (0,1) \times (0,T),$$

$$y(0,t) = y(1,t) = 0 \quad t \in (0,T),$$

$$y(x,0) = y^0(x) \quad x \in (0,1),$$

$$J_k(u) = \frac{1}{2} \|u\|_{L^2(0,T)} + \frac{k}{2} \|y(T, \cdot; u) - y_d\|_{L^2(0,1)}$$

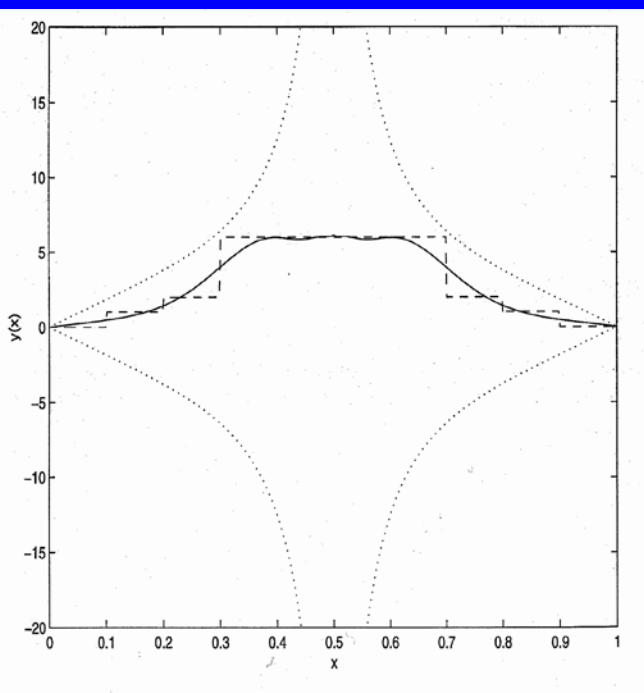
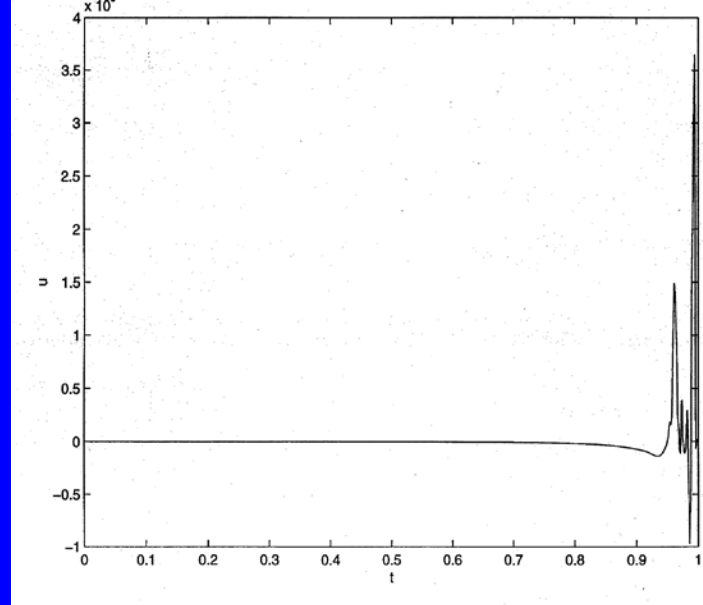
$$k = 10^{12}$$

The cost of control decreases with complexity:

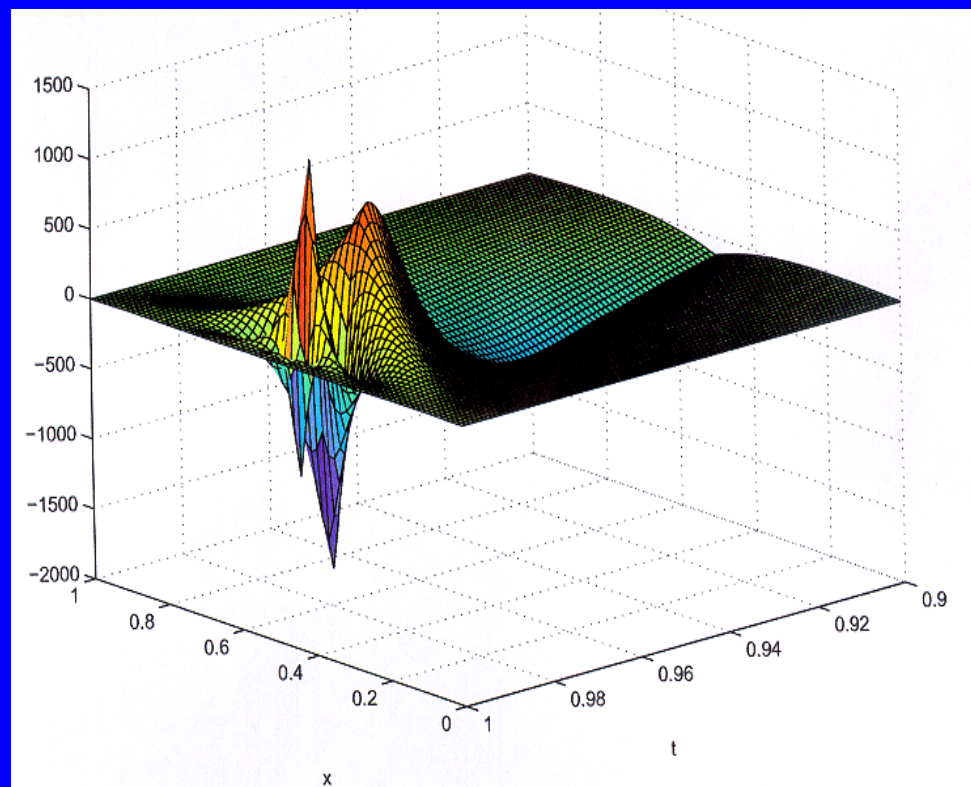
J.I. Díaz and J.L. Lions (Semilinear heat equation with blow-up)

CRAS, 2000, IFIP (Chemonix, 2000: Birkhauser)

control



State



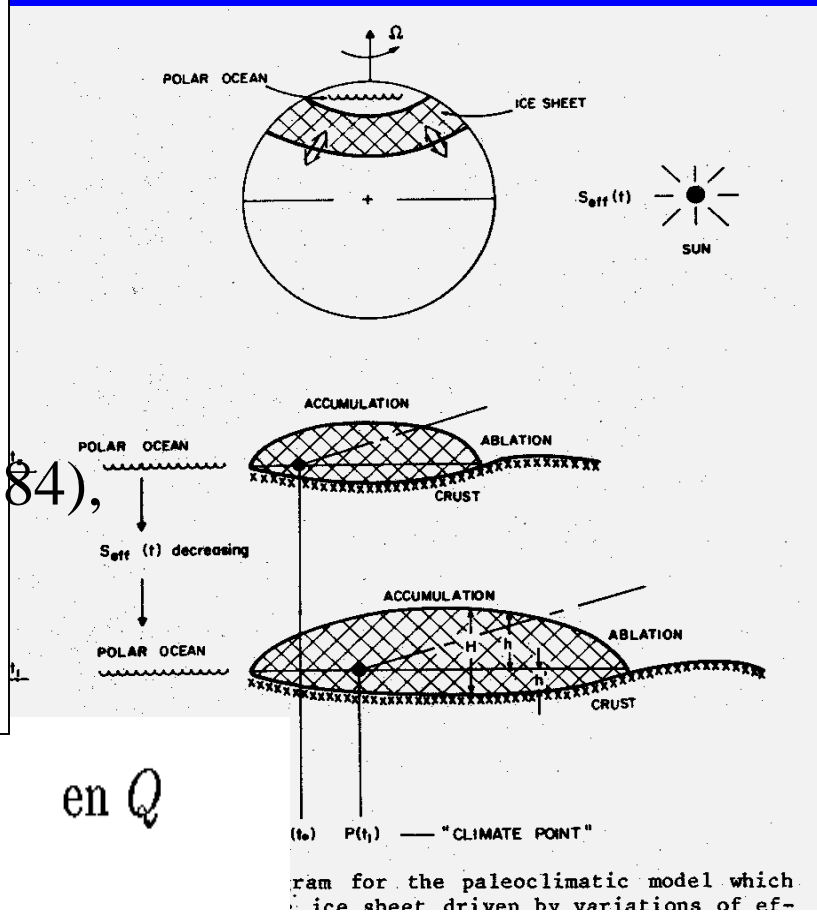
Target state

Coupling with ice sheets

Hutter (1983), Fowler (1992),
 Díaz-Schiavi (1995),
 Calvo-Durany-Vazquez (2000)

Solar Insolation

Imbrie-Imbrie (1980), Pollard (1982),
 Peltier-Hyde(1984), Watts-Hayder (1984),
 Oerlemans (1980), Denton (1988),
 Birchfield-Weertman (1978),...



$$\left\{ \begin{array}{l} h_t = \left(h^{n+2} |h_x|^{n-1} h_x \right)_x + a(t, x) \quad \text{en } Q \\ h = 0 \quad \text{en } \{S_-(t)\} \cup \{S_+(t)\} \quad t \in (0, T) \\ h(0, x) = h_0(x) \quad \text{en } D \end{array} \right.$$

- [21] J. I. DÍAZ AND E. SCHIAVI, *Tratamiento matemático de una ecuación parabólica cuasilínea degenerada en Glaciología*, in Electronic Proceedings of the XIV CEDYA-IV Congreso de Matemática Aplicada, Vic, Barcelona, 1995, <http://www.ma1.upc.es/cedya/cedya.html>.
- [22] J. I. DÍAZ AND E. SCHIAVI, *On a degenerate parabolic/hyperbolic system in glaciology giving rise to a free boundary*, *Nonlinear Anal.*, 38 (1999), pp. 649–673.

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ON A DOUBLY NONLINEAR PARABOLIC OBSTACLE PROBLEM MODELLING ICE SHEET DYNAMICS*

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Abstract. This paper deals with the weak formulation of a free (moving) boundary problem arising in theoretical glaciology. Considering shallow ice sheet flow, we present the mathematical



Milankovitch (1879-1958)

CICLOS DE MILANKOVITCH



Una oscilación como de peonza del eje de la Tierra crea un ciclo de 23.000 años.

Las variaciones en la órbita de la Tierra alrededor del Sol siguen un ciclo de 100.000 años.

